LANDAU DAMPING IN SOBOLEV SPACES FOR THE VLASOV-HMF MODEL

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Abstract. We consider the Vlasov-HMF (Hamiltonian Mean-Field) model. We consider solutions starting in a small Sobolev neighborhood of a spatially homogeneous state satisfying a linearized stability criterion (Penrose criterion). We prove that these solutions exhibit a scattering behavior to a modified state, which implies a nonlinear Landau damping effect with polynomial rate of damping.

1. Introduction

In this paper we consider the Vlasov-HMF model. This model has received much interest in the physics literature for many reasons: It is a simple ideal toy model that keeps several features of the long range interactions, it is a simplification of physical systems like charged or gravitational sheet models and it is rather easy to make numerical simulations on it. We refer for example to [1], [14], [2], [7], [8] for more details.

We shall study the long time behavior of solutions to this model for initial data that are small perturbations in a weighted Sobolev space to a spatially homogeneous stationary state satisfying a Penrose type stability condition. We shall prove that the solution scatters when times goes to infinity towards a modified state close to the initial data in a Sobolev space of lower order. This result implies a nonlinear Landau damping effect with polynomial rate for the solution which converges weakly towards a modified spatially homogeneous state. In the case of analytic or Gevrey regularity, this result has been shown to hold for a large class of Vlasov equations that contains the Vlasov-Poisson system by Mouhot and Villani [13] (see also the recent simplified proof [5]). Some earlier partial results were obtained in [6], [9]. The related problem of the stability of the Couette flow in the two-dimensional Euler equation has been also studied recently [4]. The question left open in these papers is the possibility of nonlinear Landau damping for Sobolev perturbations. In this case, one cannot hope for an exponential damping, but we can wonder if it could occur by allowing polynomial rates. For the Vlasov-Poisson system, it was proven in [11] that this is false with rough Sobolev regularity due to the presence of arbitrarily close travelling BGK states. Nevertheless, this obstruction disappears for sufficiently high Sobolev regularity as also proven in [11]. These arguments can also be extended to a large class of Vlasov equations and in particular the HMF model. Note that the regularity of the interaction kernel does not play an essential part in the argument of [13] (though the decay in Fourier space provided by the Coulomb interaction seems critical), it is more the nonlinear "plasma-echo" effect which is crucial to handle. Besides the physical interest of the HMF model, it is thus mathematically interesting to study the possibility of nonlinear Landau-damping in Sobolev spaces for this simple model.

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1.1. **The Vlasov-HMF model.** The Vlasov-HMF model reads

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) = \partial_x\Big(\int_{\mathbb{R}} P(x-y)f(t,y,u)\mathrm{d}u\mathrm{d}y\Big)\partial_v f(t,x,v), \tag{1.1}$$

where $(x,v) \in \mathbb{T}\times\mathbb{R}$ and the kernel $P(x)$ is given by $P(x) = \cos(x)$. Note that the main difference with the Vlasov-Poisson equation is the regularity of the kernel: in this latter case, $P(x) = \sum_{k\geq 0} k^{-2}\cos(kx)$ is the kernel associated with the inverse of the Laplace operator. The HMF model is thus the simplest nonlinear model with the structure (1.1). We consider initial data under the form $f_0(x,v) = \eta(v) + \varepsilon r_0(x,v)$ where ε is a small parameter and r_0 is of size one (in a suitable functional space). This means that we study small perturbations of a stationary solution $\eta(v)$. We shall thus write the solution at time t under the form

$$f(t,x,v) = \eta(v) + \varepsilon r(t,x,v).$$

We are interested in the study of the behavior of f when time goes to infinity. To filter the effect of the free transport, it is convenient to introduce (as in [13], [5]) the unknown $g(t,x,v) = r(t,x+tv,v)$ that is solution of the equation

$$\partial_t g = \{\phi(t,g),\eta\} + \varepsilon\{\phi(t,g),g\}. \tag{1.2}$$

where

$$\phi(t,g) = \int_{\mathbb{R}} (\cos(x-y+t(v-u)))g(t,y,u)\mathrm{d}u\mathrm{d}y \tag{1.3}$$

and $\{f,g\} = \partial_x f\partial_v g - \partial_v f\partial_x g$ is the usual microcanonical Poisson bracket. We shall usually write $\phi(t)$ when the dependence in g is clear.

We shall work in the following weighted Sobolev spaces, for $m_0 > 1/2$ be given, we set

$$\|f\|^2_{\mathcal{H}^n} = \sum_{|p|+|q|\leqslant n} \int_{\mathbb{T}\times\mathbb{R}} (1+|v|^2)^{m_0}|\partial_x^p\partial_v^q f|^2\mathrm{d}x\mathrm{d}v, \tag{1.4}$$

and we shall denote by $\mathcal{H}^n$ the corresponding function space. Note that compared to the usual Sobolev space H^n, there is also a fixed weight $(1+|v|^2)^{m_0}$ in physical space. The interest of the weight is that functions in $\mathcal{H}^0$ are in L^1 and thus it allows to get a pointwise control in Fourier (see Lemma 2.1 below). We do not include the dependence in m_0 in the notation since m_0 will be fixed. We shall denote by $\hat{\cdot}$ or $\mathcal{F}$ the Fourier transform on $\mathbb{T}\times\mathbb{R}$ given by

$$\hat{f}_k(\xi) = \frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}} f(x,v)e^{-ikx-i\xi v}\mathrm{d}x\mathrm{d}v.$$

Note that due to the regularity of the interaction kernel and the conservation of the L^p norms, it is very easy to prove the global well-posedness of the Vlasov-HMF model in $\mathcal{H}^s$ for every $s \geq 0$. Nevertheless, in order to study the asymptotic behaviour of g, the regularity of the kernel is not of obvious help. Indeed, when performing energy estimate on (1.2), it costs one positive power of t each time one puts a v derivative on the kernel.

