Subspace-based fault detection algorithms for vibration monitoring ⋆

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Abstract

We address the problem of detecting faults modeled as changes in the eigenstructure of a linear dynamical system. This problem is of primary interest for structural vibration monitoring. The purpose of the paper is to describe and analyze new fault detection algorithms, based on recent stochastic subspace-based identification methods and the statistical local approach to the design of detection algorithms. A conceptual comparison is made with another detection algorithm based on the instrumental variables identification method, and previously proposed by two of the authors.

Key words: Fault detection, subspace-based stochastic identification methods, statistical local approach, structural vibration monitoring.

1 Introduction

Before outlining the paper, we describe our motivations and introduce the models which we use throughout.

1.1 Motivations

The problem of fault detection and isolation (FDI) is a crucial issue which has been investigated with different types of approaches, as can be seen from the survey papers (Willsky,

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1976; Frank, 1990) and the books (Patton *et al.*, 1989; Basseville and Nikiforov, 1993) among other references. Moreover, an increasing interest in condition-based maintenance has appeared in a large number of industrial applications. The key idea there is to replace regular systematic inspections by condition-based inspections, i.e. inspections decided upon the continuous monitoring of the considered system (machine, structure, process or plant), based on the sensors data, in order to prevent from a possible malfunction or damage before it happens and optimize the maintenance cost. Condition-based maintenance typically involves the monitoring of components, and not only of sensors and actuators. It has been recognized that a relevant approach to condition-based maintenance consists in the *early detection* of *slight deviations* with respect to a (parametric) characterization of the system in usual working conditions, without artificial excitation, speeding down, or stop. Processing long samples of multi-sensor output measurements is often necessary for this purpose.

It turns out that, in many applications, the FDI problem of interest is to detect changes in the eigenstructure of a linear dynamical system. An important example is structural vibration monitoring (Basseville *et al.*, 1993). Vibrating structures are classically modeled as a continuous time linear system driven by an excitation, and whose output vector is the set of accelerometer measurements. Their vibrating characteristics (modes and modal shapes) coincide with the system eigenstructure. The key issue is to identify and monitor vibrating characteristics of mechanical structures subject to *unmeasured* and *non-stationary* natural excitation. Typical examples are offshore structures subject to swell, buildings subject to wind or earthquake, bridges, dams, wings subject to flutter in flight, and turbines subject to steam turbulence, friction in bearings, and imperfect balancing.

A relevant approach to the vibration monitoring problem has been shown to be based on the modeling of modes and modal shapes through state space representations (Prevosto *et al.*, 1991), the use of *output-only* and *covariance-driven* identification methods (such as instrumental variables or balanced realization algorithms) (Benveniste and Fuchs, 1985), and the computation of specific χ^2-type tests based on the so-called instrumental statistics (Basseville *et al.*, 1987). In practice, these tests turn out to be robust w.r.t. non-stationary excitation. Moreover, the design of these χ^2-tests is a particular case of a general approach to FDI which builds a detector from a convenient estimating function (Benveniste *et al.*, 1987; Zhang *et al.*, 1994; Basseville, 1998).

On the other hand, during the last decade there has been a growing interest in subspace-based linear system identification methods (Viberg, 1995; Van Overschee and De Moor, 1996) and their relationship with instrumental variables (IV) (Viberg *et al.*, 1997). These methods are well suited for capturing the system eigenstructure. Moreover, we know from practice that this family of methods is robust w.r.t. the non-stationarity of the zero part.

Because of what has been argued above, the question arises to design FDI algorithms based on this class of identification methods and to investigate their theoretical properties. It is the purpose of this paper to address these two issues, concentrating on the *detection* problem. The interested reader is referred to (Basseville *et al.*, 1987; Basseville, 1997; Basseville, 1998) for solutions to the *isolation* problem.

1.2 System models and parameters

We consider linear multi-variable systems described by a discrete-time state space model :

$$\begin{cases} X_{k+1} = F X_k + \varepsilon_k \\ Y_k = H X_k + \nu_k \end{cases} \tag{1}$$

where state X and observed output Y have dimensions m and r respectively. The state noise process $(\varepsilon_k)_k$ is an *unmeasured* Gaussian white noise sequence with zero mean. We assume noise ε_k to be stationary, that is of constant covariance matrix. The issue of robustness with respect to non-stationary excitation is addressed in subsection 2.5. The measurement noise process $(\nu_k)_k$ is assumed to be an *unmeasured* MA(ι) Gaussian sequence with zero mean. In the sequel, we use the notational convention that $\iota = -1$ for no measurement noise, and $\iota = 0$ for white (i.i.d.) measurement noise. Note that, with this MA assumption for its structure, measurement noise does *not* affect [3] the eigenstructure of system (1).

Let $G \stackrel{\text{def}}{=} \mathbf{E}\left(X_k Y_k^T\right)$ be the cross-correlation between state X_k and observation Y_k, and let :

$$\mathcal{O}_p = \begin{pmatrix} H \\ HF \\ \vdots \\ HF^{p-1} \end{pmatrix} \quad \text{and} \quad \mathcal{C}_p = \begin{pmatrix} G & FG & \dots & F^{p-1}G \end{pmatrix} \tag{2}$$

be the pth-order observability matrix of system (1) and controllability matrix of pair (F, G), respectively. We assume that, for p large enough, both observability and controllability matrices have full rank m.

