FREE VIBRATIONS OF AXISYMMETRIC SHELLS :
PARABOLIC AND ELLIPTIC CASES

MARIE CHAUSSADE-BEAUDOUIN, MONIQUE DAUGE, ERWAN FAOU, AND ZOHAR YOSIBASH

ABSTRACT. Approximate eigenpairs (quasimodes) of axisymmetric thin elastic domains with laterally clamped boundary conditions (Lamé system) are determined by an asymptotic analysis as the thickness (2ε) tends to zero. The departing point is the Koiter shell model that we reduce by asymptotic analysis to a scalar model that depends on two parameters: the angular frequency k and the half-thickness ε. Optimizing k for each chosen ε, we find power laws for k in function of ε that provide the smallest eigenvalues of the scalar reductions. Corresponding eigenpairs generate quasimodes for the 3D Lamé system by means of several reconstruction operators, including boundary layer terms. Numerical experiments demonstrate that in many cases the constructed eigenpair corresponds to the first eigenpair of the Lamé system.

Geometrical conditions are necessary to this approach: The Gaussian curvature has to be non-negative and the azimuthal curvature has to dominate the meridian curvature in any point of the midsurface. In this case, the first eigenvector admits progressively larger oscillation in the angular variable as ε tends to 0. Its angular frequency exhibits a power law relation of the form $k = \gamma\varepsilon^{-\beta}$ with $\beta = \frac{1}{4}$ in the parabolic case (cylinders and trimmed cones), and the various βs $\frac{2}{5}$, $\frac{3}{7}$, and $\frac{1}{3}$ in the elliptic case. For these cases where the mathematical analysis is applicable, numerical examples that illustrate the theoretical results are presented.

1. INTRODUCTION

A shell is a 3D body determined by a surface $\mathcal{S} \subset \mathbb{R}^3$ and a (small) parameter ε: Such a body is obtained by thickening $\mathcal{S}$ on either side by ε along a unit normal field $\mathbf{N}$ to $\mathcal{S}$ (see (2.1) supra for a precise definition). The shell, denoted by Ω^ε is supposed to be made of a linear homogeneous isotropic material. We investigate the first free vibration mode of such a shell when its lateral boundary is clamped and $\varepsilon \to 0$. A vibration mode is the eigenpair $(\lambda, \mathbf{u})$ where λ is the square of the eigenfrequency and $\mathbf{u}$ the eigen-displacement. The mode is denoted as exact if it is an eigenpair of the 3D Lamé system in Ω^ε complemented by suitable boundary conditions, see equations (2.2)–(2.7) for an explicit formulation. The thin domain limit $\varepsilon \to 0$ pertains to "shell theory".

Shell theory consists of finding surface models, i.e., systems of equations posed on $\mathcal{S}$, approximating the 3D Lamé system when ε tends to 0. This approach was started for plates (the case when $\mathcal{S}$ is flat) by Kirchhoff, Reissner and Mindlin see for instance [24, 34, 30] respectively. When the structure is a genuine shell for which the midsurface has nonzero curvature, the problem is even more difficult and was first tackled in the seminal works of Koiter, John, Naghdi and Novozhilov in the sixties [25, 26, 27, 23, 32, 31]. A large literature developed afterwards aimed at laying

more rigorous mathematical bases to shell theory see for instance the works of Sanchez-Palencia, Sanchez-Hubert [36, 37, 38, 35], Ciarlet, Lods, Mardare, Miara [11, 13, 12, 28] and the book [9], and more recently Dauge, Faou [20, 21, 15]. Most of these works apply to the static problem, and the results strongly depend on the geometrical nature of the shell (namely parabolic, elliptic or hyperbolic according to the Gaussian curvature K of $\mathcal{S}$ being zero, positive or negative).

Much fewer works were devoted to free vibrations of thin shells. Plates were addressed before-hand, see [10, 14]. To the best of our knowledge, all theoretical works devoted to the asymptotic analysis of eigenmodes in thin elastic shells were associated with a surface model, as the Koiter model [25, 26]:

$$\mathfrak{K}(\varepsilon) = \mathfrak{M} + \varepsilon^2 \mathfrak{B}, \tag{1.1}$$

where $\mathfrak{M}$ is the membrane operator, $\mathfrak{B}$ the bending operator, and ε the half-thickness of the shell. These two operators are 3×3 systems posed on $\mathcal{S}$, thus with two tangent variables, but acting on 3-component vector fields ζ. At this point, the vector fields have to be represented in surface fitted components, the tangential and normal components ζ_α and ζ_3. The membrane operator $\mathfrak{M}$ is of order 2 on tangential components ζ_α, but of order 0 on the normal component ζ_3 (for plates, $\mathfrak{M}$ does not act on ζ_3 at all). The bending operator $\mathfrak{B}$ has a complementary role: It is order 4 on ζ_3 (for plates, $\mathfrak{B}$ is a multiple of the biharmonic operator Δ^2 acting on the sole normal component). We will briefly recall these operators in section 2.2.

In [35], the essential spectrum of the membrane operator $\mathfrak{M}$ (the set of λ's such that $\mathfrak{M} - \lambda$ is not Fredholm) was characterized in the elliptic, parabolic, and hyperbolic cases. The series of papers by Artioli, Beirão Da Veiga, Hakula and Lovadina [7, 2, 3] investigated the first eigenvalue of models like $\mathfrak{K}(\varepsilon)$. Effective results hold for *axisymmetric* shells with *clamped* lateral boundary: Defining the order α of a positive function $\varepsilon \mapsto \lambda(\varepsilon)$, continuous on $(0, \varepsilon_0]$, by the conditions

$$\forall \eta > 0, \quad \lim_{\varepsilon \to 0^+} \lambda(\varepsilon)\, \varepsilon^{-\alpha+\eta} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \lambda(\varepsilon)\, \varepsilon^{-\alpha-\eta} = \infty \tag{1.2}$$

they proved that $\alpha = 0$ in the elliptic case, $\alpha = 1$ for parabolic case, and $\alpha = \frac{2}{3}$ in the hyperbolic case.

Concentrating on axisymmetric shells, it is possible to apply a full Fourier decomposition with respect to the azimuthal angle φ, allowing all eigenpairs to be classified by their azimuthal frequency $k(\varepsilon)$. For such shells Beirao et al. and Artioli et al. [7, 2, 3] investigated by numerical simulations the azimuthal frequency $k(\varepsilon)$ of the first eigenvector of $\mathfrak{K}(\varepsilon)$: Like in the phenomenon of *sensitivity* [35], the lowest eigenvalues are associated with eigenvectors with growing angular frequencies and $k(\varepsilon)$ exhibits a negative power law of type $\varepsilon^{-\beta}$, for which [3] identifies the exponents $\beta = \frac{1}{4}$ for cylinders (see also [6] for some theoretical arguments), $\beta = \frac{2}{5}$ for a particular family of elliptic shells, and $\beta = \frac{1}{3}$ for another particular family of hyperbolic shells.

Similarly to the aforementioned publications, we consider here axisymmetric shells whose mid-surface $\mathcal{S}$ is parametrized by a smooth positive function f representing the radius as a function of the axial variable (see Figure 1):

$$\begin{aligned} F: \quad \mathcal{I} \times \mathbb{T} \quad &\longrightarrow \quad\quad\quad\quad \mathcal{S} \\ (z, \varphi) \quad &\longmapsto \quad (f(z)\cos\varphi,\; f(z)\sin\varphi,\; z). \end{aligned} \tag{1.3}$$

Here $\mathcal{I}$ is a bounded interval and $\mathbb{T}$ is the torus $\mathbb{R}/2\pi\mathbb{Z}$. We focus on cases when sensitivity shows
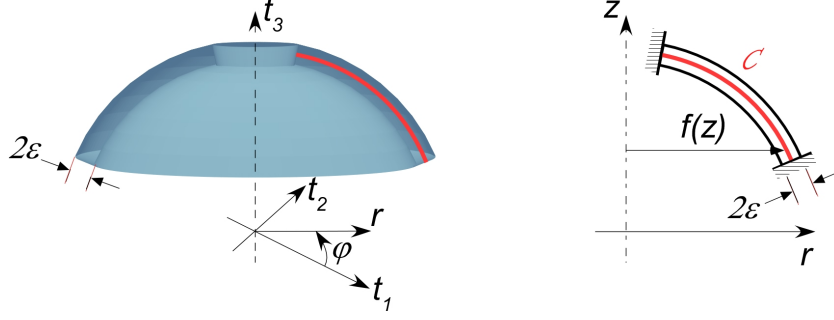
FIGURE 1. An axisymmetric shell with notations.

up, i.e., when the azimuthal frequency $k(\varepsilon)$ of the first eigenvector tends to infinity as the thickness tends to 0. As will be shown, the rules driving this phenomenon are far to be straightforward, and depend in a non trivial manner on the geometry of the shell: In the sole elliptic case, we show that there exist at least three distinct power laws for $k(\varepsilon)$. This is the expression of some *bending* effects and may sound as a paradox since for elliptic shells the membrane is an elliptic system in the sense of Agmon, Douglis and Nirenberg [1], see [22]. However, there also exist elliptic shells for which $k(\varepsilon)$ remains constant, see the computations for a spherical cap in [16].

1.1. **High frequency analysis.** Our departing point is a high frequency analysis of the membrane operator $\mathfrak{M}$ on surfaces $\mathcal{S}$ with a parametrization of type (1.3). By the Fourier decomposition naturally induced by the cylindrical symmetry, we define $\mathfrak{M}^k$ as the membrane operator acting at the frequency $k \in \mathbb{N}$ and we perform a *scalar reduction* of the eigenproblem by a special factorization in a formal series algebra in powers of the small parameter $\frac{1}{k}$. This mathematical tool, developed for cylindrical shells in the PhD thesis [5] of the first author, reduces the original eigenproblem (which is a 3×3 system) to a *scalar* eigenproblem posed on the transverse component of the displacement. This way, we can construct in a variety of parabolic and elliptic cases a new explicit scalar differential operator $\mathfrak{H}^k$ whose first eigenvalue $\mu_1(\mathfrak{H}^k)$ has a computable asymptotics as $k \to \infty$

$$\mu_1(\mathfrak{H}^k) = \mathsf{h}_0 + \mathsf{h}_1 k^{-\eta_1} + \mathcal{O}(k^{-\eta_2}), \quad 0 < \eta_1 < \eta_2. \tag{1.4}$$

In (1.4) all coefficients and exponents depend on shell's geometry, i.e. on the function f in (1.3). This leads to a *quasimode construction* for $\mathfrak{M}^k$ that is valid for all parabolic shells of type (1.3) and all elliptic shells with azimuthal curvature dominating. The operator $\mathfrak{H}^k$ strongly depends on the nature of the shell:

$$\begin{cases} \mathfrak{H}^k = k^{-4}\mathsf{H}_4 & \text{in the parabolic case (i.e., when } f'' = 0), \\ \mathfrak{H}^k = \mathsf{H}_0 + k^{-2}\mathsf{H}_2 & \text{in the elliptic case (i.e., when } f'' < 0). \end{cases} \tag{1.5}$$

with explicit operators H_0, H_2 and H_4, cf. sect. 6.1 for formulas. Let us mention at this point that for hyperbolic shells such a suitable scalar reduction $\mathfrak{H}^k$ cannot be found.

This membrane scalar reduction induces a Koiter-like scalar reduced operator $\mathfrak{A}(\varepsilon)$ for the shell that we define at the frequency k by

$$\mathfrak{A}^k(\varepsilon) = \mathfrak{H}^k + \varepsilon^2 k^4 \mathsf{B}_0 \tag{1.6}$$

where the function B_0 is positive and explicit (k^4 corresponding to the leading order in the Fourier expansion of the bending operator $\mathfrak{B}$). Then we shall show that the first eigenvalue of $\mathfrak{A}(\varepsilon)$ is the infimum on all angular frequencies of the first eigenvalues of $\mathfrak{A}^k(\varepsilon)$:

$$\mu_1(\mathfrak{A}(\varepsilon)) = \inf_{k \in \mathbb{N}} \mu_1(\mathfrak{A}^k(\varepsilon)). \tag{1.7}$$

In all relevant parabolic cases (i.e., cylinders and cones) and a variety of elliptic cases, we prove in this paper:

(i) The infimum in (1.7) is reached for $k = k(\varepsilon)$ satisfying a power law of the form[1]

$$k(\varepsilon) = \gamma \varepsilon^{-\beta}, \tag{1.8}$$

with β depending only on f and γ positive. The exponent β is calculated so to equilibrate $k^{-\eta_1}$ (cf. (1.4)) and $\varepsilon^2 k^4 \equiv k^{4-2/\beta}$, which yields:

$$\boxed{\beta = \frac{2}{4 + \eta_1}} \tag{1.9}$$

(ii) The smallest eigenvalue of the reduced scalar model $\mathfrak{A}(\varepsilon)$ has an asymptotic expansion of the form, as $\varepsilon \to 0$

$$\mu_1(\mathfrak{A}(\varepsilon)) = a_0 + a_1 \varepsilon^{\alpha_1} + \mathcal{O}(\varepsilon^{\alpha_2}), \quad 0 < \alpha_1 < \alpha_2, \tag{1.10}$$

where a_0 coincides with the coefficient h_0 present in (1.4) and α_1 is given by the formula (replace k with $\varepsilon^{-\beta}$ into the term $k^{-\eta_1}$ in (1.4))

$$\boxed{\alpha_1 = \eta_1 \beta = \frac{2\eta_1}{4 + \eta_1}} \tag{1.11}$$

(iii) The corresponding eigenvector $\zeta^1(\varepsilon)$ has a leading term that involves another exponent denoted by δ which characterizes its *concentration scale* with respect to ε. The value $\delta = 0$ means no concentration at all, and $\delta > 0$ means that $\zeta^1(\varepsilon)$ develops layers around some points $z_0 \in \overline{\mathcal{I}}$. The latter situation occurs when the reduced membrane operator $\mathfrak{H}^k$ has a singular perturbation character as $k \to \infty$.

Once the asymptotic expansions for the Koiter scalar reduced operator $\mathfrak{A}(\varepsilon)$ is resolved we construct quasimodes for the full Koiter model $\mathfrak{K}(\varepsilon)$. Then, by energy estimates linking 2D and 3D models similar to those of [15], we find an asymptotic upper bound $m_1(\varepsilon)$ for the first eigenvalue of the 3D Lamé system in the shell. This upper bound is given by the first two terms in (1.10):

$$m_1(\varepsilon) = a_0 + a_1 \varepsilon^{\alpha_1}. \tag{1.12}$$

To make the analysis more complete, we perform numerical simulations showing that, in a number of cases, the upper bound $m_1(\varepsilon)$ is actually a good approximation of the true first eigenvalue. To this end computations are performed at three different levels:

(1D) We calculate a_0, a_1 of (1.10) and γ of (1.8). We either use explicit formulas when available (such as the cylinder and specific elliptic cases), or compute numerically the spectrum of the 1D reduced operators $\mathfrak{A}^k(\varepsilon)$ by 1D finite element methods.

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[1]This equality has to be understood up to a $\mathcal{O}(1)$ term coming from rounding to the nearest integer.

(2D) Using a Fourier spectral method we compute the eigenvalues of the three-dimensional operator on the cylindrical shell by decomposing the operator in Fourier, which yields 2D operators acting on the meridian domain for all k. For a given thickness ε, we compute the lowest eigenvalue of these operators for a collection of k which provides the lowest eigenvalue of the three-dimensional operator and the value of $k(\varepsilon)$.

(3D) We compute the eigenvalues of the Lamé system using 3D finite element methods.

1.2. **Specification in the parabolic and elliptic cases.** In the Lamé system we use the engineering notations of the material parameters: E is the Young modulus and ν is the Poisson ratio. The shells to which our analysis apply are uniquely defined by f in (1.3). The inverse parametrization (the axial variable function of the radius) would not provide distinct cases where our analysis is applicable.

• The parabolic cases are those for which $f'' = 0$ on $\mathcal{I}$. So f is affine. We classify parabolic cases in two types:

   (1) **'Cylinder'** f is constant;
   (2) **'Cone'** f is affine and not constant.

• The elliptic cases are those for which $f'' < 0$ on $\mathcal{I}$. To conduct our analysis, we assume moreover that the azimuthal curvature dominates the meridian curvature (admissible cases), which amounts to

$$1 + f'^2 + ff'' \geq 0. \tag{1.13}$$

Then elliptic cases are discriminated by the behavior of the function H_0 that is the first term of the scalar reduction $\mathfrak{H}^k$, cf. (1.5),

$$\mathsf{H}_0 = E\frac{f''^2}{(1 + f'^2)^3}. \tag{1.14}$$

We classify admissible elliptic cases in three types:

   (1) **'Toroidal'** H_0 is constant.
   (2) **'Gauss'** H_0 is not constant and reaches its minimum at z_0 inside $\mathcal{I}$ and not on its boundary $\partial\mathcal{I}$, with the exception of cases for which H_0'' or $1 + f'^2 + ff''$ are zero at z_0.
   (3) **'Airy'** H_0 is not constant and reaches its minimum at z_0 in the boundary $\partial\mathcal{I}$, with the exception of cases for which H_0' or $1 + f'^2 + ff''$ are zero at z_0.

We summarize in Table 1 our main theoretical results on the exponents η_1, β, α_1, on the azimuthal frequency $k(\varepsilon)$ and the scale δ of the quasi-eigenvector, and on the quasi-eigenvalue (qev) $\mathsf{m}_1(\varepsilon)$. The exponents α of [2] are confirmed. The names of models used for numerical simulations are also mentioned in this table, whereas in Figure 2 we represent these models in their 3D version for $\varepsilon = 0.2$.

Our paper divides in three parts: The first part is devoted to the setting of the problem and to basics on shells: After the present introduction, we revisit in sect. 2 the linear shell theory in general with a brief introduction of the 3D and 2D problems, and in sect. 3 we particularize formulas for axisymmetric shells. The second part gathers our theoretical study involving the scalar reduction and its first eigenpair: In sect. 4 we set the principles of the high frequency analysis, in sect. 5 and

Type (Model)	η_1	β	α_1	a_0	a_1	$k(\varepsilon)$	δ	$\mathsf{m}_1(\varepsilon)$	α [2]
PARABOLIC									
'Cylinder' (A)	4	$\frac{1}{4}$	1	0	explicit wrt 1D ev's	$\gamma\varepsilon^{-1/4}$	0	$\mathsf{a}_1\varepsilon$	1
'Cone' (B)	4	$\frac{1}{4}$	1	0	optimization of 1D ev's	$\gamma\varepsilon^{-1/4}$	0	$\mathsf{a}_1\varepsilon$	—
ELLIPTIC									
'Toroidal' (D)	2	$\frac{1}{3}$	$\frac{2}{3}$	H_0	optimization of 1D ev's	$\gamma\varepsilon^{-1/3}$	0	$\mathsf{a}_0 + \mathsf{a}_1\varepsilon^{2/3}$	0
'Gauss' (H)	1	$\frac{2}{5}$	$\frac{2}{5}$	$\mathsf{H}_0(z_0)$	explicit	$\gamma\varepsilon^{-2/5}$	$\frac{1}{2}$	$\mathsf{a}_0 + \mathsf{a}_1\varepsilon^{2/5}$	0
'Airy' (L)	$\frac{2}{3}$	$\frac{3}{7}$	$\frac{2}{7}$	$\mathsf{H}_0(z_0)$	explicit	$\gamma\varepsilon^{-3/7}$	$\frac{2}{3}$	$\mathsf{a}_0 + \mathsf{a}_1\varepsilon^{2/7}$	0

TABLE 1. Summary of exponents η_1, β, α_1, frequency $k(\varepsilon)$, scale δ, and qev $\mathsf{m}_1(\varepsilon)$
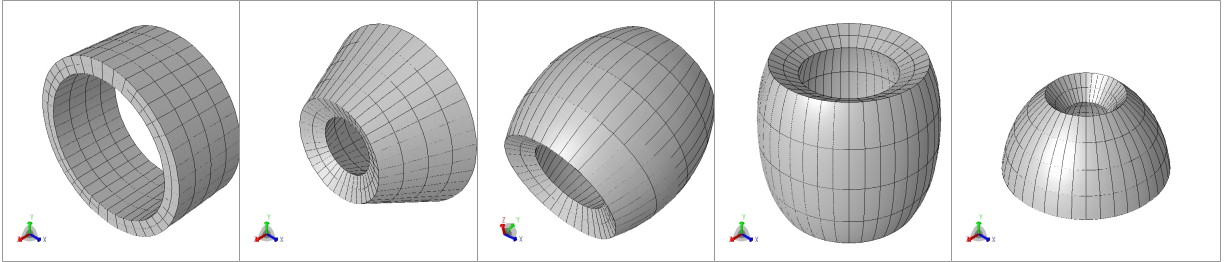
FIGURE 2. The five models A, B, D, H, L, used for computations (here $\varepsilon = 0.2$).

6 we address more particularly the parabolic and elliptic cases, respectively. Finally in the last part we present numerical experiments (sect. 7) addressing a model for each of the five main types described above. We conclude in sect. 8. We provide in Appendix A details on the factorization in formal series leading to the scalar reduction and in Appendix B variational formulations in the meridian domain after Fourier decomposition of the 3D Lamé system.

1.3. **Overview of main notation.** To relieve the complexity of notation, we gather here some definitions relating to coordinate systems, operators, and spectrum.

1.3.1. *Coordinates.* We use three systems of coordinates:

• Cartesian coordinates $\boldsymbol{t} = (t_1, t_2, t_3) \in \mathbb{R}^3$ with coordinate vectors $\mathbf{E}_{t_1}$, $\mathbf{E}_{t_2}$, $\mathbf{E}_{t_3}$.

• Cylindrical coordinates $(r, \varphi, \tau) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}$ related to Cartesian coordinates by relations

$$(t_1, t_2, t_3) = \mathscr{T}(r, \varphi, \tau) \quad \text{with} \quad t_1 = r\cos\varphi, \; t_2 = r\sin\varphi, \; t_3 = \tau. \qquad (1.15)$$

The coordinate vectors associated with the transformation $\mathscr{T}$ are $\mathbf{E}_r = \partial_r\mathscr{T}$, $\mathbf{E}_\varphi = \partial_\varphi\mathscr{T}$, and $\mathbf{E}_\tau = \partial_\tau\mathscr{T}$. We have

$$\mathbf{E}_r = \mathbf{E}_{t_1}\cos\varphi + \mathbf{E}_{t_2}\sin\varphi, \quad \mathbf{E}_\varphi = -r\mathbf{E}_{t_1}\sin\varphi + r\mathbf{E}_{t_2}\cos\varphi, \quad \text{and} \quad \mathbf{E}_\tau = \mathbf{E}_{t_3}. \qquad (1.16)$$

• Normal coordinates (x_1, x_2, x_3), specified as (z, φ, x_3) in our case. Such coordinates are related to the surface $\mathcal{S}$ and a chosen unit normal field $\mathbf{N}$ to $\mathcal{S}$. The variable x_3 is the coordinate along $\mathbf{N}$.