1.2. **The Penrose criterion.** We shall need a stability property of the reference state η in order to control the linear part of the Vlasov equation (1.2). Let us denote by η, the spatially homogeneous stationary state and let us define the function

$$K(n,t) = -np_n\, nt\,\hat{\eta}_0(nt)\mathbb{1}_{t\geq 0}, \quad t\in\mathbb{R}, \quad n\in\mathbb{Z},$$

where $(p_k)_{k\in\mathbb{Z}}$ are the Fourier coefficients of the kernel $P(x)$. We shall denote by $\hat{K}(n,\tau)$ the Fourier transform of $K(n,\cdot)$. We shall assume that η satisfies the following condition

$$(1.5)\qquad \textbf{(H)}\quad (1+v^2)\eta(v)\in\mathcal{H}^5 \quad\text{and}\quad \exists\,\kappa>0,\quad \inf_{\tau\in\mathbb{R}}|1-\hat{K}(n,\tau)|\geq\kappa,\quad n=\pm1.$$

Note that here, the assumption is particularly simple due to the fact that for our kernel, there are only two non-zero Fourier modes. This assumption is very similar to the one used in [13], [5] and can be related to the standard statement of the Penrose criterion. In particular it is verified for the states $\eta(v)=\rho(|v|)$ with ρ non-increasing which are also known to be Lyapounov stable for the nonlinear equation (see [12]).

1.3. **Main result.** In the evolution of the solution $g(t,x,v)$ of (1.2), an important role is played by the quantity

$$(1.6)\qquad \zeta_k(t)=\hat{g}_k(t,kt),\quad k\in\{\pm1\},$$

such that

$$\phi(t,g)=\frac{1}{2}\sum_{k\in\{\pm1\}}e^{ikx}e^{iktv}\zeta_k(t).$$

Note that for $k\neq0$, $\zeta_k(t)$ is the Fourier coefficient in x of the density $\rho(t,x)=\int_{\mathbb{R}}f(t,x,v)\,dv$. This quantity also plays a key part in the analysis of [13], [5]. Note that here we need only to control two Fourier modes due to our simple interaction kernel.

Let us define for every $s\geq4$ and $T\geq0$ the weighted norm

$$Q_{T,s}(g)=\sup_{t\in[0,T]}\frac{\|g(t)\|_{\mathcal{H}^s}}{\langle t\rangle^3}+\sup_{t\in[0,T]}\sup_{k\in\{\pm1\}}\langle t\rangle^{s-1}|\zeta_k(t)|+\sup_{t\in[0,T]}\|g(t)\|_{\mathcal{H}^{s-4}}.$$

Our main result is:

Theorem 1.1. *Let us fix $s\geq7$ and $R_0>0$ such that $Q_{0,s}(g)\leq R_0$ and assume that $\eta\in\mathcal{H}^{s+4}$ satisfies the assumption* (**H**). *Then there exists $R>0$ and $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0]$ and for every $T\geq0$, we have the estimate*

$$Q_{T,s}(g)\leq R.$$

As a consequence, we obtain the following scattering result:

Corollary 1.2. *Under the assumption of Theorem 1.1, there exists a constant C and $g^\infty(x,v)\in\mathcal{H}^{s-4}$ such that for all $r\leq s-4$ and $r\geq1$,*

$$(1.7)\qquad \forall\,t\geq0,\quad \|g(t,x,v)-g^\infty(x,v)\|_{\mathcal{H}^r}\leq\frac{C}{\langle t\rangle^{s-r-3}}.$$

The consequence of such results is the following nonlinear Landau damping effect: as $g(t,x,v)$ is bounded in $\mathcal{H}^{s-4}$, the solution $f(t,x,v)=\eta(v)+\varepsilon r(t,x,v)=\eta(v)+\varepsilon g(t,x-tv,v)$ satisfies

$$\forall\,n\in\mathbb{Z}^*,\quad \forall\xi\in\mathbb{R},\quad \forall\,\alpha+\beta=s-4,\quad |\hat{f}_n(t,\xi)|=\varepsilon|\hat{g}_n(t,\xi+nt)|\leq\frac{C\varepsilon}{\langle\xi+nt\rangle^\alpha\langle n\rangle^\beta}.$$

The last estimate being a consequence of the elementary embedding Lemma 2.1. This yields that for every $n\neq0$, $\hat{f}_n(t,\xi)$ tends to zero with a polynomial rate.

Moreover, by setting

$$\eta^\infty(v) := \eta(v) + \frac{\varepsilon}{2\pi}\int_{\mathbb{T}} g^\infty(x,v)\mathrm{d}x,$$

we have by the previous corollary (and again Lemma 2.1) that for $r \le s-4$,

$$\forall\, \xi \in \mathbb{R}, \quad |\hat{f}_0(\xi) - \hat{\eta}_0^\infty(\xi)| \le \frac{C}{\langle \xi\rangle^r \langle t\rangle^{s-r-3}}$$

In other words, $f(t,x,v)$ converges weakly towards $\eta^\infty(v)$.

The remaining of the paper is devoted to the proof of Theorem 1.1. We shall obtain Corollary 1.2 in section (4) as an easy consequence. As pointed out in [13] the control of the "plasma echoes" that can be seen as kind of resonances is crucial to prove nonlinear Landau damping. These resonances occur when $nt = k\sigma$ in the last integral term of (2.6). The main structural property of the Vlasov-HMF model that makes possible the following short proof of nonlinear Landau damping in Sobolev spaces is that the resonances are easy to analyze, the only possibility is when $n = k = \pm 1$ and $t = \pm\sigma$. Moreover, the structure of the nonlinearity then allows to control it without loss of decay. Making an analogy with dispersive equations (see [10] for example), the nonlinearity of the Vlasov-HMF model could be thought as a nonlinearity with null structure. Our approach also allows to handle the case of a kernel with a finite number of modes which allows more resonances, we briefly sketch the modification in section 5. Nevertheless, it is still unclear if this can be done for the Vlasov-Poisson equation.