The problem we consider is to monitor the observed system eigenstructure, that is the collection of m pairs (λ, ϕ_λ), where λ ranges over the set of eigenvalues of state transition matrix F, $\phi_\lambda = H\varphi_\lambda$ and φ_λ is the corresponding eigenvector. In all what follows, we assume that the system has no multiple eigenvalues, and thus that the λ's and φ_λ's are pairwise complex conjugate. In particular, 0 is *not* an eigenvalue of state transition matrix F. We stress that the collection of (λ, ϕ_λ) provides us with a canonical parameterization of the pole part of system (1). In particular, it does not depend on the state space basis (Benveniste and Fuchs, 1985). In the sequel, referring to vibration monitoring (Prevosto *et al.*, 1991), such a pair (λ, ϕ_λ) is called a mode. The set of the m modes is considered as the system

[3] The same would not hold true with an AR assumption for the measurement noise structure.

parameter θ :

$$\theta \stackrel{\text{def}}{=} \begin{pmatrix} \Lambda \\ \text{vec}\, \Phi \end{pmatrix} \tag{3}$$

In (3), Λ is the vector whose elements are the eigenvalues λ, Φ is the matrix whose columns are the ϕ_λ's, and vec is the column stacking operator. Parameter θ has size $(r+1)m$. The problem is to detect changes in parameter vector θ.

This detection problem can also be addressed using an input-output ARMA re-writing of state space model (1) (Akake, 1974) :

$$Y_k = \sum_{i=1}^{p} A_i \; Y_{k-i} + \sum_{j=0}^{p+\iota} B_j \; E_{k-j} \tag{4}$$

where $(E_k)_k$ is a Gaussian white noise sequence with zero mean and identity covariance matrix, and where the AR coefficients A_i are related to the pair (H, F) of model (1) via :

$$HF^p = \sum_{i=1}^{p} A_i \; HF^{p-i} \tag{5}$$

Note that the state noise process $(\varepsilon_k)_k$ in (1) is reflected only in the MA coefficients (Benveniste and Fuchs, 1985; Prevosto *et al.*, 1991). The problem is then to detect changes in the eigenstructure of a multi-variable ARMA process with unknown MA part.

We stress that models (1) and (4) are fully relevant w.r.t. the physics of vibration monitoring (Basseville *et al.*, 1987), and should not be considered as black-box models for this application.

1.3 Outline of the paper

Section 2 is devoted to the ideal case where the true system order is known and used in designing the tests. We first discuss system theoretic issues, we describe how subspace-based tests can be built, we properly formalize this design, based on the statistical local approach. Finally, we exhibit several particular tests, and discuss robustness w.r.t. nonstationary zeroes. Section 3 points out the remarkable and somewhat unexpected fact that all subspace-based FDI algorithms with known system order are equivalent. Section 4 provides additional elements for implementing the tests. Then, section 5 discusses how subspace tests should be adapted in the more realistic case of model reduction. Relation to other works is discussed in section 6, and section 7 concludes.

2 Subspace-based FDI algorithms - Known system order

Throughout this section, we assume that the true system order is known. Of course, this is somewhat unrealistic, but it allows us to introduce the key tools of our design of subspace-based tests. The more realistic case where the true system order is not known is discussed in section 5.

2.1 System theoretic issues

The arguments presented here are well known (Viberg *et al.*, 1997), we just state them here for the sake of clarity and completeness. We are given a sequence of covariances : $R_j \stackrel{\text{def}}{=} \mathbf{E}\left(Y_{k+j}\, Y_k^T\right)$ of output Y_k of a state space model (1). For $q \geq p+1$, let $\mathcal{H}_{p+1,q}$ be the block-Hankel matrix :

$$\mathcal{H}_{p+1,q} = \begin{pmatrix} R_{\iota+1} & R_{\iota+2} & \dots & R_{\iota+q} \\ R_{\iota+2} & R_{\iota+3} & \dots & R_{\iota+q+1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{\iota+p+1} & \dots & \dots & R_{\iota+p+q} \end{pmatrix} \tag{6}$$

As mentioned above, integer ι reflects the assumed correlation in the measurement noise sequence $(\nu_k)_k$. It should be considered as a design parameter for the proposed algorithms.

Choosing the eigenvectors of F as a basis[4] for the state space of model (1) yields the following particular representation of the observability matrix introduced in (2) :

$$\mathcal{O}_{p+1}(\theta) = \begin{pmatrix} \Phi \\ \Phi\Delta \\ \vdots \\ \Phi\Delta^p \end{pmatrix} \tag{7}$$

where diagonal matrix Δ is defined as $\Delta = \text{diag}(\Lambda)$, and Λ and Φ are as in (3). For any other state basis, the observability matrix $\mathcal{O}_{p+1}$ can be written as :

$$\mathcal{O}_{p+1} = \mathcal{O}_{p+1}(\theta)\, T \tag{8}$$

[4] This is called the modal basis in the vibration monitoring application (Basseville *et al.*, 1987).

for a suitable $m \times m$ invertible matrix T. Because of the definition of $\mathcal{H}_{p+1,q}$, $\mathcal{O}_p$ and $\mathcal{C}_q$ in (6) and (2), a direct computation of the R_j's from the model equations : $R_{\iota+j+1} = H\ F^{\iota+j+1}\ G$, for $j \geq 0$, leads to the following well known factorization property :

$$\mathcal{H}_{p+1,q} = \mathcal{O}_{p+1}\ \left(F^{\iota+1}\ \mathcal{C}_q\right) \tag{9}$$

From (9), (7) and (8), we get that the following property characterizes whether a nominal parameter θ_0 agrees with a given output covariance sequence $(R_j)_j$ (Viberg *et al.*, 1997) :

$$\mathcal{O}_{p+1}(\theta_0) \text{ and } \mathcal{H}_{p+1,q} \text{ have the same left kernel space}^{5} \text{ with co-rank } m. \tag{10}$$

Property (10) can be checked as follows :

(1) From θ_0 as in (3), form $\mathcal{O}_{p+1}(\theta_0)$, and pre-multiply it by some invertible weighting matrix W_1.