The variables (x_1, x_2), specified as (z, φ) in our case, parametrize the surface. The full transformation $\mathscr{F} : (z, \varphi, x_3) \mapsto (t_1, t_2, t_3)$ sends the product $\mathcal{I} \times \mathbb{T} \times (-\varepsilon, \varepsilon)$ onto the shell Ω^ε and is explicitly given by

$$t_1 = \left(f(z) + x_3 \tfrac{1}{s(z)}\right) \cos \varphi, \quad t_2 = \left(f(z) + x_3 \tfrac{1}{s(z)}\right) \sin \varphi, \quad t_3 = z - x_3 \tfrac{f'(z)}{s(z)}, \qquad (1.17)$$

where $s = \sqrt{1 + f'^2}$. The restriction of $\mathscr{F}$ on the surface $\mathcal{S}$ (corresponding to $x_3 = 0$) gives back F (1.3). The coordinate vectors associated with the transformation $\mathscr{F}$ are $\partial_z \mathscr{F} =: \mathbf{E}_z$, $\partial_\varphi \mathscr{T}$ that coincides with $\mathbf{E}_\varphi$ above, and $\partial_3 \mathscr{F} =: \mathbf{E}_3$. On the surface $\mathcal{S}$, $x_3 = 0$ and $\mathbf{E}_3$ coincides with $\mathbf{N}$, whereas $\mathbf{E}_z$ and $\mathbf{E}_\varphi$ are tangent to $\mathcal{S}$.

These three systems of coordinates determine the contravariant components of a displacement $\mathbf{u}$ in each of these systems by identities

$$\mathbf{u} = \mathsf{u}^{t_1} \mathbf{E}_{t_1} + \mathsf{u}^{t_2} \mathbf{E}_{t_2} + \mathsf{u}^{t_3} \mathbf{E}_{t_3} = \mathsf{u}^r \mathbf{E}_r + \mathsf{u}^\varphi \mathbf{E}_\varphi + \mathsf{u}^\tau \mathbf{E}_\tau = \mathsf{u}^z \mathbf{E}_z + \mathsf{u}^\varphi \mathbf{E}_\varphi + \mathsf{u}^3 \mathbf{E}_3 . \qquad (1.18)$$

The cylindrical and normal systems of coordinates are suitable for angular Fourier decomposition $\mathbb{T} \ni \varphi \mapsto k \in \mathbb{Z}$. The Fourier coefficient of rank k of a function u is denoted by u^k

$$\mathsf{u}^k = \frac{1}{2\pi} \int_0^{2\pi} \mathsf{u}(\varphi) \, e^{-ik\varphi} \, \mathrm{d}\varphi.$$

For functions on Ω^ε, the Fourier coefficients are defined on the meridian domain $\omega^\varepsilon \subset \mathbb{R}^2$ of Ω^ε.

For a 3D displacement $\mathbf{u}$ defined on Ω^ε or a surface displacement $\boldsymbol{\zeta}$ defined on $\mathcal{S}$, we have first to expand them in a suitable system of coordinates (cylindric or normal) and then calculate Fourier coefficients of their components, for instance

$$\mathbf{u}^k = (\mathsf{u}^r)^k \mathbf{E}_r + (\mathsf{u}^\varphi)^k \mathbf{E}_\varphi + (\mathsf{u}^\tau)^k \mathbf{E}_\tau \quad \text{with} \quad (\mathsf{u}^r)^k(r, \tau) = \frac{1}{2\pi} \int_0^{2\pi} \mathsf{u}^r(r, \varphi, \tau) \, e^{-ik\varphi} \, \mathrm{d}\varphi, \quad \text{etc...}$$

or

$$\boldsymbol{\zeta}^k = (\zeta^z)^k \mathbf{E}_z + (\zeta^\varphi)^k \mathbf{E}_\varphi + (\zeta^3)^k \mathbf{N} \quad \text{with} \quad (\zeta^z)^k(z) = \frac{1}{2\pi} \int_0^{2\pi} \zeta^z(z, \varphi) \, e^{-ik\varphi} \, \mathrm{d}\varphi, \quad \text{etc...}$$

1.3.2. *Operators.* We manipulate a collection of operators and their Fourier symbols. On the shell Ω^ε we have the Lamé system $\mathfrak{L}^\varepsilon$ acting on 3D displacements $\mathbf{u}$. After angular Fourier decomposition, we obtain the family of 3×3 operators $(\mathfrak{L}^\varepsilon)^k$ defined on the meridian domain ω^ε. On the surface $\mathcal{S}$ we have the membrane, bending and Koiter operators $\mathfrak{M}$, $\mathfrak{B}$ and $\mathfrak{K}(\varepsilon)$. They act on 3-component surface displacements $\boldsymbol{\zeta}$. On the meridian curve $\mathcal{C}$ of $\mathcal{S}$, we have the corresponding families $\mathfrak{M}^k$, $\mathfrak{B}^k$ and $\mathfrak{K}^k(\varepsilon)$. Finally, on the meridian curve $\mathcal{C}$, we have our scalar reductions $\mathfrak{H}^k$ and $\mathfrak{A}^k(\varepsilon)$ acting on functions η. We go from a higher model to a lower one by reduction, and the converse way by reconstruction. For instance we go from $\mathbf{u}$ to $\boldsymbol{\zeta}$ by restriction to $\mathcal{S}$. The converse way uses the reconstruction operator U (2.14). We go from $\boldsymbol{\zeta}^k$ to η^k by selecting the normal component of $\boldsymbol{\zeta}^k$. The converse way uses the reconstruction operators $\mathbf{V}[k]$ that we will construct.

1.3.3. *Spectrum.* We denote by $\sigma(\mathfrak{A})$ and $\sigma_{\mathsf{ess}}(\mathfrak{A})$ the spectrum and the essential spectrum of a selfadjoint operator $\mathfrak{A}$, respectively, which means the set of λ's such that $\mathfrak{A} - \lambda$ is not invertible and not Fredholm, respectively.

2. ESSENTIALS ON SHELL THEORY

Recall that Cartesian coordinates of a point $\mathbf{P} \in \mathbb{R}^3$ are denoted by $\boldsymbol{t} = (t_1, t_2, t_3)$. A shell Ω^ε is a three-dimensional object defined by its midsurface $\mathcal{S}$ and its thickness parameter ε in the following way: We assume that $\mathcal{S}$ is smooth and orientable, so that there exists a smooth unit normal field $\mathbf{P} \mapsto \mathbf{N}(\mathbf{P})$ on $\mathcal{S}$ and so that for $\varepsilon > 0$ small enough the following map is one to one and smooth

$$
\begin{aligned}
\Phi: \quad \mathcal{S} \times (-\varepsilon, \varepsilon) &\rightarrow \qquad \Omega^\varepsilon \\
(\mathbf{P}, x_3) &\mapsto \quad \boldsymbol{t} = \mathbf{P} + x_3\, \mathbf{N}(\mathbf{P}).
\end{aligned} \tag{2.1}
$$

The boundary of Ω^ε has two parts:

(1) Its lateral boundary $\partial_0 \Omega^\varepsilon := \Phi\big(\partial\mathcal{S} \times (-\varepsilon, \varepsilon)\big)$,
(2) The rest of its boundary (natural boundary) $\partial_1 \Omega^\varepsilon := \partial\Omega^\varepsilon \setminus \partial_0\Omega^\varepsilon$.

2.1. **3D vibration modes.** On the domain Ω^ε, we consider the Lamé operator associated with an isotropic and homogeneous material with Young coefficient E and Poisson coefficient ν. This means that the material tensor is given by

$$
A^{ijk\ell} = \frac{E\nu}{(1+\nu)(1-2\nu)} \delta^{ij}\delta^{k\ell} + \frac{E}{2(1+\nu)}(\delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}). \tag{2.2}
$$

For *clamped* boundary conditions the variational space is

$$
V(\Omega^\varepsilon) := \{\mathbf{u} = (\mathsf{u}_{t_1}, \mathsf{u}_{t_2}, \mathsf{u}_{t_3}) \in H^1(\Omega^\varepsilon)^3, \quad \mathbf{u} = 0 \quad \text{on} \quad \partial_0\Omega^\varepsilon\}. \tag{2.3}
$$

For a given displacement field $\mathbf{u}$ let $e_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i \mathsf{u}_{t_j} + \partial_j \mathsf{u}_{t_i})$ be the strain tensor, where ∂_i stands for the partial derivative with respect to t_i. The energy scalar product between two displacements $\mathbf{u}$ and $\mathbf{u}^*$ is given by

$$
a^\varepsilon_{\mathrm{3D}}(\mathbf{u}, \mathbf{u}^*) = \int_{\Omega^\varepsilon} A^{ijk\ell} e_{ij}(\mathbf{u})\, e_{k\ell}(\mathbf{u}^*)\, \mathrm{d}\Omega^\varepsilon, \tag{2.4}
$$

using the summation convention of repeated indices. The three-dimensional modal problem can be written in variational form as:

Find $(\mathbf{u}, \lambda)$ *in* $V(\Omega^\varepsilon) \times \mathbb{R}$ *with* $\mathbf{u} \neq 0$ *such that*

$$
\forall \mathbf{u}^* \in V(\Omega^\varepsilon), \quad a^\varepsilon_{\mathrm{3D}}(\mathbf{u}, \mathbf{u}^*) = \lambda \int_{\Omega^\varepsilon} \mathsf{u}^{t_i}\mathsf{u}^*_{t_i}\, \mathrm{d}\Omega^\varepsilon. \tag{2.5}
$$

The strong formulation of (2.5) can be written as

$$
\mathfrak{L}^\varepsilon \mathbf{u} = \lambda \mathbf{u}^\varepsilon, \tag{2.6}
$$

where $\mathfrak{L}^\varepsilon$ is the Lamé system

$$
\mathfrak{L}^\varepsilon = -\frac{E}{2(1+\nu)(1-2\nu)}\Big((1-2\nu)\Delta + \nabla\,\mathrm{div}\Big) \quad \text{on} \quad \Omega^\varepsilon, \tag{2.7}
$$

associated with Dirichlet BC's on $\partial_0\Omega^\varepsilon$ and natural BC's on the rest of the boundary. Its spectrum $\sigma(\mathfrak{L}^\varepsilon)$ is discrete and positive. So its eigenvalues in (2.5) can be ordered as an increasing sequence $(\lambda^\varepsilon_n)_{n \geq 1}$

$$
0 < \lambda^\varepsilon_1 \leq \lambda^\varepsilon_2 \leq \cdots \leq \lambda^\varepsilon_n \leq \cdots
$$

It would be desired to obtain an asymptotic series as $\varepsilon \to 0$ for any chosen $n \geq 1$ of λ_n^ε, at least in the axisymmetric case. This has been realized for plates [14]:

$$\lambda_n^\varepsilon = \varepsilon^2 \Lambda_n + \mathcal{O}(\varepsilon^3) \tag{2.8}$$

where Λ_n is the n-th bending eigenvalue. Presently we cannot achieve this goal in other situations. We thus concentrate on estimates for the first eigenvalue that is obtained by the minimum Rayleigh quotient

$$\lambda_1^\varepsilon = \min_{\mathbf{u} \in V(\Omega^\varepsilon)} \frac{a_{3\mathrm{D}}^\varepsilon(\mathbf{u}, \mathbf{u})}{\|\mathbf{u}\|_{L^2(\Omega^\varepsilon)}^2} \,.$$

In this paper we perform two tasks:

[A] Obtain asymptotic upper bounds for λ_1^ε via the construction of quasimodes for reduced models and reconstruction operators,

[B] Compute numerical approximation of the smallest eigenpair in (2.5) by finite element methods.

2.2. **2D shell models.** The 2D reduction to the midsurface $\mathcal{S}$, namely the membrane and bending operators, are defined via intrinsic geometrical objects attached to $\mathcal{S}$. To introduce them, we need generic parametrizations $F : (x_\alpha)_{\alpha \in 1,2} \to \boldsymbol{t}$ acting from maps neighborhoods $\mathcal{V}$ into the midsurface $\mathcal{S}$. Associated tangent coordinate vector fields are

$$\mathbf{E}_\alpha = \partial_\alpha \mathscr{F}, \quad \alpha = 1, 2, \quad \text{with} \quad \partial_\alpha = \frac{\partial}{\partial x_\alpha}.$$

Completed by the unit normal field $\mathbf{N}$ they form a basis $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{N}\}$ in each point of $\mathcal{S}$. The metric tensor $(a_{\alpha\beta})$ and the curvature tensor $(b_{\alpha\beta})$ are given by

$$a_{\alpha\beta} = \langle \mathbf{E}_\alpha, \mathbf{E}_\beta \rangle \quad \text{and} \quad b_{\alpha\beta} = \langle \partial_{\alpha\beta} \mathscr{F}, \mathbf{N} \rangle \,.$$

Denoting by $(a^{\alpha\beta})$ the inverse of $(a_{\alpha\beta})$, the curvature (symmetric) matrix is defined by

$$(b_\beta^\alpha) \quad \text{with} \quad b_\beta^\alpha = a^{\alpha\gamma} b_{\gamma\beta}.$$

The eigenvalues κ_1 and κ_2 of the matrix (b_β^α) are called the *principal curvatures* of $\mathcal{S}$ and their product is the *Gaussian curvature* K. Here comes the classification of shells: If $K \equiv 0$, the shell is *parabolic*, if $K > 0$, the shell is *elliptic*, if $K < 0$, the shell is *hyperbolic*. Finally let R denote the minimal radius of curvature of $\mathcal{S}$

$$R = \inf_{\mathbf{P} \in \mathcal{S}} \left\{ \min\{|\kappa_1(\mathbf{P})|^{-1}, |\kappa_2(\mathbf{P})|^{-1}\} \right\}. \tag{2.9}$$

The basis $\{\mathbf{E}_\alpha, \mathbf{N}\}$ determines contravariant components (ζ^α, ζ^3) of a vector field $\boldsymbol{\zeta}$ on $\mathcal{S}$:

$$\boldsymbol{\zeta} = \zeta^{t_i} \mathbf{E}_{t_i} = \zeta^\alpha \mathbf{E}_\alpha + \zeta^3 \mathbf{N} \,.$$

The covariant components are (ζ_α, ζ_3) with $\zeta_\alpha = a_{\alpha\beta} \zeta^\beta$ and $\zeta_3 = \zeta^3$. The 2D rigidity tensor on $\mathcal{S}$ is given by

$$M^{\alpha\beta\sigma\delta} = \frac{\nu E}{1 - \nu^2} a^{\alpha\beta} a^{\sigma\delta} + \frac{E}{2(1 + \nu)} (a^{\alpha\sigma} a^{\beta\delta} + a^{\alpha\delta} a^{\beta\sigma}).$$

Note that, even if $\mathcal{S}$ is flat ($a^{\alpha\beta} = \delta^{\alpha\beta}$), M is different than the 3D rigidity tensor A.

2.2.1. *Membrane operator.* The variational space associated with the membrane operator is
$$V_{\mathsf{M}}(\mathcal{S}) = H_0^1(\mathcal{S}) \times H_0^1(\mathcal{S}) \times L^2(\mathcal{S}). \tag{2.10}$$
For an element $\boldsymbol{\zeta} = (\zeta_\alpha, \zeta_3)$ in $V_{\mathsf{M}}(\mathcal{S})$, the change of metric tensor $\boldsymbol{\gamma} = \gamma_{\alpha\beta}(\boldsymbol{\zeta})$ is given by
$$\gamma_{\alpha\beta}(\boldsymbol{\zeta}) = \tfrac{1}{2}(\mathrm{D}_\alpha\zeta_\beta + \mathrm{D}_\beta\zeta_\alpha) - b_{\alpha\beta}\zeta_3,$$
where D_α is the covariant derivative on $\mathcal{S}$, see [18, 19, 39]. The membrane energy scalar product is defined as
$$a_{\mathsf{M}}(\boldsymbol{\zeta}, \boldsymbol{\zeta}^*) = \int_{\mathcal{S}} M^{\alpha\beta\sigma\delta}\gamma_{\alpha\beta}(\boldsymbol{\zeta})\,\gamma_{\sigma\delta}(\boldsymbol{\zeta}^*)\,\mathrm{d}\mathcal{S}\,.$$
Here the volume form $\mathrm{d}\mathcal{S}$ is $\sqrt{|\det(a_{\alpha\beta})|}\,\mathrm{d}x_1\mathrm{d}x_2$. The variational formulation of the modal problem associated with the membrane operator $\mathfrak{M}$ is given by

Find $(\boldsymbol{\zeta}, \Lambda)$ with $\boldsymbol{\zeta} \in V_{\mathsf{M}}(\mathcal{S}) \setminus \{0\}$ and $\Lambda \in \mathbb{R}$ such that for all $\boldsymbol{\zeta}^* \in V_{\mathsf{M}}(\mathcal{S})$,
$$a_{\mathsf{M}}(\boldsymbol{\zeta}, \boldsymbol{\zeta}^*) = \Lambda \int_{\mathcal{S}} (\zeta^\beta \zeta_\beta^* + \zeta^3 \zeta_3^*)\,\mathrm{d}\mathcal{S}. \tag{2.11}$$

2.2.2. *Bending operator and Koiter model.* The variational space associated with the bending operator is
$$V_{\mathsf{B}}(\mathcal{S}) = H_0^1(\mathcal{S}) \times H_0^1(\mathcal{S}) \times H_0^2(\mathcal{S}). \tag{2.12}$$
For an element $\boldsymbol{\zeta} = (\zeta_\alpha, \zeta_3)$ in $V_{\mathsf{B}}(\mathcal{S})$, the change of curvature tensor $\boldsymbol{\rho} = \rho_{\alpha\beta}(\boldsymbol{\zeta})$ is given by
$$\rho_{\alpha\beta}(\boldsymbol{\zeta}) = \mathrm{D}_\alpha\mathrm{D}_\beta\zeta_3 + \mathrm{D}_\alpha(b_\beta^\delta\zeta_\delta) + b_\alpha^\delta\mathrm{D}_\beta\zeta_\delta - b_\alpha^\delta b_{\beta\delta}\zeta_3.$$
The bending operator $\mathfrak{B}$ acts on the variational space $V_{\mathsf{B}}(\mathcal{S})$ and its energy scalar product is
$$a_{\mathsf{B}}(\boldsymbol{\zeta}, \boldsymbol{\zeta}^*) = \frac{1}{3}\int_{\mathcal{S}} M^{\alpha\beta\sigma\delta}\rho_{\alpha\beta}(\boldsymbol{\zeta})\rho_{\sigma\delta}(\boldsymbol{\zeta}^*)\,\mathrm{d}\mathcal{S}\,.$$
For any positive ε, the Koiter operator $\mathfrak{K}(\varepsilon)$ is defined as $\mathfrak{M} + \varepsilon^2\mathfrak{B}$. It can be shown, see [8], that $\mathfrak{K}(\varepsilon)$ is elliptic with multi-order on $V_{\mathsf{B}}(\mathcal{S})$ in the sense of Agmon-Douglis-Nirenberg [1]. The corresponding physical energy scalar product is
$$a_{2\mathsf{D}}^\varepsilon(\boldsymbol{\zeta}, \boldsymbol{\zeta}^*) = 2\varepsilon\,a_{\mathsf{M}}(\boldsymbol{\zeta}, \boldsymbol{\zeta}^*) + 2\varepsilon^3 a_{\mathsf{B}}(\boldsymbol{\zeta}, \boldsymbol{\zeta}^*). \tag{2.13}$$

2.3. **Reconstruction operators from 2D to 3D.** The parametrizations of the midsurface induce local system of normal coordinates (x_α, x_3) inside the shell and, correspondingly, the covariant components u_α and u_3 of a displacement $\mathbf{u}$. The rationale of the shell theory is to deduce by an explicit procedure a solution $\mathbf{u}$ of the 3D Lamé system posed on the shell from a 2D solution $\boldsymbol{\zeta}$ of the Koiter model posed on the midsurface. This is done via a *reconstruction operator* U, cf [26, 27] and [15]. With any 2D displacement $\boldsymbol{\zeta}(x_\alpha)$ defined on the midsurface $\mathcal{S}$, U associates a 3D displacement $\mathbf{u}$ depending on the three coordinates (x_α, x_3) in Ω^ε. The operator U is defined by
$$\mathsf{U} = \mathsf{T} \circ \mathsf{W} \tag{2.14}$$
where W is the shifted reconstruction operator
$$\mathsf{W}\boldsymbol{\zeta} = \begin{cases} \zeta_\sigma - x_3(\mathrm{D}_\sigma\zeta_3 + b_\sigma^\alpha\zeta_\alpha), \\[2mm] \zeta_3 - \frac{\nu}{1-\nu}\,x_3\,\gamma_\alpha^\alpha(\boldsymbol{\zeta}) + \frac{\nu}{2-2\nu}\,x_3^2\,\rho_\alpha^\alpha(\boldsymbol{\zeta}), \end{cases} \tag{2.15}$$

and $\mathsf{T} : \boldsymbol{\zeta} \mapsto \mathsf{T}\boldsymbol{\zeta}$ is the shifter defined as $(\mathsf{T}\boldsymbol{\zeta})_\sigma = \zeta_\sigma - x_3 b_\sigma^\alpha \zeta_\alpha$ and $(\mathsf{T}\boldsymbol{\zeta})_3 = \zeta_3$, see [31]. The 2D elastic energy of $\boldsymbol{\zeta}$ is a good approximation of the 3D elastic energy of $\mathsf{U}\boldsymbol{\zeta}$, cf. [15, Theorem A.1]: For any $\boldsymbol{\zeta} \in (H^2 \times H^2 \times H^3) \cap V_B(\mathcal{S})$, there holds, with non-dimensional constant A

$$\left| a_{2\mathrm{D}}^\varepsilon(\boldsymbol{\zeta}, \boldsymbol{\zeta}) - a_{3\mathrm{D}}^\varepsilon(\mathsf{U}\boldsymbol{\zeta}, \mathsf{U}\boldsymbol{\zeta}) \right| \le A \, a_{2\mathrm{D}}^\varepsilon(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \left(\frac{\varepsilon}{R} + \frac{\varepsilon^2}{L^2} \right), \tag{2.16}$$

where R is the minimal radius of curvature (2.9) of $\mathcal{S}$, and L is the *wave length* for $\boldsymbol{\zeta}$ defined as the largest constant such that the following "inverse estimates" hold

$$L \, |\boldsymbol{\gamma}|_{H^1(\mathcal{S})} \le \|\boldsymbol{\gamma}\|_{L^2(\mathcal{S})} \quad \text{and} \quad L \, |\boldsymbol{\rho}|_{H^1(\mathcal{S})} \le \|\boldsymbol{\rho}\|_{L^2(\mathcal{S})} . \tag{2.17}$$