2. A priori estimates

In this section, we shall study a priori estimates for the solution of (1.2). Let us fix $s \ge 7$ and introduce the weighted norms:

$$N_{T,s}(g) = \sup_{t\in[0,T]} \frac{\|g(t)\|_{\mathcal{H}^s}}{\langle t\rangle^3}, \quad M_{T,\gamma}(\zeta) = \sup_{t\in[0,T]}\sup_{k\in\{\pm1\}} \langle t\rangle^\gamma |\zeta_k(t)| \tag{2.1}$$

so that

$$Q_{T,s}(g) = N_{T,s}(g) + M_{T,s-1}(\zeta) + \sup_{[0,T]} \|g(t)\|_{\mathcal{H}^{s-4}}. \tag{2.2}$$

Let us take $R_0 > 0$ such that $Q_{0,s}(g) \le R_0$. Our aim is to prove that when ε is sufficiently small we can choose R so that we have

$$Q_{T,s}(g) \le R$$

for every $T \ge 0$.

In the following a priori estimates, C stands for a number which may change from line to line and which is independent of R_0, R, ε and T.

We shall make constant use of the following elementary lemma.

Lemma 2.1. *For every α, β, $n \in \mathbb{N}$ with $\alpha+\beta = n$ we have the following inequality:*

$$\forall\, k \in \mathbb{Z}, \quad \forall\, \xi \in \mathbb{R}, \quad |\hat{f}_k(\xi)| \leqslant 2^{n/2} C(m_0) \langle k\rangle^{-\alpha} \langle \xi\rangle^{-\beta} \|f\|_{\mathcal{H}^n}, \tag{2.3}$$

where $C(m_0)$ depends only on m_0 and where $\langle x\rangle = (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}$.

Proof. We have by using the Cauchy-Schwarz inequality that

$$\begin{aligned} \left|k^\alpha \xi^\beta \hat{f}_k(\xi)\right| &= \frac{1}{2\pi}\left|\int_{\mathbb{T}\times\mathbb{R}} \partial_x^\alpha \partial_v^\beta f(x,v) e^{-ikx} e^{-iv\xi} \mathrm{d}x\mathrm{d}v\right| \\ &\leqslant C\|f\|_{\mathcal{H}^n}\Big(\int_{\mathbb{R}} (1+|v|^2)^{-m_0}\mathrm{d}v\Big)^{1/2}. \end{aligned}$$

The previous inequality with $\alpha=\beta=0$ yields the result when $k=0$ or $|\xi|\leqslant 1$ and we conclude by using $\langle x\rangle \leqslant 2^{\alpha/2}|x|^2$ for $|x|>1$. ☐

2.1. **Estimate of $M_{T,s-1}(\zeta)$.** Towards the proof of Theorem 1.1, we shall first estimate $\zeta_k(t)$, $k=\pm 1$.

Proposition 2.2. *Assuming that $\eta\in\mathcal{H}^{s+2}$ verifies the assumption* **(H)***, then there exists $C>0$ such that for every $T>0$, every solution of (1.2) such that $Q_{T,s}(g)\le R$ enjoys the estimate*

$$M_{T,s-1}(\zeta)\le C\big(R_0+\varepsilon R^2\big). \tag{2.4}$$

Proof. We first note that in Fourier space, the equation (1.2) can be written after integration in time,

$$\begin{aligned} \hat{g}_n(t,\xi) = \hat{g}_n(0,\xi) + \int_0^t p_n\zeta_n(\sigma)\hat{\eta}_0(\xi-n\sigma)(n^2\sigma-n\xi)\mathrm{d}\sigma \\ + \varepsilon\sum_{k\in\mathbb{Z}} p_k\int_0^t \zeta_k(\sigma)\hat{g}_{n-k}(\sigma,\xi-k\sigma)(nk\sigma-k\xi)\mathrm{d}\sigma, \end{aligned} \tag{2.5}$$

for all $(n,\xi)\in\mathbb{Z}\times\mathbb{R}$, with $p_k=\frac{1}{2}$ for $k\in\{\pm 1\}$ and $p_k=0$ for $k\neq\pm 1$, and where the $\zeta_k(t)$ are defined by (1.6). Setting $\xi=nt$ in (2.5), the equation satisfied by $(\zeta_n(t))_{n=\pm1}$ can be written under the almost closed form

$$\begin{aligned} \zeta_n(t) = \hat{g}_n(0,nt) - \int_0^t p_n\zeta_n(\sigma)\eta_0(n(t-\sigma))n^2(t-\sigma)\mathrm{d}\sigma \\ - \varepsilon\sum_{k\in\{\pm1\}} p_k\int_0^t \zeta_k(\sigma)\hat{g}_{n-k}(\sigma,nt-k\sigma)kn(t-\sigma)\mathrm{d}\sigma. \end{aligned} \tag{2.6}$$

To study the equation (2.6), we shall first consider the corresponding linear equation, that is to say that we shall first see

$$F_n(t) := \hat{g}_n(0,nt) - \varepsilon\sum_{k\in\{\pm1\}} p_k\int_0^t \zeta_k(\sigma)\hat{g}_{n-k}(\sigma,nt-k\sigma)kn(t-\sigma)\mathrm{d}\sigma \tag{2.7}$$

as a given source term and we shall study the linear integral equation

$$\zeta_n(t) = \int_0^t K(n,t-\sigma)\zeta_n(\sigma)\,\mathrm{d}\sigma + F_n(t) \quad n=\pm1 \tag{2.8}$$

where the kernel $K(n,t)$ has been introduced in section 1.2.

For this linear equation, we have the estimate:

Lemma 2.3. *Let $\gamma \geq 0$, and assume that $\eta \in \mathcal{H}^{\gamma+3}$ satisfies* (**H**). *Then, there exists $C > 0$ such for every $T \geq 0$, we have*

$$M_{T,\gamma}(\zeta) \leq C M_{T,\gamma}(F).$$

Let us postpone the proof of the Lemma and finish the proof of Proposition 2.2.