(2) Pick an orthonormal basis of the left kernel space [6] of matrix $W_1\mathcal{O}_{p+1}(\theta_0)$, in terms of the columns of some matrix S of co-rank m such that :

$$S^T\ S = I_s \tag{11}$$

$$S^T\ W_1\ \mathcal{O}_{p+1}(\theta_0) = 0 \tag{12}$$

Matrix S has dimensions $(p+1)r \times s$, where $s = (p+1)r - m$, and it is not unique; it can be obtained, for example, by the SVD-factorization of $W_1\mathcal{O}_{p+1}(\theta_0)$. Two such matrices are related through a post-multiplication with an orthonormal matrix U. Also, the choice of such a matrix depends on the weighting matrix W_1. We comment further on this below. We stress that, because of (12), matrix S depends implicitly on parameter θ_0. We bring the reader's attention to the fact that, even though S is not unique, in the sequel we treat it as a function of parameter θ, denoted by $S(\theta)$, which we fully justify in section 3.

(3) The parameter θ_0 which actually corresponds to the output covariance sequence $(R_j)_j$ of model (1) is characterized by :

$$S^T(\theta_0)\ W_1\ \mathcal{H}_{p+1,q}\ W_2^T = 0 \tag{13}$$

where W_2 is another invertible weighting matrix for choice. This matrix is introduced here by reference to usual subspace identification methods (Van Overschee and De Moor, 1996).

The choice of weighting matrix W_1 only influences the particular basis of the left kernel space of $\mathcal{O}_{p+1}(\theta_0)$, which is given by the rows of $S^T W_1$ where S is orthonormal (11). However, characterization (10) does not depend, of course, on the particular weighting matrices W_1 and W_2. We stress that the above stands only in the case of known system order. How this should be modified in the (more realistic) case of model reduction is discussed in section 5.

[6] The left kernel space of matrix M is the kernel space of matrix M^T.

2.2 *Subspace-based tests : a simple derivation*

We now explain how subspace-based detectors can be designed.

Assume we have at hand a nominal model θ_0, and newly collected data $Y_1, \ldots, Y_n$. Compute the empirical covariance sequence : $\widehat{R}_j \stackrel{\text{def}}{=} 1/(n-j)\ \sum_{k=1}^{n-j} Y_{k+j}\ Y_k^T$. Then, perform steps 1 and 2 of subsection 2.1, and replace step 3 by :

(3) Define the residual vector :

$$\zeta_n(\theta_0) \stackrel{\text{def}}{=} \sqrt{n}\ \text{vec}\left(S^T(\theta_0)\, W_1\, \widehat{\mathcal{H}}_{p+1,q}\, W_2^T\right) \tag{14}$$

where $\widehat{\mathcal{H}}_{p+1,q}$ is the empirical block-Hankel matrix obtained by substituting $\widehat{R}_j$ for R_j in (6).

From (13) we already know the following. Let θ denote the actual value of the parameter, for the system which generated the new data sample. Then :

$$\mathbf{E}_\theta\ (\zeta_n(\theta_0)) = 0 \quad \text{iff} \quad \theta = \theta_0 \tag{15}$$

Thus, it is natural to replace criterion (13) by the requirement that statistics $\zeta_n(\theta_0)$ should have zero mean, and consequently plays the role of a residual. Based on a data sample, testing if this hypothesis is valid requires the knowledge of the statistical properties of the residual $\zeta_n(\theta_0)$. This is addressed next.

2.3 *Subspace-based tests : theoretical design*

We need to know the distribution of $\zeta_n(\theta_0)$ when the actual parameter for the new data sample is θ. Unfortunately, this distribution is unknown, in general. One manner to circumvent this difficulty is to use a local approach, that is to assume close hypotheses :

$$(\textit{Safe system})\ \mathbf{H}_0 :\ \theta = \theta_0 \quad \text{and} \quad (\textit{Faulty system})\ \mathbf{H}_1 :\ \theta = \theta_0 + \Upsilon/\sqrt{n} \tag{16}$$