Note that for $\boldsymbol{\zeta} \in V_{\mathrm{B}}(\mathcal{S})$, the first two components of $\mathsf{U}\boldsymbol{\zeta}$ satisfy the Dirichlet condition on $\partial_0 \Omega_\varepsilon$, whereas the third one does not need to satisfy it. In order to remedy that, we add a corrector term $\mathbf{u}^{\mathrm{cor}}$ to $\mathsf{W}\boldsymbol{\zeta}$ to compensate for the nonzero trace $g = -\frac{\nu}{1-\nu} x_3 \gamma_\alpha^\alpha(\boldsymbol{\zeta}) + \frac{\nu}{2-2\nu} x_3^2 \rho_\alpha^\alpha(\boldsymbol{\zeta})\big|_{\partial\mathcal{S}}$. This corrector term is constructed and its energy estimated in [15, sect. 7]. It has a simple tensor product form and exhibits the typical 3D boundary layer scale d/ε with $d = \mathrm{dist}(\mathbf{P}, \partial_0 \Omega^\varepsilon)$:

$$\mathbf{u}^{\mathrm{cor}} = \left(0, 0, g\, \chi\!\left(\frac{d}{\varepsilon}\right) \right)^\top \quad \text{with} \quad \chi \in C_0^\infty(\mathbb{R}), \ \chi(0) = 1.$$

"True" boundary layer terms live at the same scale, decay exponentially, but have a non-tensor form in variables (d, x_3), see [17, 14] for plates and [21] for elliptic shells. Nevertheless this expression for $\mathbf{u}^{\mathrm{cor}}$ suffices to obtain good estimates: There holds

$$a_{3\mathrm{D}}^\varepsilon(\mathbf{u}^{\mathrm{cor}}, \mathbf{u}^{\mathrm{cor}}) \le A \, a_{2\mathrm{D}}^\varepsilon(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \left(\frac{\varepsilon}{\ell} + \frac{\varepsilon^3}{\ell^3} \right),$$

for ℓ the lateral wave length of $\boldsymbol{\zeta}$ defined as the largest constant such that

$$\ell |\boldsymbol{\gamma}|_{L^2(\partial\mathcal{S})}^2 + \ell^3 |\boldsymbol{\gamma}|_{H^1(\partial\mathcal{S})}^2 \le \|\boldsymbol{\gamma}\|_{L^2(\mathcal{S})}^2 \quad \text{and} \quad \ell |\boldsymbol{\rho}|_{L^2(\partial\mathcal{S})}^2 + \ell^3 |\boldsymbol{\rho}|_{H^1(\partial\mathcal{S})}^2 \le \|\boldsymbol{\rho}\|_{L^2(\mathcal{S})}^2 . \tag{2.18}$$

Example 2.1. Let G in $H^2(\mathbb{R}_+)$ be such that $G \equiv 0$ for $t \ge 1$. Let $k \in \mathbb{N}$ and $0 < \tau < \tau_0$ for τ_0 small enough. The function $g(z, \varphi)$ defined on $\mathcal{S}$ as

$$g(z, \varphi) = e^{ik\varphi} G\!\left(\frac{d}{\tau}\right)$$

satisfies the estimates $L\, |g|_{H^1(\mathcal{S})} \le \|g\|_{L^2(\mathcal{S})}$ and $\ell |g|_{L^2(\partial\mathcal{S})}^2 + \ell^3 |g|_{H^1(\partial\mathcal{S})}^2 \le \|g\|_{L^2(\mathcal{S})}^2$ for L and ℓ larger than $c(G) \min\{\tau, k^{-1}\}$ where the positive constant $c(G)$ is independent of τ and k.

In the present work, we are interested in comparing 2D and 3D Rayleigh quotients so we introduce the following notations

$$Q_{3\mathrm{D}}^\varepsilon(\mathbf{u}) = \frac{a_{3\mathrm{D}}^\varepsilon(\mathbf{u}, \mathbf{u})}{\|\mathbf{u}\|_{L^2(\Omega^\varepsilon)}^2}, \quad \text{and} \quad Q_{2\mathrm{D}}^\varepsilon(\boldsymbol{\zeta}) = \frac{a_{2\mathrm{D}}^\varepsilon(\boldsymbol{\zeta}, \boldsymbol{\zeta})}{2\varepsilon \|\boldsymbol{\zeta}\|_{L^2(\mathcal{S})}^2} .$$

By similar inequalities as in [15] we can prove the following relative estimate

Theorem 2.2. (i) *For all* $\boldsymbol{\zeta} \in (H^2 \times H^2 \times H^3) \cap V_B(\mathcal{S})$ *and with* U *defined in* (2.14) *we set*

$$\overset{\circ}{\mathsf{U}}\boldsymbol{\zeta} = \mathsf{U}\boldsymbol{\zeta} - \mathbf{u}^{\mathrm{cor}}.$$

Then $\overset{\circ}{\mathsf{U}}\boldsymbol{\zeta}$ *belongs to the 3D variational space* $V(\Omega^\varepsilon)$. *With* L *and* ℓ *the wave lengths* (2.17) *and* (2.18), *let us assume* $\varepsilon \le L$ *and* $\varepsilon \le \ell$. *We also assume* $Q_{2\mathrm{D}}^\varepsilon(\boldsymbol{\zeta}) \le EM$ *for a chosen constant*

$M \geq 1$ *independent of ε. Then we have the relative estimates between Rayleigh quotients for ε small enough*

$$\left| Q^\varepsilon_{2D}(\boldsymbol{\zeta}) - Q^\varepsilon_{3D}(\overset{\circ}{\mathsf{U}}\boldsymbol{\zeta}) \right| \leq A'\, Q^\varepsilon_{2D}(\boldsymbol{\zeta}) \left(\frac{\varepsilon}{R} + \frac{\varepsilon^2}{L^2} + \left(\frac{\varepsilon}{\ell} \right)^{1/2} + \varepsilon\sqrt{M} \right), \tag{2.19}$$

with a constant A' independent of ε and $\boldsymbol{\zeta}$.

(ii) *If $\boldsymbol{\zeta}$ belongs to $(H_0^2 \times H_0^2 \times H_0^3)(\mathcal{S})$, the boundary corrector $\mathbf{u}^{\text{cor}}$ is zero and the above estimates do not involve the term $\sqrt{\varepsilon/\ell}$ any more.*

This theorem allows to find upper bounds for the first 3D eigenvalue λ_1^ε if we know convenient energy minimizers $\boldsymbol{\zeta}^\varepsilon$ for the Koiter model $\mathfrak{K}(\varepsilon)$ and if we have the relevant information about their wave lengths.

3. AXISYMMETRIC SHELLS

An axisymmetric shell is invariant by rotation around an axis that we may choose as t_3. Recall that $(r, \varphi, \tau) \in \mathbb{R}^+ \times \mathbb{T} \times \mathbb{R}$ denote associated cylindrical coordinates satisfying relations (1.15) and coordinate vectors are $\mathbf{E}_r$, $\mathbf{E}_\varphi$, and $\mathbf{E}_\tau$ given by (1.16). Accordingly, the (contravariant) *cylindrical components* of a displacement $\mathbf{u} = \mathsf{u}^{t_i}\mathbf{E}_{t_i}$ are $(\mathsf{u}^r, \mathsf{u}^\varphi, \mathsf{u}^\tau)$ so that $\mathbf{u} = \mathsf{u}^r\mathbf{E}_r + \mathsf{u}^\varphi\mathbf{E}_\varphi + \mathsf{u}^\tau\mathbf{E}_\tau$. In particular the *radial component* of $\mathbf{u}$ is given by

$$\mathsf{u}^r = \mathsf{u}^{t_1}\cos\varphi + \mathsf{u}^{t_2}\sin\varphi. \tag{3.1}$$

The components u^φ and u^τ are called azimuthal and axial, respectively.

An axisymmetric domain $\Omega \subset \mathbb{R}^3$ is associated with a meridian domain $\omega \subset \mathbb{R}^+ \times \mathbb{R}$ so that

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^3, \quad (r, \tau) \in \omega \text{ and } \varphi \in \mathbb{T}\}. \tag{3.2}$$

3.1. Axisymmetric parametrization.

For a shell Ω^ε that is axisymmetric, let ω^ε be its meridian domain. The midsurface $\mathcal{S}$ of Ω^ε is axisymmetric too. Let $\mathcal{C}$ be its meridian domain. We have a relation similar to (2.1)

$$\Phi: \quad \mathcal{C} \times (-\varepsilon, \varepsilon) \ni \big((r, \tau), x_3\big) \quad \longmapsto \quad (r, \tau) + x_3\mathbf{N}(r, \tau) \in \omega^\varepsilon. \tag{3.3}$$

The meridian midsurface $\mathcal{C}$ is a curve in the halfplane $\mathbb{R}^+ \times \mathbb{R}$.

Assumption 3.1. Let $\mathcal{I}$ denote any bounded interval and let z be the variable in $\mathcal{I}$.
(i) The curve $\mathcal{C}$ can be parametrized by one map defined on $\mathcal{I}$ by a smooth function f:

$$\begin{array}{ccc} \mathcal{I} & \longrightarrow & \mathcal{C} \\ z & \longmapsto & (r, \tau) = (f(z), z) \end{array} \quad \text{with} \quad f : z \mapsto r = f(z). \tag{3.4}$$

(ii) The shells are disjoint from the rotation axis, i.e., there exists $R_{\min} > 0$ such that $f \geq R_{\min}$.

Remark 3.2. We impose condition (ii) to avoid technical difficulties due to the singularity at the origin. We have observed that, if we keep this condition, the inverse parametrization $z = g(r)$ does not bring new examples in the framework that we investigate in this paper. For instance *annular plates* pertain to this inverse parametrization, but they fall in [14] that provides a complete eigenvalue asymptotics.

The parametrization (3.4) of the meridian curve $\mathcal{C}$ provides a parametrization of the meridian domain ω_ε by $\mathcal{I} \times (-\varepsilon, \varepsilon)$: Let us introduce the arc-length

$$s(z) = \sqrt{1 + f'(z)^2}, \quad z \in \mathcal{I}. \tag{3.5}$$

The unit normal vector $\mathbf{N}$ to $\mathcal{C}$ at the point $(r, \tau) = (f(z), z)$ is given by $(\frac{1}{s(z)}, -\frac{f'(z)}{s(z)})$ and the parametrization by

$$\mathcal{I} \times (-\varepsilon, \varepsilon) \ni (z, x_3) \;\longmapsto\; \left(f(z) + x_3 \, \tfrac{1}{s(z)}, z - x_3 \, \tfrac{f'(z)}{s(z)} \right) \in \omega^\varepsilon.$$

The parametrization (3.4) also induces the parametrization F (1.3) of the midsurface $\mathcal{S}$ by the variables $(z, \varphi) \in \mathcal{I} \times \mathbb{T}$. The unit normal vector $\mathbf{N}$ to $\mathcal{S}$ at the point $F(z, \varphi)$ is given by

$$\mathbf{N} = s(z)^{-1}(\mathbf{E}_r - f'(z)\mathbf{E}_\tau)$$

while tangent coordinate vectors are $\mathbf{E}_z = \partial_z F$ and $\mathbf{E}_\varphi = \partial_\varphi F$, i.e.

$$\mathbf{E}_z = f'(z)\mathbf{E}_r + \mathbf{E}_\tau$$

while $\mathbf{E}_\varphi$ coincides the coordinate vector of same name corresponding to cylindrical coordinates (1.16). The metric tensor $(a_{\alpha\beta})$ is given by $\langle \mathbf{E}_\alpha, \mathbf{E}_\beta \rangle$ with $\alpha, \beta \in \{z, \varphi\}$, i.e.

$$\begin{pmatrix} a_{zz} & a_{z\varphi} \\ a_{\varphi z} & a_{\varphi\varphi} \end{pmatrix}(z) = \begin{pmatrix} s(z)^2 & 0 \\ 0 & f(z)^2 \end{pmatrix}. \tag{3.6}$$

The curvature tensor and Gaussian curvature K are respectively given by

$$\begin{pmatrix} b_z^z & b_\varphi^z \\ b_z^\varphi & b_\varphi^\varphi \end{pmatrix}(z) = \begin{pmatrix} f''(z)s(z)^{-3} & 0 \\ 0 & -f(z)^{-1}s(z)^{-1} \end{pmatrix} \quad \text{and} \quad K(z) = -\frac{f''(z)}{f(z)s(z)^4}. \tag{3.7}$$

So the curvature tensor is in diagonal form, and K is simply the product of its diagonal elements.

Definition 3.3. We call b_z^z the *meridian curvature* and b_φ^φ the *azimuthal curvature*.

Since we have assumed that $f \geq R_0 > 0$, all terms are bounded and we find that

(1) If $f'' \equiv 0$, i.e. f is affine, the shell is (nondegenerate) parabolic. If f is constant, the shell is a cylinder, if not it is a truncated cone (without conical point!).
(2) If $f'' < 0$, the shell is elliptic.
(3) If $f'' > 0$, the shell is hyperbolic.

3.2. **2D axisymmetric models in normal coordinates.** Relations (1.3) and (2.1) define *normal coordinates* (z, φ, x_3) in the thin shell Ω^ε. For example when the midsurface $\mathcal{S}$ is a cylinder (f constant), the normal coordinates are a permutation of standard coordinates: $(z, \varphi, x_3) = (\tau, \varphi, r)$. The associate (contravariant) decomposition of surface displacement fields $\boldsymbol{\zeta}$ is written as $\boldsymbol{\zeta} = \zeta^z \mathbf{E}_z + \zeta^\varphi \mathbf{E}_\varphi + \zeta^3 \mathbf{N}$, where ζ^3 is the component of the displacement in the normal direction $\mathbf{N}$ to the midsurface, ζ^z and ζ^φ the meridian and azimuthal components respectively, defined so that there holds

$$\zeta^{t_1}\mathbf{E}_{t_1} + \zeta^{t_2}\mathbf{E}_{t_2} + \zeta^{t_3}\mathbf{E}_{t_3} = \zeta^z \mathbf{E}_z + \zeta^\varphi \mathbf{E}_\varphi + \zeta^3 \mathbf{N}.$$

Note that the azimuthal component is the same as defined by cylindrical coordinates. The covariant components are

$$\zeta_z = s^2 \zeta^z, \quad \zeta_\varphi = f^2 \zeta^\varphi, \quad \text{and} \quad \zeta_3 = \zeta^3.$$

The change of metric tensor $\gamma_{\alpha\beta}(\boldsymbol{\zeta})$ has the expression in normal coordinates

$$
\begin{aligned}
\gamma_{zz}(\boldsymbol{\zeta}) &= \partial_z \zeta_z - \frac{f' f''}{s^2}\zeta_z - \frac{f''}{s}\zeta_3 \\
\gamma_{z\varphi}(\boldsymbol{\zeta}) &= \frac{1}{2}(\partial_z \zeta_\varphi + \partial_\varphi \zeta_z) - \frac{f'}{f}\zeta_\varphi \\
\gamma_{\varphi\varphi}(\boldsymbol{\zeta}) &= \partial_\varphi \zeta_\varphi + \frac{f f'}{s^2}\zeta_z + \frac{f}{s}\zeta_3 \,,
\end{aligned}
\tag{3.8}
$$

while the change of curvature tensor $\rho_{\alpha\beta}(\boldsymbol{\zeta})$ is written as

$$
\begin{aligned}
\rho_{zz}(\boldsymbol{\zeta}) &= \partial_z^2 \zeta_3 - \frac{f''^2}{s^4}\zeta_3 + \frac{2f''}{s^3}\partial_z \zeta_z + \frac{f''' s^2 - 5 f' f''^2}{s^5}\zeta_z \\
\rho_{\varphi\varphi}(\boldsymbol{\zeta}) &= \partial_\varphi^2 \zeta_3 - \frac{1}{s^2}\zeta_3 - \frac{2}{fs}\partial_\varphi \zeta_\varphi - \frac{2f'}{s^3}\zeta_z \\
\rho_{z\varphi}(\boldsymbol{\zeta}) &= \partial_{z\varphi}\zeta_3 + \frac{f''}{s^3}\partial_\varphi \zeta_z - \frac{1}{fs}\partial_z \zeta_\varphi + \frac{2f'}{f^2 s}\zeta_\varphi \,.
\end{aligned}
\tag{3.9}
$$

4. PRINCIPLES OF CONSTRUCTION: HIGH FREQUENCY ANALYSIS

The construction is based on the following postulate:

Postulate 4.1. The eigenmodes associated with the smallest vibrations are strongly oscillating in the angular variable φ and this oscillation is dominating.

This means that if this postulate happens to be true for certain families of shells, our construction will provide rigorous quasimodes and, moreover, these quasimodes are candidates to be associated with lowest energy eigenpairs.

We may notice that Postulate 4.1 is wrong for planar shells. But it appears to be true for nondegenerate parabolic shells and some subclasses of elliptic shells.

4.1. **Angular Fourier decomposition.** We can perform a discrete Fourier decomposition in the shell $\Omega^\varepsilon \equiv \omega^\varepsilon \times \mathbb{T}$ and in its midsurface $\mathcal{S} \equiv \mathcal{C} \times \mathbb{T} \cong \mathcal{I} \times \mathbb{T}$. For a function $\mathsf{u}(z, \varphi)$ in $L^2(\mathcal{I} \times \mathbb{T})$ we denote its Fourier coefficients by

$$\mathsf{u}^k(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathsf{u}(z, \varphi)\, e^{-ik\varphi}\, \mathrm{d}\varphi, \quad k \in \mathbb{Z}, \ z \in \mathcal{I}.$$

This Fourier decomposition diagonalizes any operator L with coefficient independent of φ with respect to the angular modes $e^{ik\varphi}$, $k \in \mathbb{Z}$, due to the relation:

$$(\mathsf{Lu})^k = \mathsf{L}^k \mathsf{u}^k.$$

Here the operators of interest are the Lamé system $\mathfrak{L}^\varepsilon$ written in normal components, together with the membrane and bending operators $\mathfrak{M}$ and $\mathfrak{B}$ defined on the spaces $V_{\mathsf{M}}(\mathcal{S})$ and $V_{\mathsf{B}}(\mathcal{S})$,

composing the Koiter operator $\mathfrak{K}(\varepsilon)$. We denote by $(\mathfrak{L}^\varepsilon)^k$, $\mathfrak{M}^k$, $\mathfrak{B}^k$ and $\mathfrak{K}^k(\varepsilon)$, $k \in \mathbb{Z}$, the angular Fourier decomposition of $\mathfrak{L}^\varepsilon$, $\mathfrak{M}$, $\mathfrak{B}$ and $\mathfrak{K}(\varepsilon)$, respectively.

The (non decreasing) collections of the eigenvalues $\lambda_j^{\varepsilon,k}$ of $(\mathfrak{L}^\varepsilon)^k$ for all $k \in \mathbb{Z}$ gives back all eigenvalues of $\mathfrak{L}^\varepsilon$. Note that since $\mathfrak{L}^\varepsilon$ is real valued, the eigenvalues for k and $-k$ are identical. Thus $\lambda_1^\varepsilon = \inf_{k \in \mathbb{N}} \lambda_1^{\varepsilon,k}$ and we denote by $k(\varepsilon)$ the smallest natural integer k such that (if exists)

$$\lambda_1^\varepsilon = \lambda_1^{\varepsilon,k(\varepsilon)} .$$

Postulate 4.1 means that $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

4.2. **High frequency analysis of the membrane operator.** The eigenmode membrane equation (2.11) at azimuthal frequency k takes the form

$$\mathfrak{M}^k \zeta^k = \Lambda^k \mathbf{A} \zeta^k \tag{4.1}$$

where $\mathbf{A}$ is the mass matrix

$$\mathbf{A} = \begin{pmatrix} a^{zz} & 0 & 0 \\ 0 & a^{\varphi\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s^{-2} & 0 & 0 \\ 0 & f^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.2}$$

We construct quasimodes for $\mathfrak{M}^k$ as $k \to \infty$, i.e. pairs $(\breve{\Lambda}^k, \breve{\zeta}^k)$ with $\breve{\zeta}^k$ in the domain of the operator $\mathfrak{M}^k$ and satisfying the estimates

$$\| (\mathfrak{M}^k - \breve{\Lambda}^k) \breve{\zeta}^k \|_{L^2(\mathcal{S})} \le \delta(k) \| \breve{\zeta}^k \|_{L^2(\mathcal{S})} \quad \text{with} \quad \delta(k)/\breve{\Lambda}^k \to 0 \text{ as } k \to \infty.$$

Now we consider the membrane operator as a formal series with respect to k

$$\mathfrak{M}^k = k^2 \mathbf{M}_0 + k\mathbf{M}_1 + \mathbf{M}_2 \equiv \mathbf{M}[k], \quad \text{with} \quad \mathbf{M}[k] = k^2 \sum_{n \in \mathbb{N}} k^{-n} \mathbf{M}_n , \tag{4.3}$$

and try to solve (4.1) in the formal series algebra:

$$\mathbf{M}[k]\zeta[k] = \Lambda[k]\mathbf{A}\zeta[k]. \tag{4.4}$$

Here the multiplication of formal series is the Cauchy product: For two formal series $a[k] = \sum_n k^{-n} a_n$ and $b[k] = \sum_n k^{-n} b_n$, the coefficients of the series $a[k] \, b[k] = \sum_n k^{-n} c_n$ are given by $c_n = \sum_{\ell+m=n} a_\ell b_m$.

The director $\mathbf{M}_0$ of the series $\mathbf{M}[k]$ is given in parametrization $r = f(z)$ by

$$\mathbf{M}_0 = \frac{E}{1 - \nu^2} \begin{pmatrix} \frac{1-\nu}{2f^2 s^2} & 0 & 0 \\ 0 & \frac{1}{f^4} & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.5}$$

Its kernel is given by all triples ζ of the form $(0,0,\zeta_3)^\top$. This is the reason why we look for a reduction of the eigenvalue problem for $\mathbf{M}$ to a scalar eigenvalue problem set on the normal component ζ_3. The key is a factorization process in the formal series algebra proved in [5, Chap.3],

$$\mathbf{M}[k]\mathbf{V}[k] - \Lambda[k]\mathbf{A}\mathbf{V}[k] = \mathbf{V}_0 \circ (\mathsf{H}[k] - \Lambda[k]) . \tag{4.6}$$

Here $\mathbf{V}(k)$ is a (formal series of) reconstruction operators whose first term $\mathbf{V}_0$ is the embedding $\mathbf{V}_0\eta = (0,0,\eta)^\top$ in the kernel of $\mathbf{M}_0$, and $\mathsf{H}[k]$ is the scalar reduction.