From the previous Lemma and (2.3), we first get that

$$M_{T,s-1}(\zeta) \leq C\big(\|g(0)\|_s + \varepsilon M_{T,s-1}(F^1) + \varepsilon M_{T,s-1}(F^2)\big) \tag{2.9}$$

with where F^1 corresponds to the term with $k = -n$ in (2.7) and F^2 corresponds to the term with $k = n$, hence

$$F^1_n(t) = -n^2 p_{-n} \int_0^t \zeta_{-n}(\sigma)\hat{g}_{2n}(\sigma, n(t+\sigma))(t-\sigma)\,\mathrm{d}\sigma, \quad n = \pm 1,$$

$$F^2_n(t) = n^2 p_n \int_0^t \zeta_n(\sigma)\hat{g}_0(\sigma, n(t-\sigma))(t-\sigma)\,\mathrm{d}\sigma, \quad n = \pm 1.$$

Let us estimate F^1_n, by using again (2.3) and the definition (2.1) of $N_{\sigma,s}$, we get that

$$|F^1_n(t)| \leq C \int_0^t \frac{(t-\sigma)\langle\sigma\rangle^3 M_{\sigma,s-1}(\zeta) N_{\sigma,s}(g)}{\langle\sigma\rangle^{s-1}\langle t+\sigma\rangle^s}\,\mathrm{d}\sigma \leq C\frac{R^2}{\langle t\rangle^{s-1}} \int_0^{+\infty} \frac{1}{\langle\sigma\rangle^{s-4}}\,\mathrm{d}\sigma \leq C\frac{R^2}{\langle t\rangle^{s-1}}$$

provided $s \geq 6$. This yields that for all $T \geq 0$

$$M_{T,s-1}(F^1) \leq CR^2.$$

To estimate F^2_n, we split the integral into two parts: we write

$$F^2_n(t) = I^1_n(t) + I^2_n(t)$$

with

$$I^1_n(t) = n^2 p_n \int_0^{\frac{t}{2}} \zeta_n(\sigma)\hat{g}_0(\sigma, n(t-\sigma))(t-\sigma)\,\mathrm{d}\sigma, \quad n = \pm 1,$$

$$I^2_n(t) = n^2 p_n \int_{\frac{t}{2}}^t \zeta_n(\sigma)\hat{g}_0(\sigma, n(t-\sigma))(t-\sigma)\,\mathrm{d}\sigma, \quad n = \pm 1.$$

For I^1_n, we proceed as previously,

$$|I^1_n(t)| \leq CR^2 \int_0^{\frac{t}{2}} \frac{\langle\sigma\rangle^3 (t-\sigma)}{\langle\sigma\rangle^{s-1}\langle t-\sigma\rangle^s}\,\mathrm{d}\sigma \leq \frac{CR^2}{\langle t\rangle^{s-1}} \int_0^{+\infty} \frac{1}{\langle\sigma\rangle^{s-4}}\,\mathrm{d}\sigma$$

and hence since $s \geq 6$, we have

$$M_{T,s-1}(I^1) \leq CR^2.$$

To estimate I^2_n, we shall rather use the last factor in the definition of $Q_{s,T}$ in (2.2). By using again (2.3), we write

$$|I^2_n(t)| \leq \int_{\frac{t}{2}}^t \frac{M_{\sigma,s-1}(\zeta)}{\langle\sigma\rangle^{s-1}} \frac{\|g(\sigma)\|_{\mathcal{H}^{s-4}}}{\langle t-\sigma\rangle^{s-5}}\,\mathrm{d}\sigma \leq \frac{CR^2}{\langle t\rangle^{s-1}} \int_0^{+\infty} \frac{1}{\langle\tau\rangle^{s-5}}\,\mathrm{d}\sigma \leq \frac{CR^2}{\langle t\rangle^{s-1}}$$

and hence since $s \geq 7$, we find again

$$M_{T,s-1}(I^2) \leq CR^2.$$

By combining the last estimates and (2.9), we thus obtain (2.4). This ends the proof of Proposition 2.2. □

It remains to prove Lemma 2.3.

Proof of Lemma 2.3. Let us take $T > 0$, and let us set for the purpose of the proof $K(t) = K(n,t)$, $F(t) = F_n(t)\mathbb{1}_{0\le t\le T}$. Since we only consider the cases $n = \pm 1$, we do not write down anymore explicitly the dependence in n. We consider the equation

$$y(t) = K * y(t) + F(t), \quad t \in \mathbb{R} \tag{2.10}$$

setting $y(t) = 0$ for $t \le 0$. Note that the solution of this equation coincides with $\zeta_n(t)$ on $[0,T]$ since the modification of the source term for $t \ge T$ does not affect the past. By taking the Fourier transform in t (that we still denote by $\hat{\ }$), we obtain

$$\hat{y}(\tau) = \hat{K}(\tau)\hat{y}(\tau) + \hat{F}(\tau), \quad \tau \in \mathbb{R}, \tag{2.11}$$

with $\hat{K}(\tau) = \hat{K}(n,\tau)$. Under the assumption (**H**), the solution of (2.11) is given explicitely by the formula

$$\hat{y}(\tau) = \frac{\hat{F}(\tau)}{1 - \hat{K}(\tau)}. \tag{2.12}$$

Let us observe that since $(1 + v^2)\eta_0 \in \mathcal{H}^5$, we have by (2.3) that for $\alpha \le 2$ and for $t > 0$

$$|\partial_t^\alpha K(t)| \le \frac{C}{\langle t\rangle^4} \in L^1(\mathbb{R}_+). \tag{2.13}$$

Note that by definition of $K(t)$, the function $K(t)$ is continuous in $t = 0$, but not C^1. Using an integration by parts on the definition of the Fourier transform, we then get that

$$|\partial_\tau^\alpha \hat{K}(\tau)| \le \frac{C}{\langle \tau\rangle^2}, \quad \alpha \le 2. \tag{2.14}$$

To get this, we have used that the function $t\,\hat{\eta}_0(t)$ vanishes at zero.