In (16), vector Υ is unknown, but fixed. Note that for large n, hypothesis $\mathbf{H}_1$ corresponds to *small* deviations in θ. As formally stated below, it turns out that residual ζ_n in (14) is asymptotically Gaussian distributed, under both $\mathbf{H}_0$ and $\mathbf{H}_1$, making easy its evaluation. More precisely, let $\mathbf{E}_\theta$ and cov_θ be the expectation and the covariance, respectively, when the actual system parameter is θ. We define the mean residual sensitivity :

$$M(\theta_0) \stackrel{\text{def}}{=} -\ 1/\sqrt{n}\ \ \partial/\partial\theta\ \mathbf{E}_{\theta_0}\ \zeta_n(\theta)\ |_{\theta=\theta_0} \tag{17}$$

$$= +\ 1/\sqrt{n}\ \ \partial/\partial\theta\ \mathbf{E}_\theta\ \zeta_n(\theta_0)|_{\theta=\theta_0} \tag{18}$$

and the residual covariance matrix : $\Sigma(\theta_0) \stackrel{\text{def}}{=} \lim_{n\to\infty} \mathbf{E}_{\theta_0} \left(\zeta_n\, \zeta_n^T\right)$. Matrix M is a Jacobian matrix, matrix Σ captures the uncertainty in ζ_n, (18) holds true because of (15), and the limit in the definition of Σ exists. The following central limit theorem (CLT) holds (Basawa, 1985; Benveniste *et al.*, 1987; Benveniste *et al.*, 1990; Zhang *et al.*, 1994; Delyon *et al.*, 1997).

Theorem 2.1 (CLT.) *Provided that $\Sigma(\theta_0)$ is positive definite, the residual ζ_n in (14) is asymptotically Gaussian distributed with the same covariance under both hypotheses in (16); that is :*

$$\zeta_n(\theta_0) \xrightarrow[n\to\infty]{} \begin{cases} \mathcal{N}(\ 0,\ \Sigma(\theta_0)\ ) \text{ under } \mathbf{H}_0 \\ \mathcal{N}(\ M(\theta_0)\,\Upsilon,\ \Sigma(\theta_0)\ ) \text{ under } \mathbf{H}_1 \end{cases}$$

A deviation in the system parameter θ is thus reflected into a change in the mean value of ζ_n. Note that matrices $M(\theta_0)$ and $\Sigma(\theta_0)$ depend on neither the sample size n nor the fault vector Υ in hypothesis $\mathbf{H}_1$. Thus we do not need to re-estimate them when testing the hypotheses, they can be estimated prior to testing, using data on the safe system (exactly as the reference parameter θ_0). As shown in section 4, Jacobian matrix M can be estimated from a data sample using a sample average. The estimation of covariance matrix Σ is more tricky (Zhang *et al.*, 1994). We recommend to use an empirical estimate based on a simple version of the jackknife method (Politis, 1998; Basseville *et al.*, 1993; Gach, 1991).

Let $\widehat{M}, \widehat{\Sigma}$ be consistent estimates of M, Σ. The detection problem, namely deciding that residual ζ_n is *significantly* different from zero, can be solved as follows [7].

Theorem 2.2 (χ^2-test.) *Assume additionally that Jacobian matrix $M(\theta_0)$ is full column rank (f.c.r.). Then the test between the hypotheses $\mathbf{H}_0$ and $\mathbf{H}_1$ defined in (16) is achieved through :*

$$\chi_n^2 \stackrel{\text{def}}{=} \zeta_n^T\, \widehat{\Sigma}^{-1}\, \widehat{M} \left(\widehat{M}^T\, \widehat{\Sigma}^{-1}\, \widehat{M}\right)^{-1} \widehat{M}^T\, \widehat{\Sigma}^{-1}\, \zeta_n \tag{19}$$

which should be compared to a threshold. In (19), the dependence on θ_0 has been dropped for simplicity. Test statistics χ_n^2 is asymptotically distributed as a χ^2-variable, with $\operatorname{rank}(M)$ *degrees of freedom, and with non-centrality parameter under $\mathbf{H}_1$: $\Upsilon^T\, M^T\, \Sigma^{-1}\, M\, \Upsilon$.*

2.4 IV-, BR- and CVA-based FDI algorithms

We now illustrate the above design of subspace-based FDI algorithms. First, we recall a FDI algorithm previously proposed by two of the authors (Basseville *et al.*, 1987), which is based

[7] Theorem 2.1 is also a basis for performing isolation (Benveniste *et al.*, 1987; Basseville, 1998; Basseville, 1997).

on the IV approach to eigenstructure identification. Then, we give two other examples of the family of subspace-based FDI algorithms introduced above, namely the BR and CVA-algorithms.

IV-based FDI algorithm. The statistics ζ_n based on the input-output representation (4), introduced in (Basseville *et al.*, 1987), falls in the category of the subspace-based FDI algorithms defined above. Actually, it is well known that the identification of the AR parameters of a multi-dimensional ARMA model (4) can be achieved by solving, in least squares sense if $q > p$, the following system of delayed Yule-Walker equations :

$$\begin{pmatrix} A^T & -I_r \end{pmatrix} \ \widehat{\mathcal{H}}_{p+1,q} = 0 \tag{20}$$

where $A^T \stackrel{\text{def}}{=} (A_p \ \ldots \ A_1)$. From the estimated AR matrices, a modal parameter θ (3) is deduced by eigen-analysis of (5). We stress that matrix A is an implicit function of θ, defined by:

$$\begin{pmatrix} A^T(\theta) & -I_r \end{pmatrix} \ \mathcal{O}_{p+1}(\theta) = 0 \tag{21}$$

which can easily be shown to write under the form (12) with a matrix S satisfying (11). To this end, we consider the QR-decomposition :

$$\begin{pmatrix} A(\theta) \\ -I_r \end{pmatrix} = \begin{pmatrix} S & \widetilde{S} \end{pmatrix} \ \begin{pmatrix} B \\ 0 \end{pmatrix} = S \ B \tag{22}$$