Theorem 4.2. *Let be a formal series with real coefficients :*

$$\Lambda[k] = \sum_{n \geq 0} k^{-n}\Lambda_n.$$

For $n \geq 1$, there exist operators $\mathsf{V}_{n,z}, \mathsf{V}_{n,\varphi} : C^\infty(\overline{\mathcal{I}}) \to C^\infty(\overline{\mathcal{I}})$ of order $n - 1$, polynomial in Λ_j, for $j \leq n - 3$, and for $n \geq 0$ scalar operators $\mathsf{H}_n : C^\infty(\overline{\mathcal{I}}) \to C^\infty(\overline{\mathcal{I}})$ of order n, polynomial in Λ_j, for $j \leq n - 2$ such that if we set :

$$\mathbf{V}[k] = \sum_{n \geq 0} k^{-n}\mathbf{V}_n \quad with \quad \mathbf{V}_n = (\mathsf{V}_{n,z}, \mathsf{V}_{n,\varphi}, 0)^\top, \quad and \quad \mathsf{H}[k] = \sum_{n \geq 0} k^{-n}\mathsf{H}_n$$

we have (4.6) *in the sense of formal series.*

See Appendix A for more details on this theorem.

With the scalar reduction $\mathsf{H}[k]$ is associated the formal series problem

$$\mathsf{H}[k]\eta[k] = \Lambda[k]\eta[k] \tag{4.7}$$

where $\eta[k] = \sum_{n \geq 0} k^{-n}\eta_n$ is a scalar formal series. The previous theorem shows that any solution to (4.7) provides a solution $\boldsymbol{\zeta}[k] = \mathbf{V}[k]\eta[k]$ to (4.4).

The cornerstone of our quasimodes construction for $\mathfrak{M}^k$ as $k \to \infty$ is to construct a solution $\eta[k]$ of the problem (4.7). This relies on the possibility to extract an elliptic operator $\mathfrak{H}^k$ with compact resolvent from the first terms of the series $\mathsf{H}[k]$ as we describe in several geometrical situations later on.

Remark 4.3. The essential spectrum $\sigma_{\mathsf{ess}}(\mathfrak{M}^k)$ of the membrane operator $\mathfrak{M}^k$ at frequency k can be determined explicitly thanks to [4, Th.4.5]. It depends only on its principal part, which coincides with the (multi-degree) principal part of $\mathbf{M}_2$, and is given by the range of $\frac{E}{f(z)^2 s(z)^2}$ for $z \in \mathcal{I}$, see [5, sect. 2.7] for details. With formula (3.7), we note the relation with the azimuthal curvature

$$\sigma_{\mathsf{ess}}(\mathfrak{M}^k) = \left\{ E\, b_\varphi^\varphi(z)^2 \,, \quad z \in \mathcal{I} \right\}. \tag{4.8}$$

As a consequence of Assumption 3.1, the minimum of $\sigma_{\mathsf{ess}}(\mathfrak{M}^k)$ is positive.

4.3. **High frequency analysis of the Koiter operator.** Similar to the membrane operator $\mathbf{M}[k]$, the bending operator expands as

$$\mathfrak{B}^k = k^4\mathbf{B}_0 + \sum_{n=1}^{4} k^{4-n}\mathbf{B}_n \equiv \mathbf{B}[k],$$

with first term

$$\mathbf{B}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathsf{B}_0 \end{pmatrix} \quad with \quad \mathsf{B}_0 = \frac{E}{1-\nu^2}\frac{1}{3f^4}. \tag{4.9}$$

We notice that we have the commutation relation

$$\mathbf{B}_0\mathbf{V}[k] = \mathbf{V}_0\mathsf{B}_0\,.$$

Therefore the identity (4.6) implies for all ε the identity

$$\big(\mathbf{M}[k] + \varepsilon^2 k^4 \mathbf{B}_0\big)\mathbf{V}[k] - \Lambda[k]\mathbf{A}\mathbf{V}[k] = \mathbf{V}_0 \circ \big(\mathsf{H}[k] + \varepsilon^2 k^4 \mathsf{B}_0 - \Lambda[k]\big)\,. \qquad (4.10)$$

Thus the same factorization as for the membrane operator will generate the quasimode constructions for the Koiter operator as soon as the higher order terms of $\mathfrak{B}^k$ correspond to perturbation terms. This is related to Postulate 4.1. The identity (4.10) motivates the formula (1.6) defining the reduced Koiter operator $\mathfrak{A}^k(\varepsilon) = \mathfrak{H}^k + \varepsilon^2 k^4 \mathsf{B}_0$. In the following two sections we provide $\mathfrak{A}^k(\varepsilon)$ and its lowest eigenvalues in several well defined cases.

5. NONDEGENERATE PARABOLIC CASE.

We assume in addition to Assumption 3.1

$$f(z) = Tz + R_0, \quad z \in \mathcal{I}, \quad \text{with} \quad R_0 > 0, \ T \in \mathbb{R}\,. \qquad (5.1)$$

If $T = 0$, the corresponding surface $\mathcal{S}$ is a cylinder of radius R_0 and the minimal radius of curvature R (2.9) equals to R_0. So we write $f = R$ in the cylinder case. If $T \neq 0$, the surface $\mathcal{S}$ is a truncated cone. The arc length (3.5) is $s = \sqrt{1 + T^2}$.

5.1. Membrane scalar reduction in the parabolic case.
The first terms H_n of the scalar formal series reduction of the membrane operator have been explicitly calculated in [5] in the cylindrical case $T = 0$ and have the following expression in the general parabolic case:

$$\mathsf{H}_0 = \mathsf{H}_1 = \mathsf{H}_2 = \mathsf{H}_3 = 0 \quad \text{and} \quad \mathsf{H}_4(z,\partial_z) = E\Big(\frac{f^2}{s^6}\partial_z^4 + \frac{6f'f}{s^6}\partial_z^3 + \frac{6f'^2}{s^6}\partial_z^2\Big)\,. \qquad (5.2)$$

It is relevant to notice that H_4 is selfadjoint on $H_0^2(\mathcal{I})$ with respect to the natural measure $\mathrm{d}\mathcal{I} = f(z)s(z)\,\mathrm{d}z$, since there holds

$$\langle \mathsf{H}_4\eta, \eta^* \rangle_{\mathcal{I}} = \frac{E}{(1+T^2)^3} \int_{\mathcal{I}} f(z)^2\, \partial_z^2\eta\, \partial_z^2\eta^*\, \mathrm{d}\mathcal{I}\,. \qquad (5.3)$$

This also proves that H_4 is positive. The Dirichlet boundary conditions $\eta = \partial_z\eta = 0$ on $\partial\mathcal{I}$ are the right conditions to implement the membrane boundary condition $\zeta_\alpha = 0$ on $\partial\mathcal{I}$ through the reconstruction operators $\mathbf{V}_n$, see (A.6)–(A.7). The eigenvalue formal series $\Lambda[k]$ starts with Λ_4 that is the first eigenvalue of H_4:

$$\Lambda_0 = \Lambda_1 = \Lambda_2 = \Lambda_3 = 0 \quad \text{and} \quad \Lambda_4 > 0. \qquad (5.4)$$

The pair $(\breve{\Lambda}^k, \breve{\zeta}^k)$

$$\breve{\zeta}^k = \sum_{0 \le n+m \le 6} k^{-n-m}\mathbf{V}_n\eta_m \quad \text{and} \quad \breve{\Lambda}^k = k^{-4}\Lambda_4 \qquad (5.5)$$

with (Λ_4, η_0) an eigenpair of H_4, and η_m $(m = 1, \ldots, 6)$ constructed by induction so that the membrane boundary conditions $\check{\zeta}^k_\alpha = 0$ are satisfied, is a quasimode for $\mathfrak{M}^k$. For instance, in the cylindrical case $f = R$, the triple $\check{\zeta}^k$ takes the form

$$
\check{\zeta}^k = \begin{pmatrix} 0 \\ 0 \\ \eta_0 \end{pmatrix} + \frac{i}{k} \begin{pmatrix} 0 \\ R\eta_0 \\ 0 \end{pmatrix} + \frac{1}{k^2} \begin{pmatrix} -R\eta_0' \\ 0 \\ \eta_2 \end{pmatrix} + \frac{i}{k^3} \begin{pmatrix} 0 \\ -\nu R^3 \eta_0'' + R\eta_2 \\ \eta_3 \end{pmatrix} - \frac{1}{k^4} \begin{pmatrix} (\nu+2)R^3\eta_0''' + R\eta_2' \\ R\eta_3 \\ \eta_4 \end{pmatrix} + \ldots
$$

(5.6)

and the boundary conditions are, for $z \in \partial\mathcal{I}$

$$
\eta_0(z) = 0, \quad \eta_0'(z) = 0, \quad \eta_2(z) = \nu R^2 \eta_0''(z), \quad \eta_2'(z) = (\nu + 2)R^2 \eta_0'''(z), \quad \eta_3(z) = 0, \ldots \quad (5.7)
$$

Recall that the minimum of the essential spectrum of $\mathfrak{M}^k$ is positive by Remark 4.3. For $|k|$ large enough, $\mathfrak{M}^k$ has therefore at least an eigenvalue $\cong \Lambda_4 k^{-4}$ under its essential spectrum and

$$
\mathrm{dist}\big(k^{-4}\Lambda_4, \sigma(\mathfrak{M}^k)\big) \lesssim k^{-5}, \quad k \to \infty. \quad (5.8)
$$

5.2. Koiter scalar reduction in the parabolic case.

The leading term of the series $H(k)$ is $\mathfrak{H}^k = k^{-4}H_4$, as mentioned in the introduction, see (1.5). So, the leading term of the scalar reduction of the Koiter operator is, cf. (4.10)

$$
\mathfrak{A}^k(\varepsilon) = k^{-4}H_4 + \varepsilon^2 k^4 B_0 = k^{-4}H_4 + \frac{\varepsilon^2}{3} \frac{E}{1-\nu^2} \frac{k^4}{f^4}. \quad (5.9)
$$

5.2.1. *Optimizing k.* The operator $\mathfrak{A}^k(\varepsilon)$ is self-adjoint on $H^2_0(\mathcal{I})$ and positive. Let $\mu_1^{\mathfrak{A}}(\varepsilon, k)$ be its smallest eigenvalue. For any chosen ε we look for $k_{\min} = k(\varepsilon)$ realizing the minimum $\mu_1^{\mathfrak{A}}(\varepsilon)$ of $\mu_1^{\mathfrak{A}}(\varepsilon, k)$ if it exists:

$$
\mu_1^{\mathfrak{A}}(\varepsilon) = \mu_1^{\mathfrak{A}}(\varepsilon, k(\varepsilon)) = \min_{k \in \mathbb{R}_+} \mu_1^{\mathfrak{A}}(\varepsilon, k).
$$

To "homogenize" the terms k^{-4} and $\varepsilon^2 k^4$ let us define $\gamma(\varepsilon)$ by setting

$$
\gamma(\varepsilon) = \varepsilon^{1/4} k(\varepsilon), \quad (5.10)
$$

so that we look equivalently for $\gamma(\varepsilon)$. There holds

$$
k(\varepsilon)^{-4}H_4 + \varepsilon^2 k(\varepsilon)^4 B_0 = \varepsilon\Big(\frac{1}{\gamma(\varepsilon)^4}H_4 + \gamma(\varepsilon)^4 B_0\Big).
$$

Therefore $\gamma(\varepsilon)$ does not depend on ε. Let $\mu_1(\gamma)$ be the first eigenvalue of the operator

$$
\frac{1}{\gamma^4}H_4 + \gamma^4 B_0. \quad (5.11)
$$

The function $\gamma \mapsto \mu_1(\gamma)$ is continuous and, since H_4 and B_0 are *positive*, it tends to infinity as γ tends to 0 or to $+\infty$. Therefore we can define $\gamma_{\min}$ as the (smallest) positive constant such that $\mu_1(\gamma)$ is minimum

$$
\mu_1(\gamma_{\min}) = \min_{\gamma \in \mathbb{R}_+} \mu_1(\gamma) =: \mathsf{a}_1. \quad (5.12)
$$

Thus $k(\varepsilon)$ satisfies a power law that yields a formula for the minimal first eigenvalue $\mu_1^{\mathfrak{A}}(\varepsilon)$:

$$
k(\varepsilon) = \varepsilon^{-1/4}\gamma_{\min} \quad \text{and} \quad \mu_1^{\mathfrak{A}}(\varepsilon) = \mathsf{a}_1\varepsilon. \quad (5.13)
$$

5.2.2. *Case of cylinders.* In the cylindrical case $T = 0$, things are easier because f is constant. So everything can be written as a function of the first Dirichlet eigenvalue μ_1^{bilap} of the bilaplacian operator Δ^2 on $H_0^2(\mathcal{I})$ as we explain now. We have

$$\mathsf{H}_4 = ER^2\,\Delta^2 \quad \text{and} \quad \mathsf{B}_0 = \frac{E}{1-\nu^2}\,\frac{1}{3R^4}\,.$$

So the eigenvalue of $\frac{1}{\gamma^4}\mathsf{H}_4 + \gamma^4\mathsf{B}_0$ is

$$\mu_1(\gamma) = \frac{1}{\gamma^4}ER^2\mu_1^{\mathsf{bilap}} + \gamma^4\frac{E}{1-\nu^2}\,\frac{1}{3R^4}\,.$$

It is minimum for $\gamma_{\min}$ such that

$$\gamma_{\min}^4 = R^3\sqrt{3(1-\nu^2)\,\mu_1^{\mathsf{bilap}}}$$

and we find that the minimum eigenvalue (5.12) is

$$\mu_1(\gamma_{\min}) = \frac{2E}{R}\sqrt{\frac{\mu_1^{\mathsf{bilap}}}{3(1-\nu^2)}} =: \mathsf{a}_1\,. \tag{5.14}$$

Thus

$$k(\varepsilon) = \varepsilon^{-1/4}R^{3/4}\big(3(1-\nu^2)\,\mu_1^{\mathsf{bilap}}\big)^{1/8}. \tag{5.15}$$

Remark 5.1. Denote by μ^{bilap} the first eigenvalue of Δ^2 on the unit interval $(0,1)$. We have the relation $\mu_1^{\mathsf{bilap}} = \mu^{\mathsf{bilap}}\,L^{-4}$ with the length L of the interval $\mathcal{I}$.

5.2.3. *2D reconstruction.* We convert the law (5.13) giving k as a function of ε into a law giving ε as a function of k

$$\varepsilon = k^{-4}\gamma_{\min}^4 \tag{5.16}$$

and insert it into the identity (4.10). We obtain

$$\big(\mathbf{M}[k] + \gamma_{\min}^8 k^{-4}\mathbf{B}_0\big)\mathbf{V}[k] - \Lambda[k]\mathbf{AV}[k] = \mathbf{V}_0 \circ \big(\mathsf{H}[k] + \gamma_{\min}^8 k^{-4}\mathsf{B}_0 - \Lambda[k]\big)\,. \tag{5.17}$$

So the series $\Lambda[k]$ starts with the first eigenvalue $\Lambda_4 = \gamma_{\min}^4\mathsf{a}_1$ of $\mathsf{H}_4 + \gamma_{\min}^8\mathsf{B}_0$. Let η_0 be an associated eigenvector. Like before, but now with this new η_0, the pair $(\breve{\Lambda}^k, \breve{\boldsymbol{\zeta}}^k)$ defined by (5.5) is a quasimode for $\mathfrak{M}^k + \gamma_{\min}^8 k^{-4}\mathbf{B}_0 = \mathfrak{M}^k + \varepsilon^2 k^4\mathbf{B}_0$ with membrane boundary conditions. To restore the dependency on ε (via law (5.16)), we write

$$\breve{\Lambda}^k = \breve{\Lambda}^k(\varepsilon) \quad \text{and} \quad \breve{\boldsymbol{\zeta}}^k = \breve{\boldsymbol{\zeta}}^k(\varepsilon), \quad \text{with} \quad k = k(\varepsilon).$$

Since with law (5.16) the terms $\varepsilon^2(\mathfrak{B}^k - k^4\mathbf{B}_0)$ are of order k^{-5} or higher, the same pair is a quasimode for the full Koiter operator $\mathfrak{K}^k(\varepsilon)$, but still with *membrane boundary conditions*.

5.2.4. *Quasimodes for the Koiter model. Bending boundary layers.* The full bending boundary conditions $\zeta_3 = 0$ and $\zeta_3' = 0$ on $\partial\mathcal{I}$ cannot be implemented in general for the quasimodes $(\breve{\Lambda}^k(\varepsilon), \breve{\boldsymbol{\zeta}}^k(\varepsilon))$. The singularly perturbed nature of the Koiter operator causes the loss of these boundary conditions between the bending and membrane operator. Solutions of the Koiter model, just as eigenvectors, incorporate boundary layer terms. In all cases investigated in this paper, these terms exist at the scale $d/\sqrt{\varepsilon}$ with $d = \mathrm{dist}(z, \partial\mathcal{I})$. Such a scaling appears in [33] in a variety of nondegenerate cases (the boundary of $\partial\mathcal{S}$ is noncharacteristic for the curvature). It is rigorously analyzed in [21] in the case of static clamped elliptic shells.

More precisely, the scaled variable is (for $\mathcal{I} = (z_-, z_+)$)

$$Z = \frac{d}{\sqrt{\varepsilon}} \quad \text{with} \quad d = z_+ - z \ \text{ or } \ z - z_-, \tag{5.18}$$

according as we consider the localization at the end $z_0 = z_+$ or $z_0 = z_-$ of the interval $\mathcal{I}$. In view of law (5.13), we can write the operator $\mathfrak{K}^k(\varepsilon)$ as a series in powers of $\varepsilon^{1/4}$. In the rapid variable Z, there holds $\partial_z G(Z) = \varepsilon^{-1/2} G'$ for any profile $G(Z)$, which provides a new formal series $\mathcal{K}[\varepsilon^{1/4}]$. Its leading term $\mathcal{K}_0$ is compatible with the full bending boundary conditions at $Z = 0$. It has the following form in the cylindrical case $f = R$

$$\mathcal{K}_0 = \frac{E}{1 - \nu^2} \begin{pmatrix} -\partial_Z^2 & 0 & \frac{\nu}{R}\partial_Z \\ 0 & -\frac{1-\nu}{2R^2}\partial_Z^2 & 0 \\ -\frac{\nu}{R}\partial_Z & 0 & \frac{1}{R^2} + \frac{1}{3}\partial_Z^4 \end{pmatrix}.$$

It allows to construct a series of exponentially decreasing vector functions $\mathbf{G}[\varepsilon^{1/4}]$ satisfying a formal series relation of the type $\mathcal{K}[\varepsilon^{1/4}]\mathbf{G}[\varepsilon^{1/4}] = \Lambda[\varepsilon^{1/4}]\mathbf{G}[\varepsilon^{1/4}]$, that compensate for the missing traces of $\breve{\zeta}^k(\varepsilon)$, see [5, Section 5.6]. Our "true" quasimode has now the form $(\breve{\Lambda}^k(\varepsilon), \overset{\circ}{\zeta}{}^k(\varepsilon))$ with

$$\overset{\circ}{\zeta}{}^k(\varepsilon)(z) = \breve{\zeta}^k(\varepsilon)(z) + \chi(d)\sum_{n=2}^{6} \varepsilon^{n/4}\mathbf{G}_n(Z) \ \text{ with the same } \ \breve{\Lambda}^k(\varepsilon) = \mu_1^{\mathfrak{A}}(\varepsilon) = \mathsf{a}_1\varepsilon. \tag{5.19}$$

Here χ is a smooth cut-off that localizes near the boundary $\partial\mathcal{I}$. The outcome is the spectral estimate

$$\mathrm{dist}\big(\mathsf{a}_1\varepsilon, \ \sigma(\mathfrak{K}^k(\varepsilon))\big) \lesssim \varepsilon^{5/4} \ \text{ with } \ k(\varepsilon) = \varepsilon^{-1/4}\gamma_{\min}, \quad \text{as} \quad \varepsilon \to 0. \tag{5.20}$$

5.3. **3D reconstruction and Rayleigh quotients.** We construct a three-component vector field on the surface $\mathcal{S}$ by setting in normal coordinates

$$\boldsymbol{\zeta}^\varepsilon(z, \varphi) = e^{ik\varphi}\boldsymbol{\zeta}^k(z) \quad \text{with} \ \boldsymbol{\zeta}^k \ \text{given in (5.19) and} \ k = k(\varepsilon) = \varepsilon^{-1/4}\gamma_{\min}.$$

By construction, $\boldsymbol{\zeta}^\varepsilon$ belongs to the variational space $V_{\mathsf{B}}(\mathcal{S})$, and by the elliptic regularity of the Koiter problem, it also belongs to $(H^2 \times H^2 \times H^3)(\mathcal{S})$. So we may apply the reconstruction operator introduced in Theorem 2.2: Set

$$\mathbf{u}^\varepsilon = \overset{\circ}{\mathsf{U}}\boldsymbol{\zeta}^\varepsilon.$$

To take advantage of the comparison (2.19) between the Rayleigh quotients of $\boldsymbol{\zeta}^\varepsilon$ and $\mathbf{u}^\varepsilon$, we have to exhibit the behavior of the wave lengths $L = L^\varepsilon$ (2.17) and $\ell = \ell^\varepsilon$ (2.18) of $\boldsymbol{\zeta}^\varepsilon$ as $\varepsilon \to 0$. Following the construction of the fields $\boldsymbol{\zeta}^\varepsilon$, we see that they all originate from an eigenfunction η_0 that does not depend on ε. The nontrivial behavior of L^ε and ℓ^ε arises from, cf. Example 2.1:

- The Koiter boundary layer terms $\mathbf{G}_n(Z) = \mathbf{G}_n(d/\varepsilon^{1/2})$ that contribute a term in $\varepsilon^{1/2}$,
- The azimuthal oscillation $e^{ik\varphi}$ that contributes a term in $k^{-1} \simeq \varepsilon^{1/4}$.