By using this, estimate on $\hat{K}$, **(H)** and that $\hat{F}(\tau) \in H^1_\tau$ (the Sobolev space in τ) since F is compactly supported in time, we easily get that y defined via its Fourier transform by (2.12) belongs to H^1_τ. This implies that $\langle t\rangle y \in L^2$ and thus that $y \in L^1_t$. These remarks, together with the uniqueness of the solution of (2.10) which is a consequence of the Gronwall Lemma, justifies the use of the Fourier transform and the equivalence between equation (2.11) and equation (2.10).

Note that a L^2-based version of Lemma 2.3 would be very easily obtained. The difficulty here is to get the uniform L^∞ in time estimate we want to prove.

We shall first prove the estimate for $\gamma = 0$. Let us take $\chi(\tau) \in [0,1]$ a smooth compactly supported function that vanishes for $|\tau| \ge 1$ and which is equal to one for $|\tau| \le 1/2$. We define $\chi_R(\tau) = \chi(\tau/R)$ and $\chi_R(\partial_t)$ the corresponding operator in t variable corresponding to the convolution with the inverse Fourier transform of $\chi_R(\tau)$. Thanks to (2.14), we have that for R large

$$\langle t\rangle^2 |(1 - \chi_R(\partial_t))K(t)| \le C \sum_{\alpha\le 2} \|\partial_\tau^\alpha((1 - \chi_R(\tau))\hat{K}(\tau))\|_{L^1(\mathbb{R})} \le C \int_{|\tau|\ge R/2} \frac{1}{\langle\tau\rangle^2} \le \frac{C}{R}$$

and hence

$$\|(1 - \chi_R(\partial_t))K(t)\|_{L^1(\mathbb{R})} \le \frac{C}{R} \le \frac{1}{2} \tag{2.15}$$

for R sufficiently large. This choice fixes R.

To estimate the solution y of (2.10), we shall write that

$$y = \chi_{2R}(\partial_t)y + (1 - \chi_{2R}(\partial_t))y =: y^l + y^h.$$

By applying $(1 - \chi_{2R}(\partial_t))$ to (2.10), we get that

$$y^h = K * y^h + (1 - \chi_{2R}(\partial_t))F = \big((1 - \chi_R(\partial_t)K\big) * y^h + (1 - \chi_{2R}(\partial_t))F$$

since $(1 - \chi_R) = 1$ on the support of $1 - \chi_{2R}$. Therefore, we obtain thanks to (2.15) and the fact that $\chi_{2R}(\partial_t)$ is a convolution operator with a L^1 function, that

$$\|y^h\|_{L^\infty} \leq \frac{1}{2}\|y^h\|_{L^\infty} + C\|F\|_{L^\infty}$$

and hence

$$\|y^h\|_{L^\infty} \leq 2C\|F\|_{L^\infty}.$$

For the low frequencies, we can use directly the form (2.10) of the equation: We can write that

$$\hat{y}^l(\tau) = \frac{\chi_{2R}(\tau)}{1 - \hat{K}(\tau)}\chi_R(\tau)\hat{F}(\tau).$$

Since the denominator does not vanish thanks to (**H**), we obtain again that y^l can be written as the convolution of an L^1 function - which is the inverse Fourier transform of $\chi_{2R}(\tau)/(1 - \hat{K}(\tau))$ - by the function $\chi_R(\partial_t)F$ which is a convolution of F by a smooth function. Thus we obtain by using again the Young inequality that

$$\|y^l\|_{L^\infty} \leq C\|F\|_{L^\infty}.$$

Since $\|y\|_{L^\infty} \leq \|y^l\|_{L^\infty} + \|y^h\|_{L^\infty}$, we get the desired estimate for $\gamma = 0$. To get the estimate for arbitrary γ, we can proceed by induction. We observe that

$$ty(t) = K * (ty) + F^1$$

with $F^1 = (tK) * y + tF$. Using the result $\gamma = 0$, we obtain that $\|ty\|_{L^\infty} \leq C\|F^1\|_{L^\infty}$. Now since $\eta_0 \in \mathcal{H}^{\gamma+3}$, for $\gamma = 1$, we obtain that $tK \in L^1$ and thus

$$\|F^1\|_{L^\infty} \leq C\big(\|tF\|_{L^\infty} + \|y\|_{L^\infty}\big) \leq C\|(1+t)F\|_{L^\infty}.$$

The higher order estimates follow easily in the same way. □

2.2. Estimate of $N_{T,s}(g)$.

Proposition 2.4. *Assuming that $\eta \in \mathcal{H}^{s+2}$ verifies the assumption* (**H**)*, then there exists $C > 0$ such that for every $T > 0$, every solution of (1.2) such that $Q_{T,s}(g) \leq R$ enjoys the estimate*

$$N_{T,s}(g) \leq C(R_0 + \varepsilon R^2)(1 + \varepsilon R)e^{C\varepsilon R}.$$

Proof. To prove Proposition 2.4, we shall use energy estimates. We set $\mathcal{L}_t[g]$ the operator

$$\mathcal{L}_t[g]f = \{\phi(t,g), f\}$$

such that g solves the equation

$$\partial_t g = \mathcal{L}_t[g](\eta + \varepsilon g).$$

For any linear operator D, we thus have by standard manipulations that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|Dg(t)\|_{L^2}^2 &= 2\varepsilon\langle Dg(t), D(\mathcal{L}_t[g]g(t))\rangle_{L^2} + 2\langle Dg(t), D(\mathcal{L}_t[g](\eta))\rangle_{L^2}\\ &= 2\varepsilon\langle Dg(t), \mathcal{L}_t[g]Dg(t)\rangle_{L^2} + 2\varepsilon\langle Dg(t), [D,\mathcal{L}_t[g]]g(t)\rangle_{L^2}\\ &\quad +2\langle Dg(t), D(\mathcal{L}_t[g](\eta))\rangle_{L^2},\end{aligned}$$

where $[D,\mathcal{L}_t]$ denotes the commutator between the two operators D and $\mathcal{L}_t$. The first term in the previous equality vanishes since $\mathcal{L}_t[g]$ is the transport operator associated with a divergence free Hamiltonian vector field. Consequently, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Dg(t)\|_{L^2}^2 \leqslant 2\varepsilon\,\|Dg(t)\|_{L^2}\|[D,\mathcal{L}_t[g]]g(t)\|_{L^2} + 2\int_0^t \|Dg(t)\|_{L^2}\|D(\mathcal{L}_t[g](\eta))\|_{L^2}. \tag{2.16}$$