where matrices $(S \ \widetilde{S})$ and B are orthonormal and upper triangular, respectively. Of course, matrices $S, \widetilde{S}, B$ depend on θ. From the first equality, we deduce that $s = r$. From the second equality, we deduce that matrices S and B are orthonormal (namely $S^T S = I_r$) and invertible, respectively, and that (21) writes: $B^T \ S^T \ \mathcal{O}_{p+1}(\theta) = 0$. Turning back to (20) and (14), we get the IV-based test statistics: $\zeta_n(\theta_0) = \sqrt{n} \ \text{vec} \left(B^T(\theta_0) \ S^T(\theta_0) \ \widehat{\mathcal{H}}_{p+1,q} \right)$, with B and S defined in (22). The dimension of ζ is now $qrs = qr^2$. Note that, thanks to the invariance property stated below in section 3, the particular choice of invertible matrix B plays no role in the resulting χ^2-test.

BR- and CVA-based FDI algorithms. We now turn to the state-space model (1) and to subspace-based test statistics ζ_n of the form (14)-(12)-(11), and we discuss two particular choices of weighting matrices W_1 and W_2. The first choice corresponds to the BR-identification method, for which weights W_1 and W_2 are identity matrices. The second choice corresponds to the CVA-identification method (Akake, 1973; Arun and Kung, 1986; Van Overschee and

De Moor, 1996; Lindquist and Picci, 1996), for which weights W_1 and W_2 are given by : $W_1 = \mathcal{T}_{p+1,+}^{-1/2}, \quad W_2 = \mathcal{T}_{q,-}^{-1/2}$, where : $\mathcal{T}_{p+1,+} = \text{cov}\,\mathcal{Y}_{k,p+1}^{+}, \quad \mathcal{T}_{q,-} = \text{cov}\,\mathcal{Y}_{k,q}^{-}$, and where :

$$\mathcal{Y}_{k,p+1}^{+} \stackrel{\text{def}}{=} \begin{pmatrix} Y_k \\ \vdots \\ Y_{k+p} \end{pmatrix}, \; \mathcal{Y}_{k,q}^{-} \stackrel{\text{def}}{=} \begin{pmatrix} Y_{k-\iota-1} \\ \vdots \\ Y_{k-\iota-q} \end{pmatrix} \tag{23}$$

are the vectors containing the future and past data, respectively. Note that, thanks to the stationarity assumption, Hankel matrix writes : $\mathcal{H}_{p+1,q} = \mathbf{E}\left(\mathcal{Y}_{k,p+1}^{+}\,\mathcal{Y}_{k,q}^{-\,T}\right)$ for all k.

The main difference with the IV-based FDI algorithm is in the dimension of orthonormal matrix S, and thus in the number of degrees of freedom of the χ^2-test. This is further discussed in section 4. Additional differences might result from numerical issues on one hand, and from the effect of model reduction on the other one. The latter issue is addressed in section 5.

2.5 Subspace-based tests : robustness w.r.t. system zeroes

As discussed in (Basseville, 1998), theorem 2.1 is a particular case of a more general result on local tests (Benveniste *et al.*, 1987; Basseville and Nikiforov, 1993). Special theorems for the test related to the IV method are found in (Moustakides and Benveniste, 1986), which show that this method for eigenstructure monitoring is robust against possible time-variations of the covariance matrix of excitation noise ε_k in model (1), or, equivalently, against possible time-variations of the MA coefficients B_j in model (4). This is an important feature for vibration monitoring applications (Basseville *et al.*, 1993). Moreover, the same robustness for IV and BR *identification* methods was proved in (Benveniste and Fuchs, 1985).

This, together with experiments in vibration monitoring (Abdelghani *et al.*, 1999; Basseville *et al.*, 1986), supports the evidence that subspace-based methods for eigenstructure monitoring are robust w.r.t. the non-stationarity of system zeroes.

3 Invariance of subspace-based FDI algorithms - Known order

We now discuss the role of the design matrices in the three steps of the construction of residual $\zeta_n(\theta_0)$ in (14), assuming known system order. The weighting matrix W_1 is introduced in step 1. In step 2, we perform the SVD of $W_1\mathcal{O}_{p+1}(\theta_0)$ and collect the left singular vectors associated with singular value 0. This specifies a particular choice for $S^T(\theta_0)$, and other matrices S satisfying (11)-(12) are then obtained by post-multiplying $S(\theta_0)$ by an orthonormal matrix U. The weighting matrix W_2 is introduced in step 3. We make these three design choices explicit in the following notation :

$$\zeta_{n;U,W_1,W_2}(\theta_0) \stackrel{\text{def}}{=} \sqrt{n} \ \text{vec}\left(U^T \ S^T(\theta_0)\, W_1 \ \widehat{\mathcal{H}}_{p+1,q} \ W_2^T\right) \tag{24}$$

A straightforward calculation shows that:

$$\zeta_{n;U,W_1,W_2} = (W_2 \otimes V_1) \ \zeta_{n;I,I,I} \ , \tag{25}$$

where V_1 is any invertible matrix [8] such that: $U^T S^T(\theta_0) W_1 = V_1 S^T(\theta_0)$, and $\otimes$ is Kronecker product. Let $\chi^2_{n;U,W_1,W_2}$ be the χ^2-test (19) associated with $\zeta_{n;U,W_1,W_2}$. From (25), we get that : $\chi^2_{n;U,W_1,W_2} = \chi^2_{n;I,I,I}$, since the invertible matrix $(W_2 \otimes V_1)$ factors out in (19). Thus, for eigenstructure monitoring, *all the proposed subspace-based methods are equivalent,* when using the true system order.