As a result we find in the nondegenerate parabolic case

$$L^\varepsilon, \ \ell^\varepsilon \gtrsim \varepsilon^{1/2}.$$

So the assumptions of Theorem 2.2 are uniformly satisfied for the family $(\boldsymbol{\zeta}^\varepsilon)_\varepsilon$ and the estimate (2.19) reads now

$$\left|Q^\varepsilon_{\mathrm{2D}}(\boldsymbol{\zeta}^\varepsilon) - Q^\varepsilon_{\mathrm{3D}}(\mathbf{u}^\varepsilon)\right| \lesssim \varepsilon^{1/4}\, Q^\varepsilon_{\mathrm{2D}}(\boldsymbol{\zeta}^\varepsilon) \lesssim \varepsilon^{5/4}\,.$$

Stricto sensu, we have at hand a family of 3D displacements $\mathbf{u}^\varepsilon$ with azimuthal frequency $k(\varepsilon) \equiv \varepsilon^{-1/4}\gamma_{\min}$ such that

$$\left|Q^\varepsilon_{\mathrm{3D}}(\mathbf{u}^\varepsilon) - \mathsf{a}_1\varepsilon\right| \lesssim \varepsilon^{5/4}.$$

Moreover the trace of the transverse component u^ε_3 on the midsurface ($x_3 = 0$) is close to the generating eigenvector η_0. We have proved the results summarized in the first two lines of Table 1. Our numerical experiments (Model A, sect. 7.1, and Model B, sect. 7.2) suggest that, in fact, $(\mathsf{a}_1\varepsilon, \mathbf{u}^\varepsilon)$ is an approximation of the first 3D eigenpair.

6. ELLIPTIC CASE (SMALL MERIDIAN CURVATURE)

The elliptic case in parametrization $r = f(z)$, $z \in \mathcal{I}$, corresponds to the situation $f'' < 0$ on $\overline{\mathcal{I}}$.

6.1. Membrane reduction in the general case.

When the parametrizing function f is not affine, i.e., when $f'' \not\equiv 0$, the scalar reduction of the membrane operator has non-vanishing first terms as follows:

$$\mathsf{H}_0(z,\partial_z) = E\frac{f''^2}{s^6}, \quad \mathsf{H}_1(z,\partial_z) = 0, \quad \mathsf{H}_2(z,\partial_z) = \mathsf{H}^{(2)}_2(z)\,\partial^2_z + \mathsf{H}^{(1)}_2(z)\,\partial_z + \mathsf{H}^{(0)}_2(z) \quad (6.1)$$

with

$$\begin{cases}
\mathsf{H}^{(2)}_2(z) = 2E\Big(\dfrac{f\,f''}{s^6} + \dfrac{f^2\,f''^2}{s^8}\Big) \\[2mm]
\mathsf{H}^{(1)}_2(z) = 2E\Big(\dfrac{2f'\,f''}{s^6} + \dfrac{f\,f'''}{s^6} - \dfrac{2f\,f'\,f''^2}{s^8} + \dfrac{2f^2\,f''\,f'''}{s^8} - \dfrac{7f^2\,f'\,f''^3}{s^{10}}\Big) \\[2mm]
\mathsf{H}^{(0)}_2(z) = E\Big(-\dfrac{10f'^2\,f''^2}{s^8} + \dfrac{4f'\,f'''}{s^6} + \dfrac{2f'^2\,f''}{f\,s^6} - \dfrac{(\nu-2)f\,f'^2\,f'''}{s^{10}} - \dfrac{5f\,f'\,f''\,f'''}{s^8} \\[2mm]
\qquad\qquad + \dfrac{f\,f^{(4)}}{s^6} + \dfrac{2f^2\,f''\,f^{(4)}}{s^8} + \dfrac{36f^2\,f'^2\,f''^4}{s^{12}} + \dfrac{(\nu-2)f\,f''^3}{s^8} - \dfrac{6f^2\,f''^4}{s^{10}} \\[2mm]
\qquad\qquad - \dfrac{20f^2\,f'\,f''^2\,f'''}{s^{10}}\Big) - \Lambda_0\Big(\dfrac{1}{s} - \dfrac{\nu f''f}{s^3}\Big)^2\,.
\end{cases} \quad (6.2)$$

The rank-3 operator in the formal series $\mathsf{H}[k]$ is given by

$$\mathsf{H}_3(z,\partial_z) = \Big(-\frac{1}{s^2} + \frac{2\nu f\,f''}{s^4} - \frac{\nu^2 f^2\,f''^2}{s^6}\Big)\Lambda_1, \quad (6.3)$$

and the rank-4 operator can be written as

$$\mathsf{H}_4(z,\partial_z) = \sum_{j=0}^4 \mathsf{H}^{(j)}_4(z)\partial^j_z, \quad \text{with} \quad \mathsf{H}^{(4)}_4(z) = E\Big(\frac{4f^3\,f''}{s^8} + \frac{3f^4\,f''^2}{s^{10}} + \frac{f^2}{s^6}\Big) \quad (6.4)$$

where the other terms $\mathsf{H}^{(j)}_4(z)$ are smooth functions of z.

So, $H_0(z, \partial_z) = H_0(z)$ is the multiplication by a function (which can be seen as a potential) and we check that H_2 is a selfadjoint operator of order 2 on $H_0^1(\mathcal{I})$ with respect to the natural measure $d\mathcal{I} = f(z)s(z)\, dz$:

$$\langle H_2\eta, \eta^* \rangle_{\mathcal{I}} = \int_{\mathcal{I}} \Big(-H_2^{(2)}(z)\, \partial_z\eta\, \partial_z\eta^* + H_2^{(0)}(z)\, \eta\, \eta^* \Big)\, d\mathcal{I} . \tag{6.5}$$

We recall from (3.7) that the principal curvatures are $b_z^z = \frac{f''}{s^3}$ and $b_\varphi^\varphi = -\frac{1}{fs}$. Note that both are negative in the elliptic case.

Remark 6.1. (i) The function H_0/E coincides with the square of the meridian curvature

$$H_0 = E\, (b_z^z)^2.$$

(ii) There holds the following relation between $H_2^{(2)}$ and the principal curvatures

$$-H_2^{(2)} = 2E\, \frac{f^2}{s^2}\, b_z^z(b_\varphi^\varphi - b_z^z). \tag{6.6}$$

(iii) Similarly

$$H_4^{(4)} = E\, \frac{f^4}{s^4}\, (b_\varphi^\varphi - 3b_z^z)(b_\varphi^\varphi - b_z^z). \tag{6.7}$$

6.2. **High frequency analysis of the membrane operator in the elliptic case.** As mentioned above, we have to select one or several terms starting the series $H[k]$ that will play the role of an engine to work out a recurrence and allow to solve the formal series problem (4.7). In the parabolic case, this engine is h^4H_4. In the elliptic case, H_0 is the multiplication by the positive function $E(b_z^z)^2$. Its spectrum is essential and its bottom determines Λ_0

$$\Lambda_0 = E\min_{z\in\overline{\mathcal{I}}}(b_z^z)^2. \tag{6.8}$$

We have to complete H_0 by further terms so that to obtain an operator with discrete spectrum close to the minimum energy Λ_0. This will be the case for the operator

$$\mathfrak{H}^k = H_0 + k^{-2}H_2 \tag{6.9}$$

if, cf. condition (1.13),

$$-H_2^{(2)} \geq 0 \quad \text{on } \overline{\mathcal{I}}, \quad \text{i.e.} \quad |b_\varphi^\varphi| \geq |b_z^z| \quad \text{on } \overline{\mathcal{I}}, \tag{6.10}$$

(use (6.6)), with strict inequalities for the values of z where H_0 attains its minimum Λ_0. It is interesting to note that the latter condition implies that, cf. (6.8) and (4.8),

$$\min_{z\in\overline{\mathcal{I}}}(b_z^z)^2 < \min_{z\in\overline{\mathcal{I}}}(b_\varphi^\varphi)^2, \quad \text{i.e.} \quad \Lambda_0 < \min\sigma_{\mathsf{ess}}(\mathfrak{M}^k),$$

which means that the expected limit at high frequency will be attained by eigenvalues below the essential spectrum.

Remark 6.2. We note that in the hyperbolic case, $f'' > 0$, so $b_z^z > 0$. Hence the coefficient $-H_2^{(2)}$ is *always negative* and our analysis *never applies* in the hyperbolic case. Besides, in this case, Λ_0 is not the membrane high frequency limit, that is indeed 0 (recall that the exponent in (1.2) is $\alpha = \frac{2}{3}$ in hyperbolic case).

From now on, we assume that (6.10) holds and we discuss the lowest eigenpairs of the operators $\mathfrak{H}^k$ defined in (6.9) and

$$\mathfrak{A}^k(\varepsilon) = \mathsf{H}_0 + k^{-2}\mathsf{H}_2 + \varepsilon^2 k^4 \mathsf{B}_0, \quad \text{where} \quad \mathsf{B}_0 = \frac{1}{3} \frac{E}{1-\nu^2} \frac{1}{f^4} \tag{6.11}$$

in relation with properties of the "potential" H_0. For simplicity we denote

$$\mathsf{g}(z) := -\mathsf{H}_2^{(2)}(z), \tag{6.12}$$

and consider successively the cases when H_0 has a non-degenerate minimum inside or on the boundary of the interval $\mathcal{I}$, or when it is constant.

6.3. Internal minimum of the potential (Gaussian case).

Besides (6.10), we assume that H_0 has a (unique) nondegenerate minimum in $z_0 \in \mathcal{I}$. Thus

$$\Lambda_0 = \mathsf{H}_0(z_0) \quad \text{and} \quad \partial_z^2 \mathsf{H}_0(z_0) > 0.$$

We assume moreover $\mathsf{g}(z_0) > 0$.

6.3.1. *High frequency analysis for the membrane operator.* Then the lowest eigenpairs of the membrane reduction $\mathfrak{A}^k = \mathsf{H}_0 + k^{-2}\mathsf{H}_2$ as $k \to \infty$ are driven by the harmonic oscillator

$$- \mathsf{g}(z_0)\, \partial_Z^2 + \frac{Z^2}{2}\, \partial_z^2 \mathsf{H}_0(z_0)\,. \tag{6.13}$$

Here, the new homogenized variable Z spans $\mathbb{R}$ and is linked to the physical variable z by the relation

$$Z = \sqrt{k}\,(z - z_0)\,. \tag{6.14}$$

This change of variable can be applied to the formal series reduction (4.6) as follows: Let $L[k] = \sum_{k \geq 0} k^{-n} L_n(z, \partial_z)$ be a formal series such that L_n is an operator of order n. By Taylor expansion around z_0, we can expand for all n the operator $L_n(z, \partial_z) = \sum_{j \geq -n} k^{-j/2} \mathcal{L}_{n,j}(Z, \partial_Z)$. By reordering the powers of $k^{-j/2}$, we thus see that we can write

$$L[k] \equiv \mathcal{L}[k^{1/2}] = \sum_{n \geq 0} k^{-n/2} \mathcal{L}_n(Z, \partial_Z),$$

where the operators $\mathcal{L}_n$ have polynomial coefficients in Z. Applying this change of variable to the formal series reduction given in Theorem 4.2, we obtain the new identity

$$\boldsymbol{\mathcal{M}}[k^{1/2}]\boldsymbol{\mathcal{V}}[k^{1/2}] - \Lambda[k^{1/2}]\mathbf{A}\boldsymbol{\mathcal{V}}[k^{1/2}] = \mathbf{V}_0 \circ (\mathcal{H}[k^{1/2}] - \Lambda[k^{1/2}])\,, \tag{6.15}$$

where $\boldsymbol{\mathcal{M}}[k^{1/2}]$, $\boldsymbol{\mathcal{V}}[k^{1/2}]$ and $\mathcal{H}[k^{1/2}]$ are the formal series induced by the formal series $\mathbf{M}[k]$, $\mathbf{V}[k]$ and $\mathsf{H}[k]$ respectively. We also agree that $\Lambda[k^{1/2}]$ is related with the old series $\Lambda_{\mathsf{old}}[k] = \sum_{n \geq 0} k^{-n}\Lambda_{n,\mathsf{old}}$ by the identities $\Lambda_n = 0$ if n is odd, and $\Lambda_n = \Lambda_{n/2,\mathsf{old}}$ if n is even. Moreover, we calculate that

$$\mathcal{H}_0 = \mathsf{H}_0(z_0), \quad \mathcal{H}_1 = 0, \quad \text{and} \quad \mathcal{H}_2 = -\mathsf{g}(z_0)\, \partial_Z^2 + \frac{Z^2}{2}\, \partial_z^2 \mathsf{H}_0(z_0).$$

Like for (4.7), the previous reduction leads to consider the formal series problem

$$\mathcal{H}[k^{1/2}]\eta[k^{1/2}] = \Lambda[k^{1/2}]\eta[k^{1/2}]\,. \tag{6.16}$$

The first equation induced by this identity is $\mathcal{H}_0\eta_0 = \Lambda_0\eta_0$, hence we have found again $\Lambda_0 = \mathcal{H}_0 = \mathsf{H}_0(z_0)$. Since for any η we have now $\mathcal{H}_0\eta = \Lambda_0\eta$, the next equations yield

$$\mathcal{H}_1\eta_0 = \Lambda_1\eta_0 \quad \text{and} \quad \mathcal{H}_2\eta_0 = \Lambda_2\eta_0.$$

Therefore $\Lambda_1 = 0$ (which is coherent with what was agreed in identity (6.15)) and η_0 is an eigenvector of the harmonic oscillator (6.13). The eigenvalues of this latter operator are

$$(2\ell - 1)\,\mathsf{c}, \quad \ell = 1, 2, \dots \quad \text{with} \quad \mathsf{c} = \frac{1}{\sqrt{2}}\sqrt{\mathsf{g}(z_0)\,\partial_z^2\mathsf{H}_0(z_0)} \tag{6.17}$$

and the corresponding eigenvectors are Gaussian functions. Taking $\eta_0(Z)$ as the first eigenmode ($\ell = 1$) we can construct the first terms $\eta_1, \eta_2, \dots$ of the formal series problem (6.16). As the coefficients of the operators $\mathcal{H}_j$ depend polynomially on Z, these terms are exponentially decreasing with respect to Z. We can then define the pair $(\breve{\Lambda}^k, \breve{\zeta}^k)$ by the formula

$$\breve{\zeta}^k = \chi(z)\Big(\mathbf{V}_0\eta_0 + \sum_{1 \le n+m \le 6} k^{-(n+m)/2}(\mathbf{V}_n\eta_m)(\sqrt{k}(z - z_0))\Big)$$
$$\breve{\Lambda}^k = \mathsf{H}_0(z_0) + k^{-1}\mathsf{c}, \tag{6.18}$$

where $\chi \in C_0^\infty(\mathcal{I})$ is identically equal to 1 in a neighborhood of z_0. This pair is a quasimode for the full membrane operator $\mathfrak{M}^k$ as $k \to \infty$, and we obtain that

$$\mathrm{dist}\big(\mathsf{H}_0(z_0) + k^{-1}\mathsf{c}\,,\, \sigma(\mathfrak{M}^k)\big) \lesssim k^{-3/2}, \quad k \to \infty. \tag{6.19}$$

Note that in this case, the boundary conditions are automatically fulfilled as the quasimode constructed is localized near z_0.

6.3.2. *High frequency analysis for Koiter and Lamé operators.*

Now we consider the operator $\mathfrak{A}^k(\varepsilon)$ defined in (6.11). We look for its smallest eigenvalue $\mu_1^{\mathfrak{A}}(\varepsilon, k)$ and for each $\varepsilon > 0$ small enough, look for $k(\varepsilon)$ such that $\mu_1^{\mathfrak{A}}(\varepsilon, k(\varepsilon))$ is minimum. Setting $\delta := \varepsilon^2 k^4$, we see that the operator $\mathfrak{A}^k(\varepsilon)$ has the form

$$W + k^{-2}\mathsf{H}_2 \quad \text{with} \quad W = \mathsf{H}_0 + \delta\mathsf{B}_0.$$

If δ is small enough, the function W has the same property as H_0, i.e., it has a (unique) nondegenerate minimum. Let $z_0(\delta)$ be the point where this minimum is attained. By implicit function theorem, the correspondence $\delta \to z_0(\delta)$ is smooth for δ small enough and there holds

$$\mu_1^{\mathfrak{A}}(\varepsilon, k) = \mathsf{H}_0(z_0(\delta)) + \delta\mathsf{B}_0(z_0(\delta)) + \frac{k^{-1}}{\sqrt{2}}\sqrt{\mathsf{g}(z_0(\delta))\,\partial_z^2(\mathsf{H}_0 + \delta\mathsf{B}_0)(z_0(\delta))} + \mathcal{O}(k^{-3/2})$$

But

$$\mathsf{H}_0(z_0(\delta)) = \mathsf{H}_0(z_0) + \mathcal{O}(\delta^2), \qquad \partial_z^2\mathsf{H}_0(z_0(\delta)) = \partial_z^2\mathsf{H}_0(z_0) + \mathcal{O}(\delta),$$
$$\mathsf{B}_0(z_0(\delta)) = \mathsf{B}_0(z_0) + \mathcal{O}(\delta), \qquad \mathsf{g}(z_0(\delta)) = \mathsf{g}(z_0) + \mathcal{O}(\delta).$$

Hence

$$\mu_1^{\mathfrak{A}}(\varepsilon, k) = \mathsf{H}_0(z_0) + \delta\mathsf{B}_0(z_0) + \frac{k^{-1}}{\sqrt{2}}\sqrt{\mathsf{g}(z_0)\,\partial_z^2\mathsf{H}_0(z_0)} + \mathcal{O}(\delta^2) + \mathcal{O}(k^{-1}\delta) + \mathcal{O}(k^{-3/2}).$$

Let us set

$$\mathsf{b} = \mathsf{B}_0(z_0) \quad \text{and} \quad \mathsf{c} = \frac{1}{\sqrt{2}}\sqrt{\mathsf{g}(z_0)\,\partial_z^2\mathsf{H}_0(z_0)}. \tag{6.20}$$

So, replacing δ by its value $\varepsilon^2 k^4$, we look for $k = k(\varepsilon)$ such that $\varepsilon^2 k^4 \mathsf{b} + k^{-1}\mathsf{c}$ is minimum and such that $\delta = \varepsilon^2 k^4$ is small[2]. We homogenize the powers of k by letting $\gamma(\varepsilon) = k(\varepsilon)\,\varepsilon^{2/5}$, and setting $\mu_1^{\mathfrak{A}}(\varepsilon) = \mu_1^{\mathfrak{A}}(\varepsilon, k(\varepsilon))$ we find

$$k(\varepsilon) = \gamma\,\varepsilon^{-2/5} \quad \text{and} \quad \mu_1^{\mathfrak{A}}(\varepsilon) = \mathsf{H}_0(z_0) + \mathsf{a}_1\,\varepsilon^{2/5} + \mathcal{O}(\varepsilon^{3/5}), \tag{6.21}$$

with the explicit constants γ and a_1:

$$\gamma = \left(\frac{\mathsf{c}}{4\mathsf{b}}\right)^{1/5} \quad \text{and} \quad \mathsf{a}_1 = (4\mathsf{b}\mathsf{c}^4)^{1/5}\Big(1 + \frac{1}{4}\Big). \tag{6.22}$$

Along the same lines as in the parabolic case, we convert the power law for k (6.21) into the power law $\varepsilon = (\gamma/k)^{5/2}$. We can then consider a formal series reduction as in (5.17) and combine it with the change of variable $Z = \sqrt{k}(z - z_0)$. The same analysis as before yields quasimodes $(\breve{\Lambda}^k(\varepsilon), \breve{\zeta}^k(\varepsilon))$. Here $\breve{\Lambda}^k(\varepsilon) = \mu_1^{\mathfrak{A}}(\varepsilon)$ and $\breve{\zeta}^k(\varepsilon)$ has a form similar to (6.18), whereas $k = k(\varepsilon)$. Note that these quasimodes remain localized around z_0 and hence bending boundary layers do not show up as they did in the parabolic case. We thus obtain

$$\mathrm{dist}\big(\mathsf{m}_1(\varepsilon)\,,\,\sigma(\mathfrak{K}(\varepsilon))\big) \lesssim \varepsilon^{3/5} \quad \text{with} \quad \mathsf{m}_1(\varepsilon) = \mathsf{H}_0(z_0) + \mathsf{a}_1\,\varepsilon^{2/5}. \tag{6.23}$$

Remark 6.3. If H_0 attains its minimum a_0 in a finite number of points $z_0^{(i)}$, we can construct quasimodes attached to each of these points of the same form as above, and with disjoint supports. The associated quantities obey to the same formulas as in (6.21)-(6.23)

$$k^{(i)}(\varepsilon) = \gamma^{(i)}\,\varepsilon^{-2/5} \quad \text{and} \quad \mathsf{m}_1^{(i)}(\varepsilon) = \mathsf{a}_0 + \mathsf{a}_1^{(i)}\,\varepsilon^{2/5},$$

with $\gamma^{(i)}$ and $\mathsf{a}_1^{(i)}$ defined by (6.22) with the values of quantities b and c at point $z_0^{(i)}$. Then $\mathsf{m}_1(\varepsilon) = \min_i \mathsf{m}_1^{(i)}(\varepsilon)$ and $k(\varepsilon) = k^{(i_0)}(\varepsilon)$ for i_0 such that the previous minimum is attained.

6.3.3. *3D reconstruction and Rayleigh quotients.* As in the parabolic case, we construct a three-component vector field on the surface $\mathcal{S}$ by setting in normal coordinates

$$\boldsymbol{\zeta}^\varepsilon(z, \varphi) = e^{ik\varphi}\boldsymbol{\zeta}^k(z) \quad \text{with} \quad \boldsymbol{\zeta}^k \text{ given in (6.18) and } k = \varepsilon^{-2/5}\gamma_{\min},$$

and by setting

$$\mathbf{u}^\varepsilon = \overset{\circ}{\mathsf{U}}\boldsymbol{\zeta}^\varepsilon.$$

Since all traces of any order of $\boldsymbol{\zeta}^\varepsilon$ vanish on $\partial\mathcal{I}$, there is no boundary corrector and we are in case *(ii)* of Theorem 2.2. So we only have to estimate the behavior of the wave length $L = L^\varepsilon$ (2.17) of $\boldsymbol{\zeta}^\varepsilon$ as $\varepsilon \to 0$. We note the influence of:

- The profiles $\mathbf{G}_n(\sqrt{k}(z - z_0))$ with $k \simeq \varepsilon^{-2/5}$, that contribute a term in $1/\sqrt{k} \simeq \varepsilon^{1/5}$,
- The azimuthal oscillation $e^{ik\varphi}$ that contributes a term in $k^{-1} \simeq \varepsilon^{2/5}$.