To get the estimates of Proposition 2.4, we shall use the previous estimates with the operator $D = D^{m,p,q}$ defined as the Fourier multiplier by $k^p\xi^q\partial_\xi^m$ for $(m,p,q)\in\mathbb{N}^{3d}$ such that $p+q\leqslant s$, $m\leqslant m_0$ and the definition (1.4) of the $\mathcal{H}^s$ norm. To evaluate the right hand-side of (2.16), we shall use

Lemma 2.5. *For $p+q\leqslant\gamma$ and $m\leqslant m_0$, and functions $h(t)$ and $g(t)$, we have the estimates*

$$\|\big[D^{m,p,q},\mathcal{L}_\sigma[g]\big]h(\sigma)\|_{L^2} \le C\big(m_{\sigma,\gamma+1}(\zeta)\|h(\sigma)\|_{\mathcal{H}^1} + m_{\sigma,2}(\zeta)\|h(\sigma)\|_{\mathcal{H}^\gamma}\big), \tag{2.17}$$

$$\|D^{m,p,q}\big(\mathcal{L}_\sigma[g]\big)h(\sigma)\|_{L^2} \le C\big(m_{\sigma,\gamma+1}(\zeta)\|h(\sigma)\|_{\mathcal{H}^1} + m_{\sigma,2}(\zeta)\|h(\sigma)\|_{\mathcal{H}^{\gamma+1}}, \tag{2.18}$$

for all σ, where ζ is still defined by $\zeta_k(t)=\hat{g}_k(t,kt)$, $k\in\{\pm1\}$, and where

$$m_{\sigma,\gamma}(\zeta) = \langle\sigma\rangle^\gamma\Big(\sup_{k\in\{\pm1\}}|\zeta_k(\sigma)|\Big),$$

with a constant C depending only on γ, and in particular, does not depend on σ.

Let us finish first the proof of Proposition 2.4. By using the previous lemma with $\gamma=s$ and (2.17) with $h=g$ and (2.18) with $h=\eta$, we obtain from (2.16) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|_{\mathcal{H}^s}^2 \le \langle t\rangle^2 m_{t,s-1}(\zeta)\big(\|\eta\|_{\mathcal{H}^1}+\varepsilon\|g(t)\|_{\mathcal{H}^1}\big)\|g(t)\|_{\mathcal{H}^s}\\ +\frac{1}{\langle t\rangle^{s-3}}m_{t,s-1}(\zeta)\|\eta\|_{\mathcal{H}^{s+1}}\|g(t)\|_{\mathcal{H}^s} + \frac{\varepsilon}{\langle t\rangle^{s-3}}m_{t,s-1}(\zeta)\|g(t)\|_{\mathcal{H}^s}^2.\end{aligned}$$

This yields using the fact that $M_{t,\gamma}(\zeta)=\sup_{\sigma\in[0,t]}m_{\sigma,\gamma}(\zeta)$,

$$\|g(t)\|_{\mathcal{H}^s} \le \|g(0)\|_{\mathcal{H}^s} + \langle t\rangle^3 M_{t,s-1}(\zeta)\big(\|\eta\|_{\mathcal{H}^{s+1}}+\varepsilon R\big) + \varepsilon R\int_0^t \frac{1}{\langle\sigma\rangle^{s-3}}\|g(\sigma)\|_s\,\mathrm{d}\sigma$$

for $t\in[0,T]$. From the Gronwall inequality, we thus obtain

$$\|g(t)\|_{\mathcal{H}^s} \le \Big(\|g(0)\|_{\mathcal{H}^s} + \langle t\rangle^3 M_{t,s-1}(\zeta)\big(\|\eta\|_{\mathcal{H}^{s+1}}+\varepsilon R\big)\Big)e^{\varepsilon R\int_0^{+\infty}\frac{\mathrm{d}\sigma}{\langle\sigma\rangle^{s-3}}}.$$

By using Proposition 2.2, this yields

$$N_{T,s}(g) \le \Big(R_0 + (R_0+\varepsilon R^2)(C+\varepsilon R)\Big)e^{C\varepsilon R}.$$

This ends the proof of Proposition 2.4.

□

Let us give the proof of Lemma 2.5.

Proof of Lemma 2.5. We give the proof of (2.17), the proof of the second estimate being slightly easier. In the Fourier side, we have for $\mathcal{L}_\sigma[g](h)$ the expression

$$(\mathcal{F}\mathcal{L}_\sigma[g]h)_n(\xi) = \sum_{k\in\{\pm 1\}} kp_k\zeta_k(\sigma)\hat{h}_{n-k}(\sigma,\xi-k\sigma)(n\sigma-\xi).$$

Consequently, we obtain that

$$\big(\mathcal{F}([D^{m,p,q},\mathcal{L}_\sigma[g]h)\big)_n(\xi) = \sum_{k\in\{\pm 1\}} kp_k\zeta_k(\sigma)\Big(n^p\xi^q\partial_\xi^m\big(\hat{h}_{n-k}(\sigma,\xi-k\sigma)(n\sigma-\xi)\big) - \big((n-k)^p(\xi-\sigma)^q\partial_\xi^m\hat{h}_{n-k}(\sigma,\xi-k\sigma)(n\sigma-\xi)\big)\Big).$$

For $k=\pm 1$, we can thus expand the above expression into a finite sum of terms under the form

$$I_n^k(\sigma,\xi) = kp_k\zeta_k(\sigma)k^{p_1}(n-k)^{p-p_1+\alpha}\big(k\sigma\big)^{q_1+\alpha}(\xi-k\sigma)^{q-q_1+\beta}\partial_\xi^{m_1}\hat{h}_{n-k}(\sigma,\xi-k\sigma)$$

where

$$0\le p_1\le p,\ 0\le q_1\le q,\quad m-1\le m_1\le m,\quad \alpha+\beta=m_1-m+1,\ \alpha,\ \beta\ge 0.$$

Moreover, if $m_1=m$, then we have $p_1+q_1>0$.