4 Estimation and rank of the Jacobian matrix

In this section, we investigate the estimation and the rank of the Jacobian matrix $M(\theta)$ of a residual of the form (24). Two methods for estimating this Jacobian can be considered, corresponding to (17) and (18), respectively.

4.1 First estimation method

From (17), we get :

$$M(\theta_0) = - \ \left(W_2 \ \mathcal{H}^T_{p+1,q} \ W_1^T \otimes U^T\right) \ \ \partial \mathcal{S}(\theta)/\partial\theta|_{\theta=\theta_0} \tag{26}$$

where $\mathcal{S} \stackrel{\text{def}}{=} \text{vec}(S^T)$, and $\partial \mathcal{S}(\theta)/\partial\theta|_{\theta=\theta_0}$ is obtained by differentiating (11)-(12) with respect to θ_0. Writing $\mathcal{O}'_{p+1}(\theta) \stackrel{\text{def}}{=} \partial \,\text{vec}\, \mathcal{O}_{p+1}(\theta)/\partial\theta$, we easily get that $\partial \mathcal{S}(\theta)/\partial\theta$ is solution of the system :

$$\begin{aligned} \left(\mathcal{O}^T_{p+1}(\theta) \ W_1^T \otimes I_s\right) \ \partial \mathcal{S}(\theta)/\partial\theta + \ \left(I_m \otimes S^T(\theta) \ W_1\right) \ \mathcal{O}'_{p+1}(\theta) &= 0 \\ \left(I_s \otimes S^T(\theta)\right) \ \partial \mathcal{S}(\theta)/\partial\theta &= 0 \end{aligned}$$

Even though a solution $\partial \mathcal{S}(\theta)/\partial\theta$ is non unique, the right hand-side of (26) is unique, since: $\text{Ker}\left(\mathcal{O}^T_{p+1}(\theta) \ W_1^T \otimes I_s\right) \subset \text{Ker}\left(W_2 \ \mathcal{H}^T_{p+1,q} \ W_1^T \otimes U^T\right)$. A consistent estimate $\widehat{M}$ for $M(\theta_0)$,

[8] For example, $V_1 = U^T \ S^T(\theta_0) \ W_1 \ S(\theta_0)$.

based on a data sample and a solution $\partial \mathcal{S}(\theta)/\partial\theta|_{\theta=\theta_0}$, is obtained by substituting $\widehat{\mathcal{H}}$ for $\mathcal{H}$ in (26). The unicity of $\widehat{M}$ is then guaranteed only asymptotically.

4.2 Second estimation method

From (18), we get : $M(\theta_0) = \partial/\partial\theta \ \text{vec}\left(U^T \ S^T(\theta_0) \, W_1 \, \mathcal{H}_{p+1,q} \, W_2^T\right)\Big|_{\theta=\theta_0}$, which also writes :

$$M(\theta_0) = \partial/\partial\theta \ \text{vec}\left(U^T \ S^T(\theta_0) \, W_1 \, \mathcal{O}_{p+1}(\theta) \, F^{\iota+1} \, \mathcal{C}_q \, W_2^T\right)\Big|_{\theta=\theta_0} \tag{27}$$

thanks to factorization (9) of the Hankel matrix. Hence :

$$\begin{aligned} M(\theta) &= \left(W_2 \, \mathcal{C}_q^T \ F^{\iota+1\,T} \otimes U^T \ S^T(\theta) \, W_1\right) \ \mathcal{O}'_{p+1}(\theta) \\ &= \left(W_2 \otimes U^T \ S^T(\theta) \, W_1\right) \ \left(\mathcal{C}_q^T \ F^{\iota+1\,T} \otimes I_{(p+1)r}\right) \ \mathcal{O}'_{p+1}(\theta) \\ &= \left(W_2 \otimes U^T \ S^T(\theta) \, W_1\right) \ \left(\mathcal{H}_{p+1,q}^T \ \mathcal{O}_{p+1}^{\dagger\,T}(\theta) \otimes I_{(p+1)r}\right) \ \mathcal{O}'_{p+1}(\theta) \end{aligned} \tag{28}$$

The last equality, where $\mathcal{O}_{p+1}^{\dagger}(\theta)$ is the pseudo-inverse of $\mathcal{O}_{p+1}(\theta)$, is obtained using factorization (9) again. A consistent estimate $\widehat{M}$, based on a data sample, is obtained by substituting $\widehat{\mathcal{H}}$ for $\mathcal{H}$ in (28). Some comments are in order on formula (28), which also apply to $\widehat{M}$. First, it is easily checked that $M(\theta)$ does *not* depend on the particular choice of the eigenvectors ϕ_λ contained in the parameter vector θ defined in (3). Multiplying the ϕ_λ's by constant complex numbers amounts to post-multiply observability matrix $\mathcal{O}_{p+1}(\theta)$ in (7) by an invertible diagonal matrix D, to post-multiply matrix $\mathcal{O}_{p+1}^{\dagger\,T}(\theta)$ by D^{-1}, and to pre-multiply matrix $\mathcal{O}'_{p+1}(\theta)$ by $(D \otimes I_{(p+1)r})$. And all the terms in D cancel out in (28). Second, matrix $\mathcal{O}'_{p+1}(\theta)$ is full rank $(r+1)m$, as can be checked directly from the following explicit formula :