As a result we find in the nondegenerate parabolic case

$$L^\varepsilon \gtrsim \varepsilon^{2/5}.$$

So the assumptions of Theorem 2.2 are uniformly satisfied for the family $(\boldsymbol{\zeta}^\varepsilon)_\varepsilon$ and the estimate (2.19) reads now

$$\big|Q_{2\mathrm{D}}^\varepsilon(\boldsymbol{\zeta}^\varepsilon) - Q_{3\mathrm{D}}^\varepsilon(\mathbf{u}^\varepsilon)\big| \lesssim \varepsilon\,Q_{2\mathrm{D}}^\varepsilon(\boldsymbol{\zeta}^\varepsilon) \lesssim \varepsilon.$$

---

[2]We check that $\delta = \mathcal{O}(\varepsilon^{2/5})$

Thus, we have exhibited a family of 3D displacements $\mathbf{u}^\varepsilon$ with azimuthal frequency $k(\varepsilon) \equiv \varepsilon^{-2/5}\gamma$ such that

$$\left| Q_{3D}^\varepsilon(\mathbf{u}^\varepsilon) - \mathsf{m}_1(\varepsilon) \right| \lesssim \varepsilon^{3/5} \quad \text{with} \quad \mathsf{m}_1(\varepsilon) = \mathsf{H}_0(z_0) + \mathsf{a}_1\,\varepsilon^{2/5}.$$

So we have proved the results summarized in the third line of Table 1. The numerical experiments (Model H, sect. 7.4) suggest that $(\mathsf{m}_1(\varepsilon), \mathbf{u}^\varepsilon)$ is indeed an approximation of the first 3D eigenpair.

6.4. Minimum of the potential on the boundary (Airy case).

We assume that H_0 attains its minimum at a point $z_0 \in \partial\mathcal{I}$ with $\partial_z\mathsf{H}_0(z_0) \neq 0$. Let us agree that z_0 is the left end of $\mathcal{I}$, i.e., $z_0 = z_-$, so that we have

$$\Lambda_0 = \mathsf{H}_0(z_0) \quad \text{and} \quad \partial_z\mathsf{H}_0(z_0) > 0.$$

We still assume $\mathsf{g}(z_0) > 0$. The analysis is somewhat similar to the previous case, though a little more tricky.

6.4.1. *High frequency analysis for the membrane operator. Membrane boundary layers.*

We meet the Airy-like operator

$$- \mathsf{g}(z_0)\partial_Z^2 + Z\,\partial_z\mathsf{H}_0(z_0) \tag{6.24}$$

on $H_0^1(\mathbb{R}^+)$ instead the harmonic oscillator (6.13). The homogenized variable Z is given by

$$Z = (z - z_0)k^{2/3}. \tag{6.25}$$

We can perform an analysis very similar to the previous case by doing a change of variable in the formal series reduction. This yields formal series problem in powers of $k^{-1/3}$ whose first terms are given by $\mathcal{H}_0 + k^{-2/3}\mathcal{H}_2$ where $\mathcal{H}_0 = \mathsf{H}_0(z_0)$ and $\mathcal{H}_2$ is the operator (6.24). The eigenvalues of the model operator (6.24) are given by

$$\mathsf{z}_{\mathrm{Airy}}^{(\ell)} \left(\mathsf{g}(z_0)\right)^{1/3} \left(\partial_z\mathsf{H}_0(z_0)\right)^{2/3}, \quad \ell = 1, 2, \dots \tag{6.26}$$

where $\mathsf{z}_{\mathrm{Airy}}^{(\ell)}$ is the ℓ-th zero of the reverse Airy function Ai. We find that the first eigenvalue of the membrane reduction $\mathfrak{H}^k$ satisfies

$$\mu_1^{\mathfrak{H}}(k) = \mathsf{H}_0(z_0) + k^{-2/3}\mathsf{c} + \mathcal{O}(k^{-1}) \quad \text{with} \quad \mathsf{c} = \mathsf{z}_{\mathrm{Airy}}^{(1)} \left(\mathsf{g}(z_0)\right)^{1/3} \left(\partial_z\mathsf{H}_0(z_0)\right)^{2/3}. \tag{6.27}$$

Using the reconstruction operators $\mathcal{V}[k^{1/3}]$ in the scaled variable allows to construct displacement $\breve{\zeta}^k$ from an eigenfunction profile $\eta_0(Z)$ of the Airy operator. However, the first terms of the reconstruction take the form:

$$\begin{pmatrix} k^{-4/3}\zeta_z^k \\ k^{-1}\zeta_\varphi^k \\ \zeta_3^k \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} \zeta_z^k \\ \zeta_\varphi^k \\ \zeta_3^k \end{pmatrix} = \begin{pmatrix} f^2(b_\varphi^\varphi - (\nu + 2)b_z^z)\partial_Z\eta_0 \\ -if^2(b_\varphi^\varphi + \nu b_z^z)\eta_0 \\ \eta_0 \end{pmatrix} + \mathcal{O}(k^{-1/3}).$$

While we can impose $\eta_0(0) = 0$ to ensure that $\zeta_\varphi^k = 0$ at first order, we see that we have in general $\zeta_z^k \neq 0$. To construct a quasimode, we have to add new boundary layer terms to $\breve{\zeta}^k$.

To determine such boundary layers near $z_0 = z_-$, we introduce the scaled variable $Z = kd$ with $d = z - z_-$. Like already seen for the Koiter operator (sect. 5.2.4), the formal series operator $\mathbf{M}[k]$

is changed to a new formal series $\mathcal{M}[k]$ whose first term is given by

$$\mathcal{M}_0 = \begin{pmatrix} -\frac{1}{s^4}\partial_Z^2 + \frac{1-\nu}{2}(b_\varphi^\varphi)^2 & -\frac{1+\nu}{2}i(b_\varphi^\varphi)^2\partial_Z & \frac{1}{s^2}(b_z^z + \nu b_\varphi^\varphi)\partial_Z \\ -\frac{1+\nu}{2}i(b_\varphi^\varphi)^2\partial_Z & -\frac{1-\nu}{2}(b_\varphi^\varphi)^2\partial_Z^2 + \frac{1}{f^4} & i\frac{1}{f^2}(b_\varphi^\varphi + \nu b_z^z) \\ -\frac{1}{s^2}(b_z^z + \nu b_\varphi^\varphi)\partial_Z & -i\frac{1}{f^2}(b_\varphi^\varphi + \nu b_z^z) & (b_\varphi^\varphi + \nu b_z^z)^2 \end{pmatrix}$$

where the quantities are evaluated in $z_0 = z_-$. We can prove that this operator yields boundary layer profiles $\mathbf{G}(Z)$ exponentially decreasing with respect to $Z = kd$, and satisfying $G_z(0) = a_z$ for any given number a_z, which allows to compensate for the trace of the first term of ζ_z^k.

We obtain a compound quasimode combining terms at scale $k^{2/3}d$ and terms at scale kd, and deduce in the end

$$\text{dist}\left(\mathsf{H}_0(z_0) + k^{-2/3}\mathsf{c}\,,\ \sigma(\mathfrak{M}^k)\right) \lesssim k^{-1}, \quad k \to \infty. \tag{6.28}$$

6.4.2. *High frequency analysis for Koiter operator.* The Koiter scalar reduction operator $\mathfrak{A}^k(\varepsilon)$, see (6.11) is still an Airy-like operator because the minimum of $\mathsf{H}_0 + \varepsilon^2 k^4$ is still z_0 for $\varepsilon^2 k^4$ small enough. We look for $k = k(\varepsilon)$ such that $\varepsilon^2 k^4 \mathsf{b} + k^{-2/3}\mathsf{c}$ is minimum. We homogenize $\varepsilon^2 k^4$ with $k^{-2/3}$. We find

$$k(\varepsilon) = \gamma\,\varepsilon^{-3/7} \quad \text{and} \quad \mu_1^{\mathfrak{A}}(\varepsilon) = \mathsf{H}_0(z_0) + \mathsf{a}_1\,\varepsilon^{2/7} + \mathcal{O}(\varepsilon^{3/7}), \tag{6.29}$$

with the explicit constants γ and a_1, with $\mathsf{b} = \mathsf{B}_0(z_0)$ and c defined in (6.27):

$$\gamma = \left(\frac{\mathsf{c}}{6\mathsf{b}}\right)^{3/14} \quad \text{and} \quad \mathsf{a}_1 = (6\mathsf{b}\mathsf{c}^6)^{1/7}(1 + \frac{1}{6}). \tag{6.30}$$

Note that in this case, two types of boundary layer terms are present: the one constructed above (membrane boundary layer) and the bending boundary layers terms associated with the Koiter operator, see sect. 5.2.4. We obtain

$$\text{dist}\left(\mathsf{m}_1(\varepsilon)\,,\ \sigma(\mathfrak{K}(\varepsilon))\right) \lesssim \varepsilon^{3/7} \quad \text{with} \quad \mathsf{m}_1(\varepsilon) = \mathsf{H}_0(z_0) + \mathsf{a}_1\,\varepsilon^{2/7}.$$

At this point, the reconstruction operator $\overset{\circ}{\mathsf{U}}$ is not precise enough to allow us to conclude as in the parabolic and Gaussian cases. Nevertheless numerical experiments (Model L, sect. 7.5) tend to confirm that the first eigenvalue of the Lamé operator $\mathfrak{L}^\varepsilon$ behaves like $\mathsf{m}_1(\varepsilon)$.

6.5. **Constant potential (toroidal case).** Let us assume that H_0 is constant. We recall that $\mathsf{H}_0 = E(b_z^z)^2$. But b_z^z coincides with the curvature of the arc $\mathcal{C}$ of equation $r = f(z)$ in the meridian plane. So, b_z^z is constant if and only if $\mathcal{C}$ is a *circular* arc. Let R be its radius and $(r_\circ, z_\circ) \in \mathbb{R}^2$ be its center. Notice that the center of the circular arc may be at negative $r_\circ$. Then, in the elliptic case $f'' < 0$,

$$f(z) = r_\circ + \sqrt{R^2 - (z - z_\circ)^2}, \tag{6.31}$$

and the principal curvatures are given by

$$b_z^z = -\frac{1}{R} \quad \text{and} \quad b_\varphi^\varphi(z) = -\frac{1}{R}\left(1 - \frac{r_\circ}{f(z)}\right). \tag{6.32}$$

So in this case, we have

$$\mathsf{H}_0 = \frac{E}{R^2} = \Lambda_0 \quad \text{and} \quad \mathsf{g} = -\mathsf{H}_2^{(2)} = -2E\,\frac{f}{s^2}\,\frac{r_\circ}{R^2}. \tag{6.33}$$

Now the Koiter scalar reduction operator $\mathfrak{A}^k(\varepsilon)$ is $\mathsf{H}_0 + k^{-2}\mathsf{H}_2 + \varepsilon^2 k^4 \mathsf{B}_0$ where H_0 is a constant function acting as a simple shift on the spectrum. In this case, no concentration occurs, and we have simply to come back to the approach used for the parabolic case *mutatis mutandis*, with k^{-2} instead of k^{-4}.

6.5.1. *Membrane scalar reduction.* We assume the sharp version of condition (6.10) (strict inequalities) that ensures that H_2 has a compact resolvent and is semibounded from below. Thanks to (6.33), we find that such condition is equivalent to

$$r_\circ < 0 . \tag{6.34}$$

Let Λ_2 be the first eigenvalue of H_2. Then the first eigenvalue of the membrane scalar reduction operator $\mathfrak{H}^k = \mathsf{H}_0 + k^{-2}\mathsf{H}_2$ is $\Lambda_0 + k^{-2}\Lambda_2$ and we can deduce that

$$\operatorname{dist}\big(\Lambda_0 + k^{-2}\Lambda_2 ,\ \sigma(\mathfrak{M}^k)\big) \lesssim k^{-5/2}, \quad k \to \infty. \tag{6.35}$$

6.5.2. *Koiter scalar reduction.* The operator $\mathfrak{A}^k(\varepsilon) = \mathsf{H}_0 + k^{-2}\mathsf{H}_2 + \varepsilon^2 k^4 \mathsf{B}_0$ is self-adjoint on $H^1_0(\mathcal{I})$. Let $\mu_1^{\mathfrak{A}}(\varepsilon, k)$ be its first eigenvalue. For any chosen ε we look for $k(\varepsilon)$ realizing the minimum of $\mu_1^{\mathfrak{A}}(\varepsilon, k)$. We set

$$\gamma(\varepsilon) = k(\varepsilon)\,\varepsilon^{1/3} \tag{6.36}$$

so that our operator becomes

$$\mathsf{H}_0 + \varepsilon^{2/3}\Big(\frac{1}{\gamma(\varepsilon)^2}\,\mathsf{H}_2 + \gamma(\varepsilon)^4\,\mathsf{B}_0\Big).$$

Therefore γ does not depend on ε. Let $\mu_1(\gamma)$ be the first eigenvalue of the operator

$$\frac{1}{\gamma^2}\,\mathsf{H}_2 + \gamma^4\,\mathsf{B}_0 . \tag{6.37}$$

The function $\gamma \mapsto \mu_1(\gamma)$ is continuous. At this point we need the following extra assumption:

$$\Lambda_2 > 0, \quad \text{i.e.} \quad \mathsf{H}_2 > 0. \tag{6.38}$$

Then the same argument as in the parabolic case allows to define $\gamma_{\min}$ as the (smallest) positive constant such that $\mu_1(\gamma)$ is minimum

$$\mu_1(\gamma_{\min}) = \min_{\gamma \in \mathbb{R}_+} \mu_1(\gamma) =: \mathsf{a}_1 . \tag{6.39}$$

As an illustration of the non-trivial behavior of the quantities Λ_2, $\gamma_{\min}$ and a_1, we plot them versus $r_\circ$ in Figure 3 (we choose $R = 2$ and $z_\circ = 0$).

Thus $k(\varepsilon)$ satisfies a power law that yields a formula for the minimal first eigenvalue $\mu_1^{\mathfrak{A}}(\varepsilon)$:

$$k(\varepsilon) = \varepsilon^{-1/3}\gamma_{\min} \quad \text{and} \quad \mu_1^{\mathfrak{A}}(\varepsilon) = \mathsf{m}_1(\varepsilon) = \mathsf{H}_0 + \varepsilon^{2/3}\mathsf{a}_1 , \tag{6.40}$$

and after adding membrane and bending boundary layer terms as in the Airy case we arrive to

$$\operatorname{dist}\big(\mathsf{m}_1(\varepsilon) ,\ \sigma(\mathfrak{K}(\varepsilon))\big) \lesssim \varepsilon \quad \text{with} \quad \mathsf{m}_1(\varepsilon) = \mathsf{H}_0 + \varepsilon^{2/3}\mathsf{a}_1.$$

We note that, in contrast with the two previous cases when H_0 is not constant, the lower order term $\mathsf{H}_2^{(0)}$ of the operator H_2 is involved in the asymptotics. Finally, like in the Airy case, we would need a more complete reconstruction operator combined with "plates" boundary layer terms to
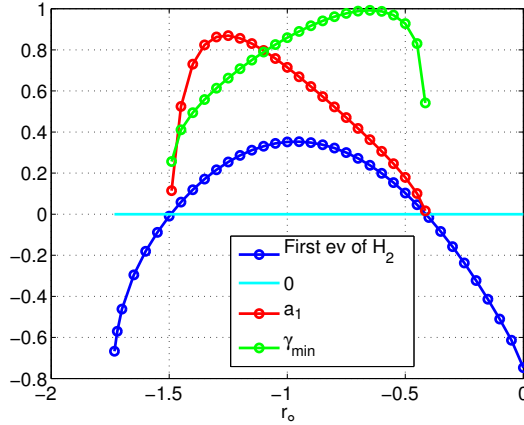
FIGURE 3. Quantities Λ_2, $\gamma_{\min}$ and a_1 vs $r_\circ$ (with $R = 2$ and $z_\circ = 0$).

conclude the construction of 3D minimizers or 3D quasimodes. Nevertheless, numerical experiments (Model D, sect. 7.3) prove that, at least for some values of $r_\circ$, the lowest 3D eigenpairs fit the asymptotics $\mathsf{m}_1(\varepsilon) = \mathsf{H}_0 + \varepsilon^{2/3}\mathsf{a}_1$ and $k(\varepsilon) = \varepsilon^{-1/3}\gamma_{\min}$.

7. MODELS

We present in this section five models: cylinders (Model A), cones (Model B), "toroidal barrels" (Model D), "Gaussian barrels" (Model H), and "Airy barrels" (Model L), representing each of the five types that we could investigate from a theoretical point of view. We choose for all models

$$E = 1 \quad \text{and} \quad \nu = 0.3$$

and perform 1D, 2D and 3D computations for each model. The 2D and 3D computations are performed with finite element codes (MÉLINA [29] for 2D and STRESSCHECK[3] for 3D) and for a finite set of values of ε ranging from 0.2 to 10^{-4} (in general this set contains the values 0.2, 0.1, and $5 \cdot 10^{-j}$, $2 \cdot 10^{-j}$, 10^{-j} for $j = 2, 3, 4$).

The 1D calculations consist in computing the coefficients a_0, a_1 of (1.10) and γ of (1.8). For Gaussian and Airy barrels we use our explicit formulas (6.22) and (6.30). For cylinders, cones and toroidal barrels, we compute with MATLAB the spectrum of the one dimensional reduced operators (5.11) and (6.37) and optimize with respect to the parameter γ.

The 2D calculations solve the Lamé system at azimuthal frequency k on meridian domains ω^ε, see the corresponding variational formulations in Appendix B. For each thickness parameter ε, any integer value of k from 0 to a certain cut-off frequency $k_{\max} = k_{\max}(\varepsilon)$ is used. The cut-off frequency $k_{\max}$ is determined so that we can observe a minimum for the first eigenvalue depending on k. This provides the numerical value for $k(\varepsilon)$. The domain ω^ε is meshed by curvilinear quadrilaterals of geometric degree 3. The meshes contain 2 elements in the thickness direction,

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[3]Stress Check 9.0 is a trade mark of Engineering Software Research and Development, Inc., St. Louis, MO 63141, U.S.A.

Model	$k(\varepsilon)$	γ	$\mathsf{m}_1(\varepsilon)$	a_0	a_1	Remainder
PARABOLIC						
A / Cylinder	$\gamma\varepsilon^{-1/4}$	2.9323126	$\mathsf{a}_1\varepsilon$	0	3.385232	$15\,\varepsilon^{3/2}$ (asymptotics)
B / Cone	$\gamma\varepsilon^{-1/4}$	2.1247294	$\mathsf{a}_1\varepsilon$	0	3.446424	$15\,\varepsilon^{3/2}$ (asymptotics)
ELLIPTIC						
D / Toroidal	$\gamma\varepsilon^{-1/3}$	0.8593528	$\mathsf{a}_0 + \mathsf{a}_1\varepsilon^{2/3}$	0.2500000	0.715000	$0.2\,\varepsilon$ (upper bound)
H / Gauss	$\gamma\varepsilon^{-2/5}$	0.7590108	$\mathsf{a}_0 + \mathsf{a}_1\varepsilon^{2/5}$	0.0625000	0.607854	$0.05\,\varepsilon^{3/5}$ (asymptotics)
L / Airy	$\gamma\varepsilon^{-3/7}$	0.8514053	$\mathsf{a}_0 + \mathsf{a}_1\varepsilon^{2/7}$	0.1780449	1.554722	$2.9\,\varepsilon^{4/7}$ (asymptotics)

TABLE 2. Numerical values for asymptotic quantities $k(\varepsilon)$ and $\mathsf{m}_1(\varepsilon)$. Observed asymptotics for the remainder $\tilde{\lambda}_1^\varepsilon - \mathsf{m}_1(\varepsilon)$ (with $\tilde{\lambda}_1^\varepsilon$ obtained by 2D computations).

and 8, 12 or 16 in the meridian direction. The polynomial interpolation degree of the FEM is 6 in each direction.

The 3D calculations solve the Lamé system $\mathfrak{L}$ on the 3D shells Ω^ε. The azimuthal frequency $k(\varepsilon)$ is observed by counting the oscillations of the radial component (3.1) of the first eigenmode.

We represent in figures 4, 7 and 10, the meridian domains ω^ε in the (r, z) plane for models A, B and D, respectively. The curve $\mathcal{C}$ is dotted. Figures 15 and 18 provide ω^ε for $\varepsilon = 0.2$ for models H and L. Figures 5, 8, 11, 13, and 16 show the lowest computed eigenvalue $\tilde{\lambda}_1^\varepsilon$ and the associated azimuthal frequency $k(\varepsilon)$ versus ε in loglog scale (in base 10). For elliptic models D, H and L, the difference $\tilde{\lambda}_1^\varepsilon - \Lambda_0$ is plotted. The 1D asymptotics is the line $\varepsilon \mapsto \mathsf{m}_1(\varepsilon)$ (1.12) (or $\varepsilon \mapsto \mathsf{m}_1(\varepsilon) - \Lambda_0$ in elliptic models). Figures 6, 9, 12, 14, and 17 show the radial component of the first 3D eigenvector for three values of ε and the five models, respectively. Figures 15 and 18 are surface plots of the first 2D eigenvectors in Gaussian and Airy barrels, respectively, exhibiting the concentration of the modes as ε decreases. For visibility, they are scaled with respect to the width in order to be represented on the meridian domain with thickness parameter $\varepsilon = 0.2$.

We summarize in Table 2 the numerical values of the asymptotic quantities $\mathsf{m}_1(\varepsilon)$ and $k(\varepsilon)$, as well as the observed asymptotics for the remainder $\tilde{\lambda}_1^\varepsilon - \mathsf{m}_1(\varepsilon)$ for each of the five models.

The formulas providing parameters γ, a_0 and a_1 are given in the following equations: (5.12)–(5.15) for models A and B, (6.31), (6.39) for model D. Concerning models H and L, the function $\mathsf{H}_0/E = \frac{f''^2}{s^6}$ is equal to $(\frac{1}{4} + \frac{3}{4}z^2)^2/(1 + (\frac{1}{4}z + \frac{1}{4}z^3)^2)^3$ and reaches its minimum in the interior point $z_0 = 0$ for model H, and in the boundary point $z_0 = 0.5$ for model L. Formulas for γ and a_1 are given in (6.22) with (6.20), and (6.30) with (6.27) for models H and L, respectively.