We have to estimate $\sum_n\int_\xi|\sum_{k\in\pm 1} I_n^k(\sigma,\xi)|^2\,d\xi$ by isometry of the Fourier transform.

We note that for a fixed $k\in\{\pm 1\}$ then for $|n-k|+|\xi-k\sigma|\le|k|\sigma$, we have

$$|I_n^k(\sigma,\xi)|\le C\sigma^{p+q+1}|\zeta_k(\sigma)||n-k||\partial_\xi^{m_1}\hat{h}_{n-k}(\sigma,\xi-k\sigma)|$$

whereas for $|n-k|+|\xi-k\sigma|\ge|k|\sigma$, we have

$$|I_n^k(\sigma,\xi)|\le C\langle\sigma\rangle^2|\zeta_k(\sigma)|(|n-k|+|\xi-k\sigma|)^\gamma|\partial_\xi^{m_1}\hat{h}_{n-k}(\sigma,\xi-k\sigma)|.$$

Consequently by taking the L^2 norm, we find that

$$\|\sum_{k\in\pm 1} I_n^k(\sigma,\xi)\|_{L^2}\le C\big(m_{\sigma,\gamma+1}(\zeta)\|h(\sigma)\|_{\mathcal{H}^1}+m_{\sigma,2}(\zeta)\|h(\sigma)\|_{\mathcal{H}^m}\big).$$

This ends the proof of the Lemma. ☐

2.3. **Estimate of $\|g\|_{\mathcal{H}^{s-4}}$.** To close the argument, it only remains to estimate $\|g\|_{\mathcal{H}^{s-4}}$.

Proposition 2.6. *Assuming that $\eta\in\mathcal{H}^{s+2}$ verifies the assumption* **(H)***, then there exists $C>0$ such that for every $T>0$, every solution of (1.2) such that $Q_{T,s}(g)\le R$ enjoys the estimate*

$$\|g(t)\|_{\mathcal{H}^{s-4}}\le C\big(R_0+\varepsilon R^2\big)e^{C\varepsilon R},\quad \forall t\in[0,T].$$

Proof. We use again (2.16) with $D=D^{m,p,q}$ but now with $p+q\le s-4$. By using Lemma 2.5, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|g(t)\|^2_{\mathcal{H}^{s-4}}\le m_{t,s-3}(\zeta)\big(\|\eta\|_{\mathcal{H}^{s-3}}\|g(t)\|_{\mathcal{H}^{s-4}}+\varepsilon\|g(t)\|^2_{\mathcal{H}^{s-4}}\big). \tag{2.19}$$

This yields

$$\|g(t)\|_{\mathcal{H}^{s-4}} \leq \|g(0)\|_{\mathcal{H}^{s-4}} + \|\eta\|_{\mathcal{H}^{s-3}} M_{t,s-1}(\zeta) \int_0^t \frac{1}{\langle\sigma\rangle^2}\, \mathrm{d}\sigma + \varepsilon M_{t,s-1}(\zeta) \int_0^t \frac{1}{\langle\sigma\rangle^2} \|g(\sigma)\|_{\mathcal{H}^{s-4}}\, \mathrm{d}\sigma.$$

By using Proposition 2.2, we thus get

$$\|g(t)\|_{\mathcal{H}^{s-4}} \leq C\big(R_0 + \varepsilon R^2\big) + \varepsilon R \int_0^t \frac{1}{\langle\sigma\rangle^2} \|g(\sigma)\|_{\mathcal{H}^{s-4}}\, \mathrm{d}\sigma.$$

From the Gronwall inequality, we finally find

$$\|g(t)\|_{\mathcal{H}^{s-4}} \leq C\big(R_0 + \varepsilon R^2\big) e^{C\varepsilon R}.$$

This ends the proof of Proposition 2.6.

□

3. Proof of Theorem 1.1

The proof of Theorem 1.1 follows from the a priori estimates in Propositions 2.2, 2.4 and 2.6 and a continuation argument. Indeed, by combining the estimates of these three propositions, we get that

$$Q_{T,s}(g) \leq C(R_0 + \varepsilon R^2)(1 + \varepsilon R) e^{C\varepsilon R}$$

assuming that $Q_{T,s}(g) \leq R$. Consequently, let us choose R such that $R > CR_0$, then for ε sufficiently small we have $R > C(R_0 + \varepsilon R^2)(1 + \varepsilon R)\Big) e^{C\varepsilon R}$ and hence by usual continuation argument, we obtain that the estimate $Q_{T,s}(g) \leq R$ is valid for all times.

4. Proof of Corollary 1.2

In view of (2.5), let us define $g^\infty(x, v)$ by

$$g^\infty(x, v) = g(0, x, v) + \int_0^{+\infty} \{\phi(\sigma, g), \eta + \varepsilon g(\sigma)\}\, \mathrm{d}\sigma.$$

Note that the integral is convergent in $\mathcal{H}^{s-4}$ since thanks to (2.18), we have

$$\|\{\phi(\sigma, g), \eta + \varepsilon g(\sigma)\}\|_{\mathcal{H}^{s-4}} \leq C(R)\big(\frac{1}{\langle\sigma\rangle^2} + \frac{\langle\sigma\rangle^{\frac{3}{4}}}{\langle\sigma\rangle^{s-3}}\big).$$

Note that for the last estimate, we have used that by interpolation

$$\|g\|_{\mathcal{H}^{s-3}} \leq C\|g\|_{\mathcal{H}^{s-4}}^{\frac{3}{4}} \|g\|_{\mathcal{H}^{s}}^{\frac{1}{4}} \leq C(R)\langle\sigma\rangle^{\frac{3}{4}}.$$