$$\mathcal{O}'_{p+1}(\theta) = \left(\begin{array}{ccc|ccc} \Lambda_1'^{(p)} \otimes \phi_1 & & 0 & \Lambda_1^{(p)} \otimes I_r & & 0 \\ & \ddots & & & \ddots & \\ 0 & & \Lambda_m'^{(p)} \otimes \phi_m & 0 & & \Lambda_m^{(p)} \otimes I_r \end{array}\right)$$

where : $\Lambda_i^{(p)\,T} \stackrel{\text{def}}{=} (\, 1 \ \lambda_i \ \lambda_i^2 \ \ldots \ \lambda_i^p \,)$, $\Lambda_i'^{(p)\,T} \stackrel{\text{def}}{=} (\, 0 \ 1 \ 2 \ \lambda_i \ \ldots \ p \ \lambda_i^{p-1} \,)$, for $1 \leq i \leq m$.

Conditions regarding the rank of $M(\theta_0)$ are as follows.

4.3 Conditions ensuring full rank

Formula (27) also writes :

$$M(\theta_0) = \underbrace{\left(W_2\, \mathcal{C}_q^T\, F^{\iota+1\,T} \otimes U^T\right)}_{A} \underbrace{\left(\partial/\partial\theta \ \mathrm{vec}(S^T(\theta_0)\, W_1\, \mathcal{O}_{p+1}(\theta))\Big|_{\theta=\theta_0}\right)}_{B}$$

$$\stackrel{\mathrm{def}}{=}$$

where matrices A and B have dimensions $qrs \times sm$ and $sm \times (r+1)m$, respectively. A necessary and sufficient condition for M to be f.c.r. is : B f.c.r. and $\mathrm{Ker}\, A \cap \mathrm{Range}\, B = 0$.

Now, for matrix B to be f.c.r., it is necessary that: $s \geq r+1$. Moreover, since matrices W_2, $\mathcal{C}_q^T F^{\iota+1\,T}$ and U are invertible, f.c.r. and orthonormal, respectively, matrix A is f.c.r. Thus, a *necessary and sufficient* condition for matrix M to be f.c.r. is : B f.c.r. A *necessary* condition is : $sm \geq (r+1)m = \mathrm{size}\ \theta$.

Note that, in the case of the IV-based FDI algorithm described in subsection 2.4, we have $s = r$, and thus Jacobian matrix M is *not* f.c.r.

5 Subspace-based FDI algorithms - Model reduction

Most practical situations correspond to the case where actual data are generated by a system of higher order than that of the nominal model θ_0. Thus, the nominal model has reduced order. Then, a new question arises : what does it mean, for a nominal model θ_0, to match a given data sample when model reduction is enforced ? Of course, system theoretic characterization (10), or (13), is no longer valid, and the same is true for the definition of the residual in (14). Other definitions are needed, which we introduce now.

5.1 Revisiting the nominal model characterization

Since $\mathrm{rank}\, \mathcal{H}_{p+1,q} > \mathrm{rank}\, \mathcal{O}_{p+1}(\theta_0) \stackrel{\mathrm{def}}{=} m$, condition (10) for perfect matching cannot be satisfied. What can be required, instead, is that the left kernel space of $\mathcal{O}_{p+1}(\theta_0)$ is orthogonal to the m-th order principal subspace [9] of $\mathcal{H}_{p+1,q}$.

Steps 1 and 2 of subsection 2.1 are unchanged, but step 3 is modified accordingly, and

[9] The m-th order principal subspace of a matrix is the subspace spanned by the left singular vectors associated with its m largest singular values.

reformulated as follows :

(3) SVD-decompose $W_1\ \mathcal{H}_{p+1,q}\ W_2^T$ as

$$W_1\ \mathcal{H}_{p+1,q}\ W_2^T = (P_m\quad \overline{P_m})\ D\ V^T \tag{29}$$

where P_m collects the first m left singular vectors. Then, (13) is replaced by :

$$\begin{aligned} S^T(\theta_0)\ P_m\ D\ V^T = 0\ , &\quad \text{equivalently rewritten as} \\ S^T(\theta_0)\ W_1\ \mathcal{H}_{p+1,q}\ W_2^T = \Big(0 &\quad S^T(\theta_0)\ \overline{P_m}\ D\ V^T\Big) \end{aligned} \tag{30}$$

Using the properties of the SVD, characterization (29)-(30) is equivalent to :

$$\theta_0 = \arg\min_{\theta}\ \sigma_{\max}\left(S^T(\theta)\ W_1\ \mathcal{H}_{p+1,q}\ W_2^T\right) \tag{31}$$

where $\sigma_{\max}(A)$ denotes the maximal singular value of matrix A. It is important to note that $\sigma_{\max}(S^T(\theta)\ W_1\ \mathcal{H}_{p+1,q}\ W_2^T)$ is independent of the particular choice of matrix S, as discussed for introducing (24).