7.1. **Model A: Cylindrical shells.** The midsurface parametrization, cf (5.1), is given by

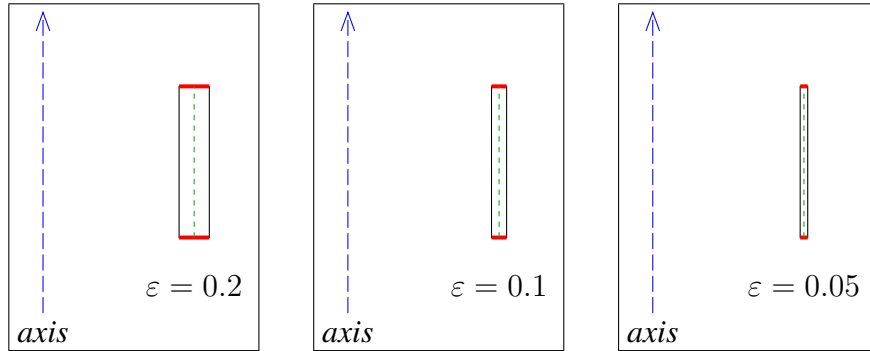$$f(z) = R, \quad z \in (-1, 1), \quad R = 2.$$

FIGURE 4. Model A: Meridian domains for several values of ε.
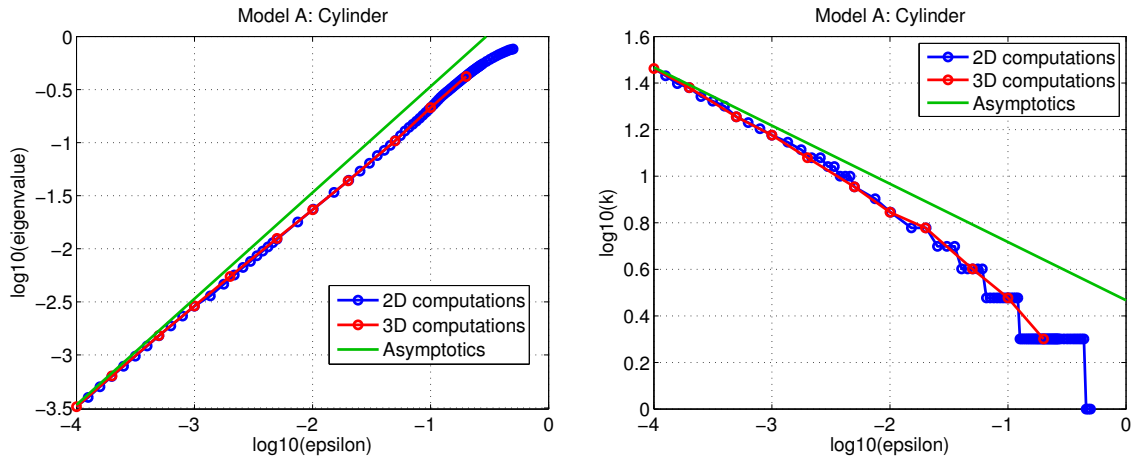

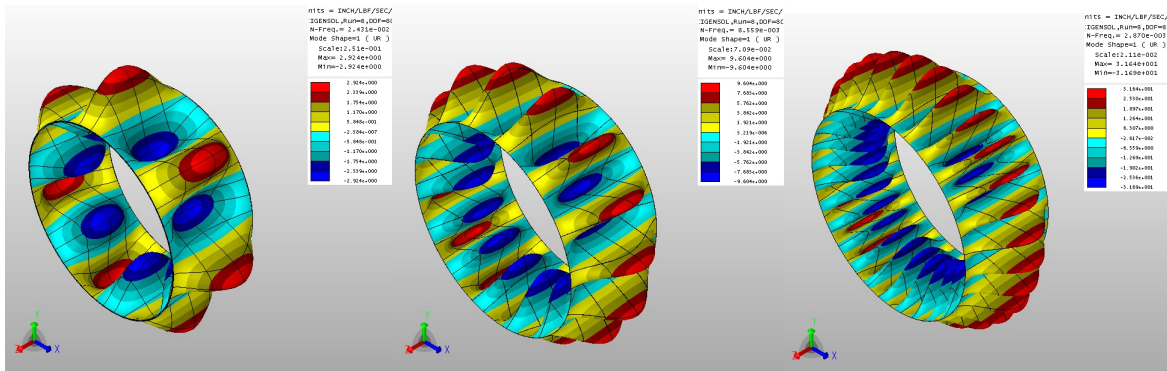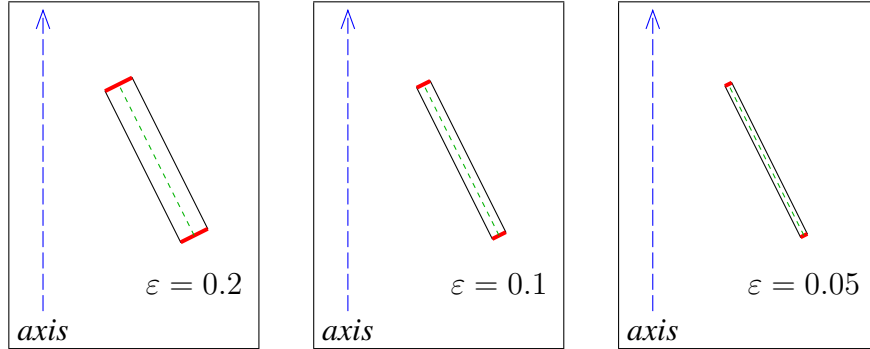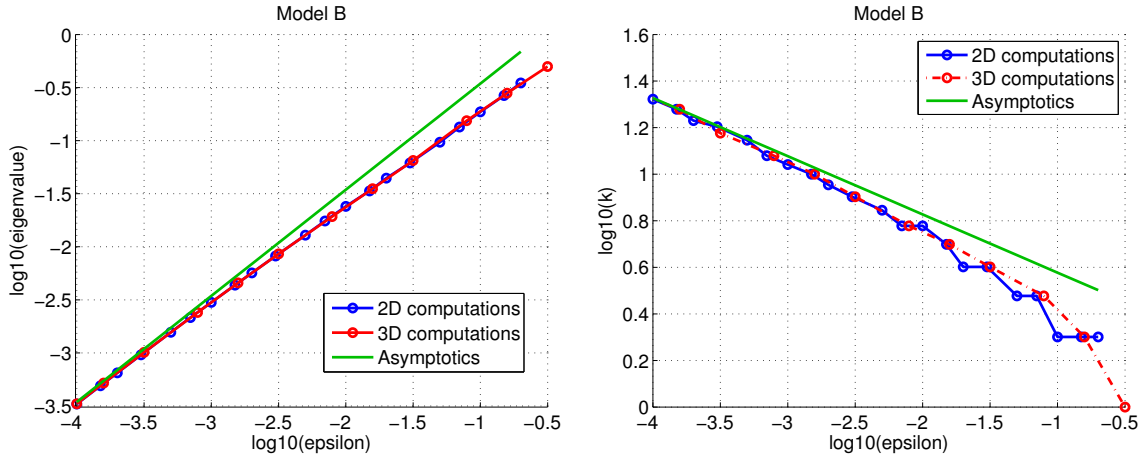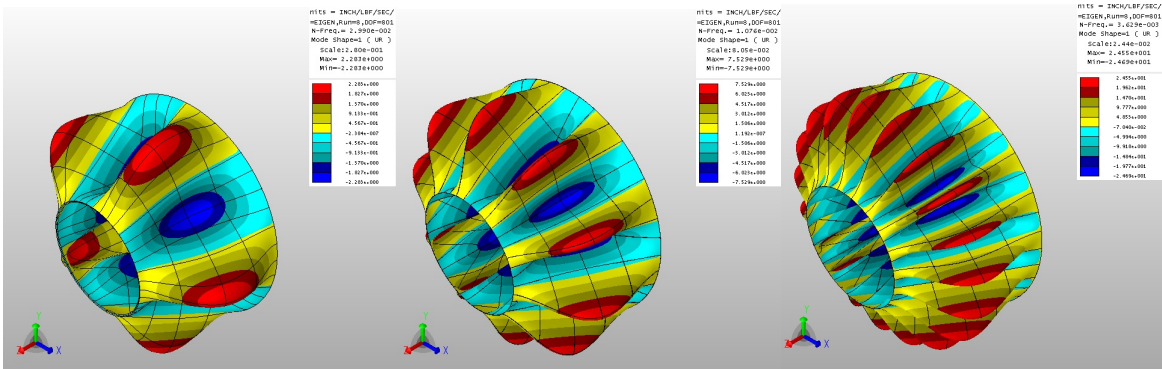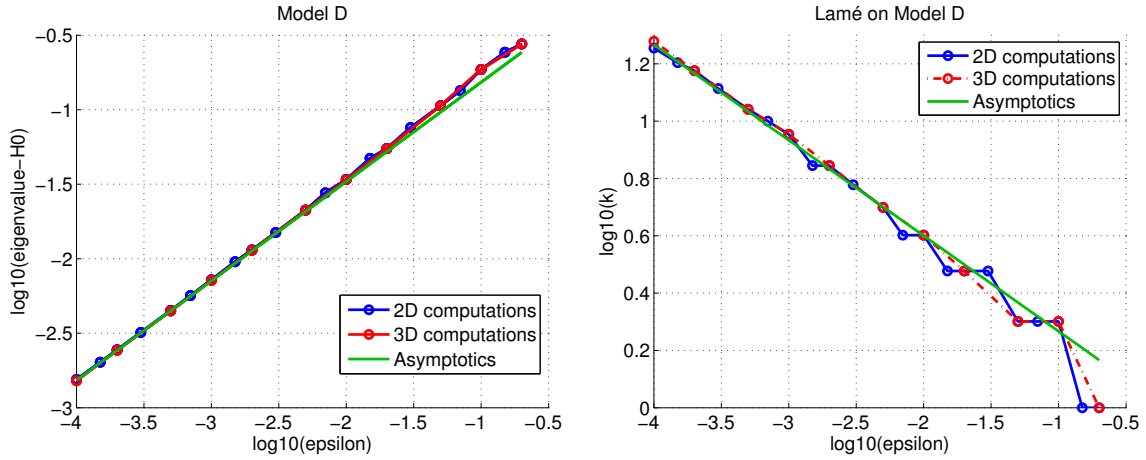FIGURE 5. Model A: First eigenvalue $\tilde{\lambda}_1^{\varepsilon}$ and associated azimuthal frequency $k(\varepsilon)$.
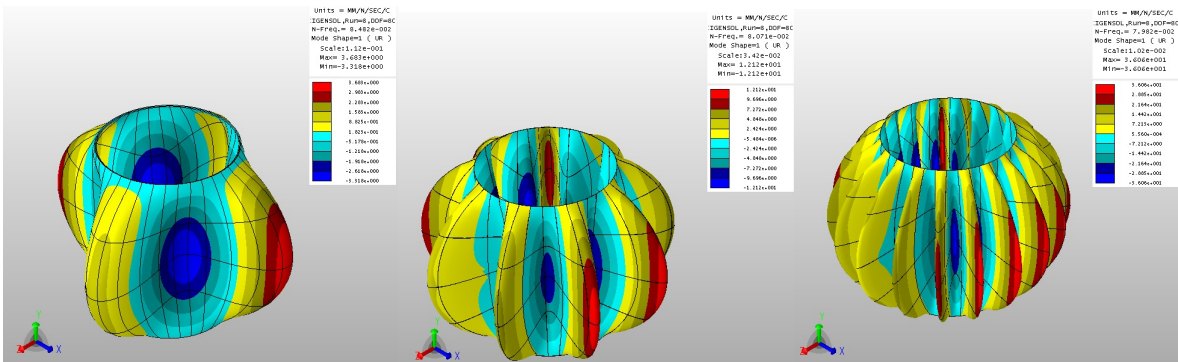

FIGURE 6. Model A: First eigenmode (radial component) for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$.

7.2. **Model B: Conical shells.** The midsurface parametrization, cf (5.1), is given by

$$f(z) = Tz + R, \quad z \in (-1, 1), \quad T = -0.5, \quad R = 1.5.$$

FIGURE 7. Model B: Meridian domains for several values of ε.

FIGURE 8. Model B: First eigenvalue $\tilde{\lambda}_1^\varepsilon$ and associated azimuthal frequency $k(\varepsilon)$.

FIGURE 9. Model B: First eigenmode (radial component) for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$.

7.3. **Model D: Toroidal barrels.** The midsurface parametrization is, cf (6.31)

$$f(z) = r_\circ + \sqrt{R^2 - z^2}, \quad z \in (-1, 1), \quad R = 2 \quad \text{and} \quad r_\circ = -1$$
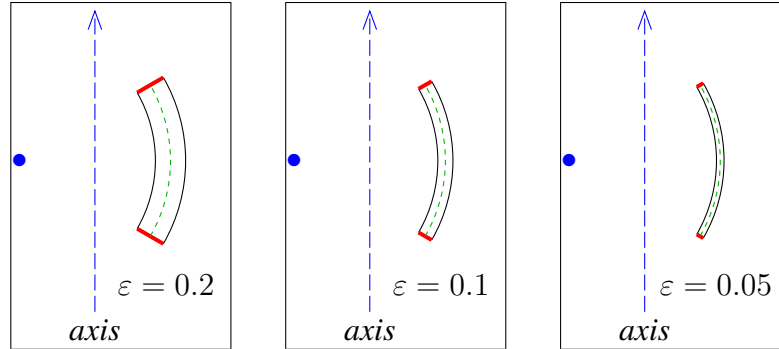
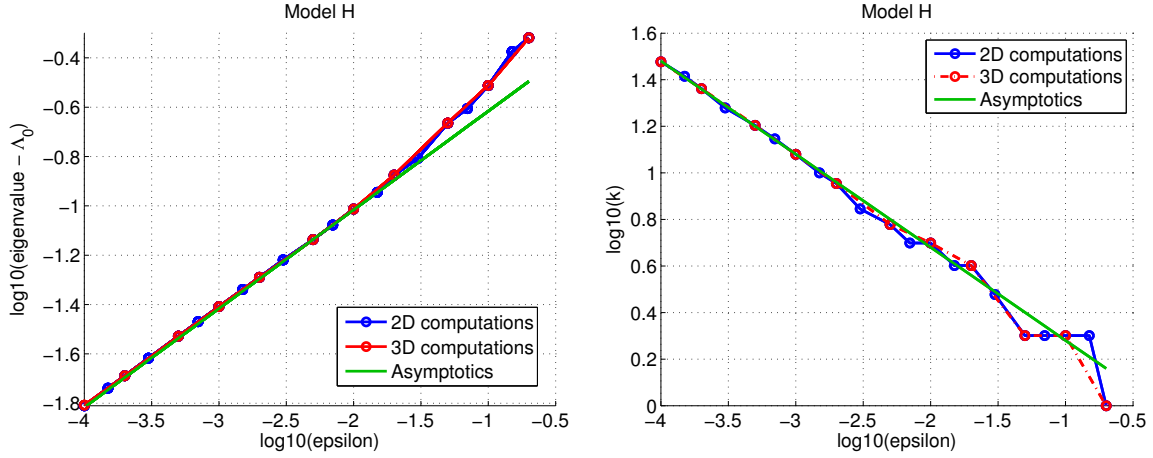FIGURE 10. Model D: Meridian domains for several values of ε.

FIGURE 11. Model D: Difference $\tilde{\lambda}_1^\varepsilon - \Lambda_0$ and associated azimuthal frequency $k(\varepsilon)$.
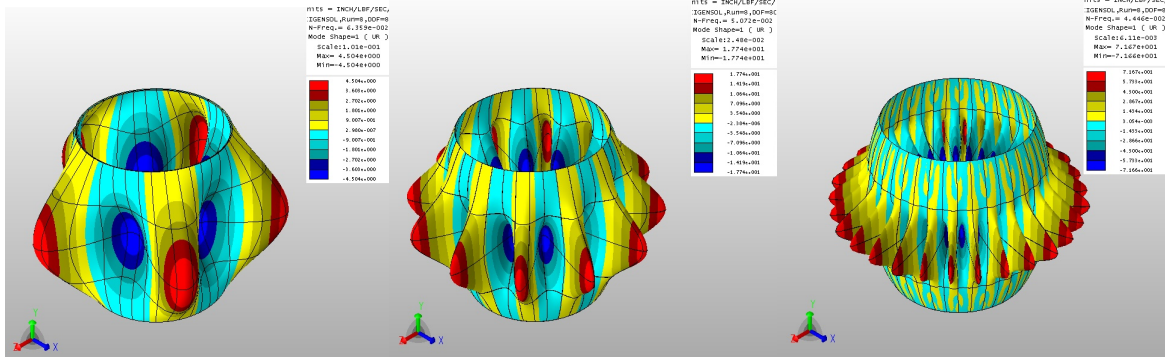

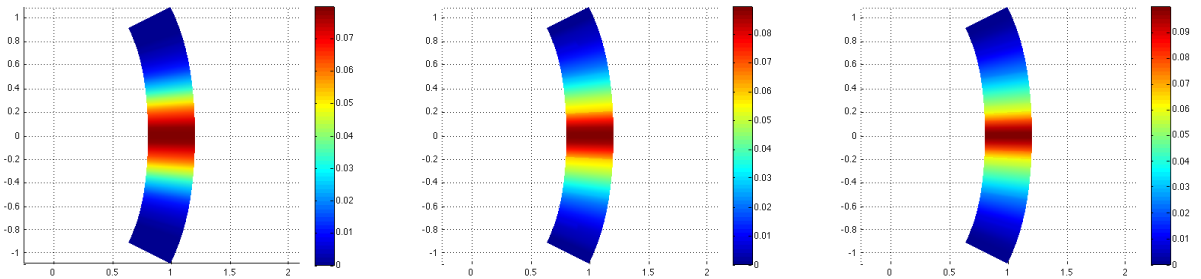FIGURE 12. Model D: First eigenmode (radial component) for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$.

7.4. Model H: Gaussian barrel. The midsurface parametrization is

$$f(z) = 1 - \frac{z^2}{8} - \frac{z^4}{16}, \quad z \in (-1, 1).$$

FIGURE 13. Model H: Difference $\tilde{\lambda}_1^\varepsilon - \Lambda_0$ and associated azimuthal frequency $k(\varepsilon)$.

FIGURE 14. Model H: First eigenmode (radial component) for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$.

FIGURE 15. Model H: 2D first eigenmode (radial component) for $\varepsilon = 10^{-3}$ and $k = 12$, $\varepsilon = 3 \cdot 10^{-4}$ and $k = 19$, $\varepsilon = 10^{-4}$ and $k = 30$. Represented on $\omega^{0.2}$.

7.5. **Model L: Airy barrel.** The midsurface parametrization is

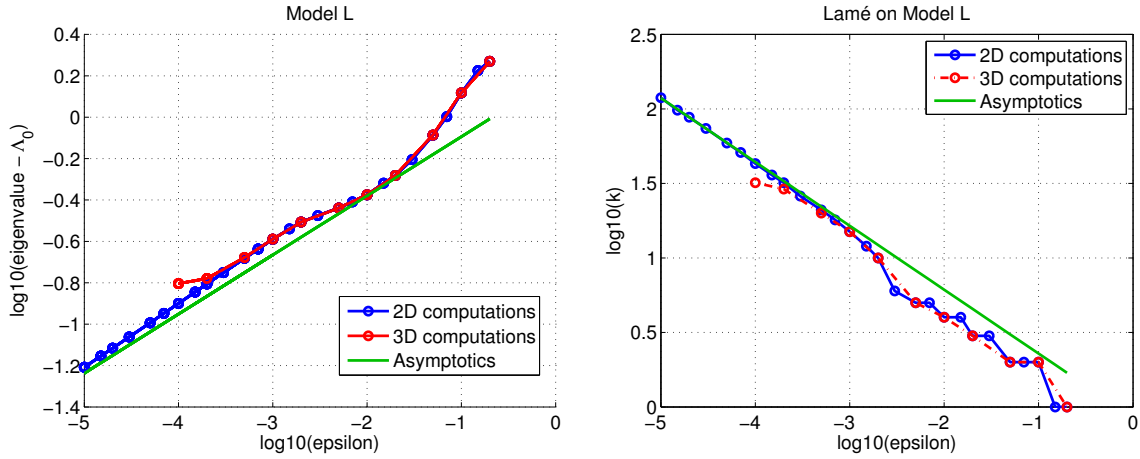$$f(z) = 1 - \frac{z^2}{8} - \frac{z^4}{16}, \quad z \in (0.5, 1.5).$$

FIGURE 16. Model L: Difference $\tilde{\lambda}_1^\varepsilon - \Lambda_0$ and associated azimuthal frequency $k(\varepsilon)$.
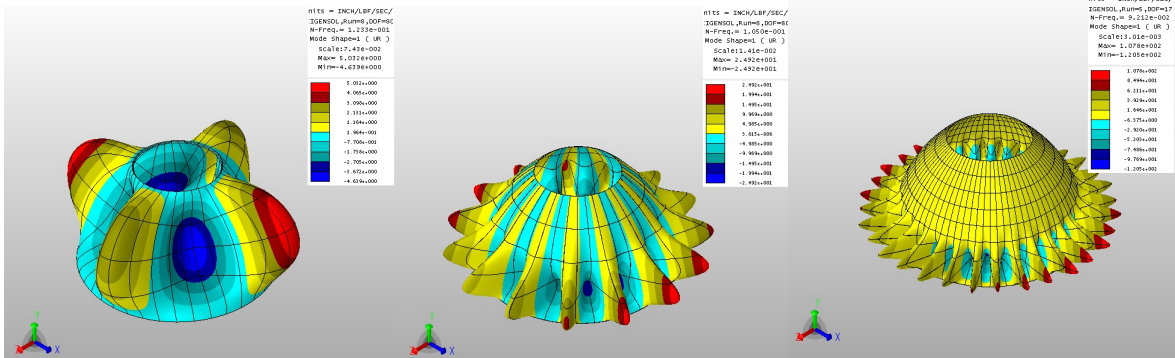

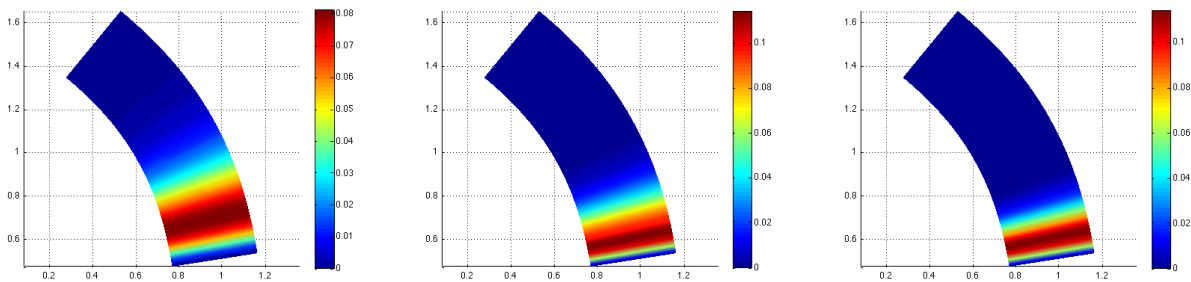FIGURE 17. Model L: 3D first eigenmode (radial component) for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$.