5.2 Revisiting the residuals

The definition of the residuals in subsection 2.2 should be modified accordingly. Point 3 of this subsection should be rephrased as follows :

(3) SVD decompose $W_1\ \widehat{\mathcal{H}}_{p+1,q}\ W_2^T$ as: $W_1\ \widehat{\mathcal{H}}_{p+1,q}\ W_2^T = (\widehat{P}_m\quad \overline{\widehat{P}_m})\ \widehat{D}\ \widehat{V}^T$, and then set :

$$\zeta_n(\theta_0) \stackrel{\text{def}}{=} \sqrt{n}\ S^T(\theta_0)\ \widehat{P}_m\ \widehat{D}\ \widehat{V}^T$$

Then, the rest of subsections 2.2 and 2.3 applies, substituting everywhere $\widehat{P}_m\widehat{D}\widehat{V}^T$ for $W_1\widehat{\mathcal{H}}_{p+1,q}W_2^T$. This includes theorems 2.1 and 2.2, which hold true without any further modification.

5.3 Revisiting the role of weighting

As formulas (29)-(30)-(31) reveal, the characterization of model-to-data matching now *depends* on the choice of the design matrices W_1 and W_2. And so does the subspace test itself. In particular, section 3 is no longer valid in the case of model reduction. Not surprisingly, selecting W_1 and W_2 as in the CVA method amounts to testing for the first m canonical correlation coefficients to be zero.

6 Relation to other works

Now we discuss how the subspace-based FDI algorithms proposed in this paper relate to the well known parity space approach to FDI (Chow and Willsky, 1984) and to another subspace-based detection algorithm recently proposed (Gustafsson, 1996).

6.1 Relation to the parity space approach

As explained in (Chow and Willsky, 1984; Mironovski, 1979; Basseville, 1997), FDI of additive *sensor* (or actuator) faults in linear dynamic systems is achieved through residuals :

$$\zeta = \mathcal{G}\, \mathcal{Y}^{+}_{k,p+1} \tag{32}$$

called parity checks, where $\mathcal{Y}^{+}_{k,p+1}$ is defined in (23) and matrix $\mathcal{G}$ is such that :

$$\mathcal{G}\, \mathcal{O}_{p+1} = 0 \tag{33}$$

Residuals ζ in (32), based on constraint (33), are basically aimed at rejecting the unknown state X in the relation : $\mathcal{Y}^{+}_{k,p+1} = \mathcal{O}_{p+1}\, X_{k-p}$+fault vector+noise vector. By definition, parity checks (32) are first order (linear) residuals – because linear combinations of the output *data*.

Now, the left-hand side of constraint (33) is exactly in the same form as (12). The (realization theory) property of identical left kernel spaces (10) could lead us to use, as often reported in the literature, the parity checks ζ defined in (32) for FDI of *system* faults also, and not only for sensor faults. But, from statistical inference theory, it is known that for performing inference about second-order characteristics – here, (modal) vibrating characteristics, or equivalently the eigenstructure of a linear state space system –, it is mandatory to use (at least) second-order statistics, that is covariances. Using linear combinations of the output data is *not* sufficient (in the statistical sense). Note that the subspace-based residuals defined in (14), and based on constraints (12), are actually linear combination of the output *covariances*, and indeed are aimed at monitoring the system eigenstructure.

6.2 Relation to another subspace-based detection algorithm

The problem addressed in (Gustafsson, 1996) is to detect changes in the pole *and* the zero parts of a transfer function from the inputs and the innovation towards the output of a multivariable system, whereas ours is to detect changes in the pole parts of the transfer function from a possibly non-stationary excitation towards the output of a multivariable system. The design of the detection algorithm is based on subspace identification methods, nicely re-written as prediction error methods. As a consequence, since the subspace-based formulation is not directly used, no analysis of the possible effect of different weighting matrices is performed, whereas in this paper we provide an invariance result in section 3.

The test statistics in (Gustafsson, 1996) are based on the correlation between this prediction error and the past outputs, whereas ours (24), which we re-write :

$$\zeta_{n;U,W_1,W_2}(\theta_0) = \sqrt{n}\ W_2\ \mathcal{Y}^{-}_{k-p,q}\ \otimes\ U^T\ S^T(\theta_0)\ W_1\ \mathcal{Y}^{+}_{k-p,p+1}$$

are based on the correlation between past outputs and a linear combination of future outputs *different* from the prediction error. In particular, for the IV-related test, the correlation between past outputs and the MA part is used.

One interesting feature of the approach in (Gustafsson, 1996) is that it does not use any canonical parameterization, whereas the modal parameterization does play a key role in our intended application to vibration monitoring. As a consequence, since no parameter is introduced, no analysis of the effect of the change on the test statistics is performed, whereas in this paper we provide an explicit formula for the mean value of the test statistics under the alternative through the CLT theorem 2.1 and the formulas for Jacobian matrix M.

7 Conclusion

In this paper, we have systematically investigated how subspace-based methods for eigenstructure identification can be turned into fault detection methods. This work is based on the general approach advocated by two of the authors for designing algorithms to detect slight deviations from nominal behavior in dynamical systems. Our experience (Basseville *et al.*, 1993; Gach, 1991; Abdelghani *et al.*, 1999; Hermans *et al.*, 1999; Mevel *et al.*, 1999) suggests that subspace-based methods for eigenstructure identification and testing are useful for in-operation modal analysis and monitoring of mechanical structures subject to vibration. An industrial version of the corresponding set of algorithms is currently under development by the company LMS-International, Leuven, Belgium.

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