FIGURE 18. Model L: 2D first eigenmode (radial component) for $\varepsilon = 10^{-3}$ and $k = 15$, $\varepsilon = 3 \cdot 10^{-4}$ and $k = 26$, $\varepsilon = 10^{-4}$ and $k = 43$. Represented on $\omega^{0.2}$.

8. CONCLUSION

For five categories of clamped axisymmetric shells, we have exhibited a scalar 1D operator that determines the asymptotic expansion of the azimuthal frequency $k(\varepsilon)$ of the first vibration mode, and a two-term asymptotic expansion for $m_1(\varepsilon) = a_0 + a_1\varepsilon^{\alpha_1}$ for the first eigenvalue of the full 3D Lamé system in the shell. These five categories are the cylinders and the trimmed cones (parabolic shells), as well as what we denote toroidal, Gaussian and Airy barrels (elliptic shells). The most striking outcome of our analysis is the extremely good agreement of the three computation methods (3D, 2D and 1D) in all the five cases described above, strengthening the relevance of our constructions. The presented methods demonstrate that the smallest eigenpairs for the Lamé system can be estimated for specific shells by the reduced 1D model. Furthermore, the spreading or the concentration of the first eigenmode can be predicted accurately: The cases for which concentration occurs are the Gauss and Airy barrels and for those shells, the first eigenmode concentrates around a ring whose location $(f(z_0), z_0)$ is analytically known.

Another interesting observation is the comparison with the computations in [3]. The elliptic case that is considered there is $f(z) = 1 - \frac{1}{2}z^2$ on the interval $\mathcal{I} = (-a, a)$ with $a = 0.892668$. For this example we find

$$g(z) = \tfrac{3}{2}z^2(1 - \tfrac{1}{2}z^2) \quad \text{and} \quad H_0(z) = (1 + z^2)^{-3}.$$

So we see that we are in our admissible 'Airy' case, predicting a behavior in $\varepsilon^{-3/7}$ for the azimuthal frequency $k(\varepsilon)$ of the first eigenvector. Noting that $\frac{3}{7} \simeq 0.43$ and $\frac{2}{5} = 0.4$, we believe that this explains what have observed the authors [3, p.55]: *"We also notice that in the elliptic case, K_t [$k(\varepsilon)$] is probably growing slightly faster than exactly $t^{-2/5}$ [$\varepsilon^{-2/5}$]"*.

APPENDIX A. HIGH FREQUENCY REDUCTION OF THE MEMBRANE OPERATOR

Recall that the membrane operator is written as $\mathbf{M}[k] = k^2\mathbf{M}_0 + k\mathbf{M}_1 + \mathbf{M}_2$. Let us give elements of the proof of Theorem 4.2. We write the reduction formula (4.6) in the form

$$\mathbf{M}[k]\mathbf{V}[k] = \mathbf{V}_0 \circ (H[k] - \Lambda[k]) + \Lambda[k]\mathbf{A}\mathbf{V}[k].$$

This formula holds in the sense of the formal series algebra: This means that it is equivalent to the collection of equations: For all $n \geq 0$,

$$\mathbf{M}_0\mathbf{V}_n + \mathbf{M}_1\mathbf{V}_{n-1} + \mathbf{M}_2\mathbf{V}_{n-2} = \mathbf{V}_0 \circ (H_{n-2} - \Lambda_{n-2}) + \sum_{p+q=n-2} \Lambda_p\mathbf{A}\mathbf{V}_q. \qquad (A.1)$$

with the convention that $\mathbf{V}_n$, H_n, and Λ_n are 0 for $n < 0$, where $\mathbf{V}_n$ and H_n are the unknown coefficients of the formal series $\mathbf{V}[k] = \sum_{n\geq0} h^n\mathbf{V}_n$ and $H[k] = \sum_{n\geq0} h^nH_n$. Here the series $\Lambda[k] = \sum_{n\geq0} h^n\Lambda_n$ is given. The mass matrix $\mathbf{A}$ is given by (4.2), and the operators $\mathbf{M}_\ell$ by

$$\mathbf{M}_0 = \frac{E}{1 - \nu^2} \begin{pmatrix} \frac{1-\nu}{2f^2s^2} & 0 & 0 \\ 0 & \frac{1}{f^4} & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \begin{pmatrix} M_0^{zz} & 0 & 0 \\ 0 & M_0^{\varphi\varphi} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (A.2)$$

$$\mathbf{M}_1 = i\,\frac{E}{1-\nu^2}\begin{pmatrix} 0 & -\frac{1+\nu}{2f^2s^2}\partial_z + \frac{2f'}{f^3s^2} & 0 \\[2mm] -\frac{1+\nu}{2f^2s^2}\partial_z & & \\ +\left(\frac{(\nu-3)f'}{2f^3s^2} + \frac{(1+\nu)f'f''}{2f^2s^4}\right) & 0 & -\frac{1}{f^3s} + \frac{\nu f''}{f^2s^3} \\[2mm] 0 & \frac{1}{f^3s} - \frac{\nu f''}{f^2s^3} & 0 \end{pmatrix} \tag{A.3}$$

and

$$\mathbf{M}_2 = \frac{E}{1-\nu^2}\begin{pmatrix} -\frac{1}{s^4}\partial_z^2 + \left(\frac{3f'f''}{s^6} - \frac{f'}{fs^4}\right)\partial_z & & \left(\frac{f''}{s^5} - \frac{\nu}{fs^3}\right)\partial_z \\ +\frac{f''^2+f'f'''}{s^6} - \frac{4f'^2f''^2}{s^8} & 0 & +\frac{f'}{f^2s^3} + \frac{f'''}{s^5} \\ -\frac{\nu f''}{fs^4} + \frac{(1+\nu)f'^2f''}{fs^6} + \frac{f'^2}{f^2s^4} & & -\frac{3f'f''^2}{s^7} + \frac{f'f''}{fs^5} \\[4mm] & -\frac{1-\nu}{2f^2s^2}\partial_z^2 & \\ 0 & +\left(\frac{(1-\nu)f'f''}{2f^2s^4} + \frac{(1-\nu)f'}{2f^3s^2}\right)\partial_z & 0 \\ & +\frac{(1-\nu)f''}{f^3s^2} - \frac{(1-\nu)f'^2f''}{f^3s^4} & \\[4mm] \left(-\frac{f''}{s^5} + \frac{\nu}{fs^3}\right)\partial_z & & \\ +\frac{f'f''^2}{s^7} + \frac{f'}{f^2s^3} - \frac{2\nu f'f''}{fs^5} & 0 & \frac{f''^2}{s^6} + \frac{1}{f^2s^2} - \frac{2\nu f''}{fs^4} \end{pmatrix}. \tag{A.4}$$

Let us emphasize that the operators $\mathbf{M}_1$ and $\mathbf{M}_2$ have the following structure:

$$\mathbf{M}_1 = \begin{pmatrix} 0 & \mathsf{M}_1^{z\varphi} & 0 \\ \mathsf{M}_1^{\varphi z} & 0 & \mathsf{M}_1^{\varphi 3} \\ 0 & \mathsf{M}_1^{3\varphi} & 0 \end{pmatrix} \quad\text{and}\quad \mathbf{M}_2 = \begin{pmatrix} \mathsf{M}_2^{zz} & 0 & \mathsf{M}_2^{z3} \\ 0 & \mathsf{M}_2^{\varphi\varphi} & 0 \\ \mathsf{M}_2^{3z} & 0 & \mathsf{M}_2^{33} \end{pmatrix}. \tag{A.5}$$

Let us now examine the collection of equation (A.1). For $n = 0$, this equation reduces to

$$\mathbf{M}_0\mathbf{V}_0 = 0,$$

which is satisfied with the choice $\mathbf{V}_0 = (0, 0, \mathrm{Id})^\top$. For $n = 1$, using the structure (A.5) of the operator $\mathbf{M}_1$, the equation is $\mathbf{M}_0\mathbf{V}_1 = -\mathbf{M}_1\mathbf{V}_0$ that can be written as the system

$$\begin{pmatrix} \mathsf{M}_0^{zz}\mathsf{V}_{1,z} \\ \mathsf{M}_0^{\varphi\varphi}\mathsf{V}_{1,\varphi} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\mathsf{M}_1^{\varphi 3} \\ 0 \end{pmatrix}.$$

Hence, solving this equation we find $\mathsf{V}_{1,z} = 0$ and $\mathsf{V}_{1,\varphi} = -(\mathsf{M}_0^{\varphi\varphi})^{-1}\mathsf{M}_1^{\varphi 3}$, i.e.,

$$\mathsf{V}_{1,z} = 0 \quad \mathsf{V}_{1,\varphi} = \frac{if}{s} - \frac{i\nu f'' f^2}{s^3}. \tag{A.6}$$

The equation for $n = 2$ is written as

$$\mathbf{M}_0\mathbf{V}_2 = -\mathbf{M}_1\mathbf{V}_1 - \mathbf{M}_2\mathbf{V}_0 + \mathbf{V}_0 \circ (\mathsf{H}_0 - \Lambda_0) + \Lambda_0\mathbf{A}\mathbf{V}_0.$$

Using the structure (A.5) and the expressions of $\mathbf{V}_0$ and $\mathbf{V}_1$, it is equivalent to the system

$$\begin{cases} \mathsf{M}_0^{zz}\mathsf{V}_{2,z} &= -\mathsf{M}_1^{z\varphi}\mathsf{V}_{1,\varphi} - \mathsf{M}_2^{z3} \\ \mathsf{M}_0^{\varphi\varphi}\mathsf{V}_{2,\varphi} &= 0 \\ 0 &= -\mathsf{M}_1^{3\varphi}\mathsf{V}_{1,\varphi} - \mathsf{M}_2^{33} + \mathsf{H}_0. \end{cases}$$

The last equation of the previous system joint with (A.6) gives the expression of the operator H_0

$$H_0 = M_2^{33} - M_1^{3\varphi}(M_0^{\varphi\varphi})^{-1}M_1^{\varphi3},$$

and using the first two equations, we can solve for $\mathbf{V}_2$ by setting

$$V_{2,z} = (M_0^{zz})^{-1}(M_1^{z\varphi}(M_0^{\varphi\varphi})^{-1}M_1^{\varphi3} - M_2^{z3}) \quad \text{and} \quad V_{2,\varphi} = (M_0^{\varphi\varphi})^{-1}a^{\varphi\varphi}\Lambda_0.$$

Thus we find that $H_0 = E\frac{f''^2}{s^6}$ and that the components of $\mathbf{V}_2$ are

$$V_{2,\varphi} = 0,$$

$$V_{2,z} = \left(-\frac{f}{s} - \frac{(\nu+2)f^2f''}{s^3}\right)\partial_z$$
$$+ \frac{f'}{s} + \frac{3(\nu+2)f^2f'f''^2}{s^5} - \frac{(\nu+2)f^2f'''}{s^3} - \frac{(2\nu+1)ff'f''}{s^3}. \tag{A.7}$$

Now let us assume that the operators L_n and $\mathbf{V}_{n+1}$ are constructed for $n \geq 1$. Then writing the equation (A.1) for $n+2$, we obtain the relation (using the fact that $-\mathbf{V}_0\Lambda_n + \Lambda_n\mathbf{A}\mathbf{V}_0 = 0$)

$$\mathbf{M}_0\mathbf{V}_{n+2} - \mathbf{V}_0 \circ H_n = -\mathbf{M}_1\mathbf{V}_{n+1} - \mathbf{M}_2\mathbf{V}_n + \sum_{p=0}^{n-1}\Lambda_p\mathbf{A}\mathbf{V}_{n-p}.$$

This equation is equivalent to the system (using the fact that $V_{n,3} = 0$ for $n \geq 1$)

$$\begin{cases} M_0^{\rho\rho}V_{n+2,z} = -M_1^{z\varphi}V_{n+1,\varphi} - M_2^{zz}V_{n,z} + \sum_{p=0}^{n-1}\Lambda_p a^{zz}V_{n-p,z} \\ M_0^{\varphi\varphi}V_{n+2,\varphi} = -M_1^{\varphi z}V_{n+1,z} - M_2^{\varphi\varphi}V_{n,\varphi} + \sum_{p=0}^{n-1}\Lambda_p a^{\varphi\varphi}V_{n-p,\varphi} \\ H_n = M_1^{3\varphi}V_{n+1,\varphi} + M_2^{3z}V_{n,z}, \end{cases}$$

which gives the existence of the operators $V_{n+2,z}$, $V_{n+2,\varphi}$ and H_n. This shows the existence of the operators $\mathbf{V}_n = (V_{n,z}, V_{n,\varphi}, 0)^\top$. Moreover, we can check that $\mathbf{V}_n$ is an operator of order $n-1$ and is polynomial in Λ_j, for $j \leq n-3$. The scalar operators H_n are of order n, polynomial in Λ_j, for $j \leq n-2$.

Expressions of the operators H_n for $n = 0, \ldots, 4$ are given by the formulas (6.1)–(6.4) and the components of the operator $\mathbf{V}_3$ are given by

$$V_{3,z} = 0,$$

$$V_{3,\varphi} = i\left(-\frac{\nu f^3}{s^3} - \frac{(1+2\nu)f^4f''}{s^5}\right)\partial_z^2$$
$$+ i\left(-\frac{(4\nu+6)f^3f'f''}{s^5} - \frac{(4\nu+2)f^4f'''}{s^5} + \frac{7(2\nu+1)f^4f'f''^2}{s^7} - \frac{f^2f'}{s^3}\right)\partial_z$$
$$+ i\left(\frac{(\nu^2+19\nu+19)f^3f'^2f''^2}{s^7} - \frac{(6\nu+6)f^3f'f'''}{s^5} - \frac{(5\nu+3)f^2f'^2f''}{s^5}\right.$$
$$- \frac{(36\nu+18)f^4f'^2f''^3}{s^9} + \frac{(20\nu+10)f^4f'f''f'''}{s^7} + \frac{\nu f''f^2}{s^3} + \frac{f'^2f}{s^3}$$
$$\left. + \frac{(6\nu+3)f^4f''^3}{s^7} - \frac{(2\nu+1)f^4f^{(4)}}{s^5} - \frac{(\nu^2+\nu+1)f^3f''^2}{s^5}\right)$$
$$+ i\frac{1-\nu^2}{E}\Lambda_0\left(\frac{f^3}{s} - \frac{\nu f^4f''}{s^3}\right). \tag{A.8}$$

APPENDIX B. VARIATIONAL FORMULATIONS ON THE MERIDIAN DOMAIN

The 3D Lamé operator is independent of the azimuthal coordinate φ if expressed in cylindrical components of displacements. The contravariant cylindrical components used in this paper are

$$
\begin{cases}
\mathsf{u}^r = \mathsf{u}^{t_1}\cos\varphi + \mathsf{u}^{t_2}\sin\varphi, & \text{(radial)} \\
\mathsf{u}^\varphi = -\mathsf{u}^{t_1}\frac{1}{r}\sin\varphi + \mathsf{u}^{t_2}\frac{1}{r}\cos\varphi, & \text{(azimuthal)} \\
\mathsf{u}^\tau = \mathsf{u}^{t_3} & \text{(axial)}.
\end{cases}
$$

For a displacement $\mathbf{u}$ defined on Ω^ε, denote by $\widehat{\mathbf{u}}_k = (\mathsf{u}_k^r, \mathsf{u}_k^\varphi, \mathsf{u}_k^\tau)$ the Fourier coefficients of these components:

$$
\mathsf{u}_k^a(r,\tau) = \frac{1}{2\pi}\int_0^{2\pi} \mathsf{u}^a(r,\varphi,\tau)\,e^{-ik\varphi}\,\mathrm{d}\varphi, \quad a \in \{r,\varphi,\tau\}, \quad k \in \mathbb{Z}, \quad (r,\tau) \in \omega^\varepsilon.
$$

The energy bilinear form $a_{\mathrm{3D}}^\varepsilon$ is decomposed in Fourier coefficients as follows

$$
a_{\mathrm{3D}}^\varepsilon(\mathbf{u},\mathbf{v}) = \sum_{k\in\mathbb{Z}} a_k^\varepsilon(\widehat{\mathbf{u}}_k, \widehat{\mathbf{v}}_k).
$$

As soon as $k \neq 0$, the energy bilinear form a_k^ε at azimuthal frequency k as non real coefficients. Nevertheless, by a simple change of components, the coefficients are back to real:

$$
\begin{aligned}
a_k^\varepsilon\big((\mathsf{u}^r, i\mathsf{u}^\varphi, \mathsf{u}^\tau), (\mathsf{v}^r, &-i\mathsf{v}^\varphi, \mathsf{v}^\tau)\big) = \\
\frac{E}{1-\nu^2}\int_{\omega^\varepsilon}\Bigg\{ & \frac{(1-\nu)^2}{1-2\nu}\left(r\partial_r\mathsf{u}^r\partial_r\mathsf{v}^r + r\partial_\tau\mathsf{u}^\tau\partial_\tau\mathsf{v}^\tau + \frac{k^2}{r^3}\mathsf{u}^\varphi\mathsf{v}^\varphi + \frac{1}{r}\mathsf{u}^r\mathsf{v}^r\right) \\
& + \frac{\nu(1-\nu)}{1-2\nu}\,r\left(\partial_r\mathsf{u}^r\partial_\tau\mathsf{v}^\tau + \partial_\tau\mathsf{u}^\tau\partial_r\mathsf{v}^r\right) \\
& + \frac{\nu(1-\nu)}{1-2\nu}\left(\partial_r\mathsf{u}^r\mathsf{v}^r + \mathsf{u}^r\partial_r\mathsf{v}^r + \mathsf{u}^r\partial_\tau\mathsf{v}^\tau + \partial_\tau\mathsf{u}^\tau\mathsf{v}^r\right) \\
& + \frac{1-\nu}{2}\frac{k^2}{r}\left(\mathsf{u}^r\mathsf{v}^r + \mathsf{u}^\tau\mathsf{v}^\tau\right) \\
& + \frac{1-\nu}{2}\frac{1}{r}\left(\partial_r\mathsf{u}^\varphi\partial_r\mathsf{v}^\varphi - \frac{2}{r}\partial_r\mathsf{u}^\varphi\mathsf{v}^\varphi - \frac{2}{r}\mathsf{u}^\varphi\partial_r\mathsf{v}^\varphi + \frac{4}{r^2}\mathsf{u}^\varphi\mathsf{v}^\varphi + \partial_\tau\mathsf{u}^\varphi\partial_\tau\mathsf{v}^\varphi\right) \\
& + \frac{1-\nu}{2}\,r\left(\partial_r\mathsf{u}^\tau\partial_\tau\mathsf{v}^r + \partial_\tau\mathsf{u}^r\partial_r\mathsf{v}^\tau + \partial_r\mathsf{u}^\tau\partial_r\mathsf{v}^\tau + \partial_\tau\mathsf{u}^r\partial_\tau\mathsf{v}^r\right) \\
& + k\Bigg[\frac{(1-\nu)^2}{1-2\nu}\frac{1}{r^2}\left(\mathsf{u}^\varphi\mathsf{v}^r + \mathsf{u}^r\mathsf{v}^\varphi\right) + \frac{1-\nu}{r^2}\left(\mathsf{u}^\varphi\mathsf{v}^r + \mathsf{u}^r\mathsf{v}^\varphi\right) \\
& + \frac{\nu(1-\nu)}{1-2\nu}\frac{1}{r}\left(\mathsf{u}^\varphi\partial_r\mathsf{v}^r + \partial_r\mathsf{u}^r\mathsf{v}^\varphi + \mathsf{u}^\varphi\partial_\tau\mathsf{v}^\tau + \partial_\tau\mathsf{u}^\tau\mathsf{v}^\varphi\right) \\
& - \frac{1-\nu}{2}\frac{1}{r}\left(\mathsf{u}^\tau\partial_\tau\mathsf{v}^\varphi + \partial_\tau\mathsf{u}^\varphi\mathsf{v}^\tau + \mathsf{u}^r\partial_r\mathsf{v}^\varphi + \partial_r\mathsf{u}^\varphi\mathsf{v}^r\right)\Bigg]\Bigg\}\,drd\tau
\end{aligned}
$$

The associate eigenproblem is: Find λ and a nonzero $(\mathsf{u}^r, \mathsf{u}^\varphi, \mathsf{u}^\tau) \in V(\omega^\varepsilon)$ such that for all $(\mathsf{v}^r, \mathsf{v}^\varphi, \mathsf{v}^\tau) \in V(\omega^\varepsilon)$

$$
a_k^\varepsilon\big((\mathsf{u}^r, i\mathsf{u}^\varphi, \mathsf{u}^\tau), (\mathsf{v}^r, -i\mathsf{v}^\varphi, \mathsf{v}^\tau)\big) = \lambda\int_{\omega^\varepsilon}\left[\mathsf{u}^r\mathsf{v}^r + \frac{1}{r^2}\mathsf{u}^\varphi\mathsf{v}^\varphi + \mathsf{u}^\tau\mathsf{v}^\tau\right] rdrd\tau.
$$

Here the variational space $V(\omega^\varepsilon)$ corresponds to $V(\Omega^\varepsilon)$

$$V(\omega^\varepsilon) := \{\widehat{\mathbf{u}} = (\mathsf{u}^r, \mathsf{u}^\varphi, \mathsf{u}^\tau) \in H^1(\omega^\varepsilon)^3, \quad \widehat{\mathbf{u}} = 0 \quad \text{on} \quad \partial_0\omega^\varepsilon\}.$$

Note that the eigenvalues of a_k^ε are the same as those of a_{-k}^ε because of the identity

$$a_{-k}^\varepsilon\big((\mathsf{u}^r, i\mathsf{u}^\varphi, \mathsf{u}^\tau), (\mathsf{v}^r, -i\mathsf{v}^\varphi, \mathsf{v}^\tau)\big) = a_k^\varepsilon\big((\mathsf{u}^r, -i\mathsf{u}^\varphi, \mathsf{u}^\tau), (\mathsf{v}^r, i\mathsf{v}^\varphi, \mathsf{v}^\tau)\big).$$

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MC, MD & EF: IRMAR, (CNRS, INRIA) UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE

ZY: BEN-GURION UNIVERSITY, DEPT. OF MECHANICAL ENGINEERING, POBOX 653, BEER-SHEVA 84105, ISRAEL

E-mail address: monique.dauge@univ-rennes1.fr

URL: http://perso.univ-rennes1.fr/monique.dauge/

E-mail address: Erwan.Faou@inria.fr

URL: http://www.irisa.fr/ipso/perso/faou/

E-mail address: zohary@bgu.ac.il

URL: http://www.bgu.ac.il/˜zohary/