From the same arguments, we also find that

$$\|g(t) - g_\infty\|_{\mathcal{H}^{s-4}} \leq C(R)\Big(\int_t^{+\infty} \frac{1}{\langle\sigma\rangle^2} + \frac{1}{\langle\sigma\rangle^{s-3-\frac{3}{4}}}\, \mathrm{d}\sigma\Big) \leq \frac{C(R)}{\langle t\rangle}.$$

In a similar way, by using again (2.18), we have for $r \leq s - 4$ and $r \geq 1$,

$$\|g(t) - g_\infty\|_{\mathcal{H}^{r}} \leq C(R)\Big(\int_t^{+\infty} \frac{1}{\langle\sigma\rangle^{s-r-2}} + \frac{1}{\langle\sigma\rangle^{s-3}}\, \mathrm{d}\sigma\Big) \leq C(R)\big(\frac{1}{\langle t\rangle^{s-r-3}} + \frac{1}{\langle t\rangle^{s-4}}\big) \leq \frac{C(R)}{\langle t\rangle^{s-r-3}}.$$

5. The case of a kernel with a finite number of modes

In this section, we briefly indicate the modifications in the case that in (1.1), the kernel P is defined by

$$P(x) = \sum_{k=1}^{M} p_k \cos(kx),$$

for some $p_k \in \mathbb{R}$ and for a fixed M. For the Penrose criterion (1.5), it suffices to consider that it holds for any n, $|n| \leq M$, $n \neq 0$.

We can use the weighted norms

$$Q_{T,s}(g) = \sup_{t\in[0,T]} \frac{\|g(t)\|_{\mathcal{H}^s}}{\langle t\rangle^{2M+1}} + \sup_{t\in[0,T]} \sup_{|k|\leq M,\, k\neq 0} \langle t\rangle^{s+1-2k}|\zeta_k(t)| + \sup_{t\in[0,T]} \|g(t)\|_{\mathcal{H}^{s-2M-2}}.$$

One can then obtain that

Theorem 5.1. *Let us fix $s \geq 4M+2$ and $R_0 > 0$ such that $Q_{0,s}(g) \leq R_0$ and assume that $\eta \in \mathcal{H}^{s+4}$ satisfies the assumption* (**H**)*. Then there exists $R > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ and for every $T \geq 0$, we have the estimate*

$$Q_{T,s}(g) \leq R.$$

It is then easy to get from this result a nonlinear damping effect as previously.

The proof of this result follows exactly the same lines as the proof of Theorem 1.1. The estimates for $\sup_{t\in[0,T]} \frac{\|g(t)\|_{\mathcal{H}^s}}{\langle t\rangle^{2M+1}}$ and for $\sup_{t\in[0,T]} \|g(t)\|_{\mathcal{H}^{s-2M-2}}$ can be obtained exactly in the same way as in Proposition 2.4 and Proposition 2.6. The only technical difference is that in Lemma 2.5, we define

$$m_{\sigma,\gamma}(\zeta) = \langle\sigma\rangle^{\gamma} \sup_{|k|\leq M,\, k\neq 0} |\zeta_k(\sigma)|.$$

The only part were we need to be careful is to estimate $\sup_{t\in[0,T]} \sup_{|k|\leq M,\, k\neq 0} \langle t\rangle^{s+1-2k}|\zeta_k(t)|$ as in Proposition 2.2 since more resonances are possible in the integral equation (2.6). By using the Volterra equation (2.8) for $|n| \leq M$, $n \neq 0$, we still get that

$$\sup_{t\in[0,T]} \langle t\rangle^{s+1-2n}|\zeta_n(t)| \leq \sup_{t\in[0,T]} \langle t\rangle^{s+1-2n}|F_n(t)|$$

and we only need to estimate the right hand side.

The only difficulty is to estimate the contribution of the integral terms

$$J_n = \sup_{t\in[0,T]} \langle t\rangle^{s+1-2n} \sum_{|k|\leq M,\, k\neq 0} |p_k| \int_0^t |\zeta_k(\sigma)|\, |\hat{g}_{n-k}(\sigma, nt - k\sigma)||kn|(t-\sigma)\mathrm{d}\sigma$$

for $|n| \leq M$.

If k and n have opposite sign, then, we can proceed as in the estimate of F_n^1 in the proof of Proposition 2.2, we find

$$\langle t\rangle^{s+1-2n}\int_0^t |\zeta_k(\sigma)|\,|\hat{g}_{n-k}(\sigma,nt-k\sigma)||kn|(t-\sigma)\mathrm{d}\sigma$$
$$\leq CQ_{t,s}(g)^2\langle t\rangle^{s+1-2n}\int_0^t \frac{\langle\sigma\rangle^{1+2M}(t-\sigma)}{\langle\sigma\rangle^{s+1-2k}\langle t\rangle^{s}}\,\mathrm{d}\sigma$$
$$\leq CQ_{t,s}(g)^2\langle t\rangle^{2-2n}\int_0^{+\infty}\frac{1}{\langle\sigma\rangle^{s-4M}}\,\mathrm{d}\sigma$$

which is uniformly bounded since $s\geq 4M+2$ and $|n|\geq 1$.

Now let us assume that k and n have the same sign. We can assume that $k\geq 1$ and $n\geq 1$, the other situation being similar. If $n>k$, then we have that $nt-k\sigma\geq(k+1)t-k\sigma\geq t$ and hence we can use the same bound as above. If $n=k$, we can proceed exactly as for the term F_n^2 in the proof of Proposition 2.2. It remains to handle the case $n<k$ which is new. For this one, we split the time integral in the region $\sigma\leq\frac{n}{2k}t$ and the region $\sigma\geq\frac{n}{2k}t$. For the first region we have $nt-k\sigma\geq nt/2$ and hence this part of the integral can be handled as previously. For the region $\sigma\geq\frac{n}{2k}t$, we estimate it by

$$\langle t\rangle^{s+1-2n}\int_0^t |\zeta_k(\sigma)|\,|\hat{g}_{n-k}(\sigma,nt-k\sigma)||kn|(t-\sigma)\mathrm{d}\sigma$$
$$\leq CQ_{t,s}(g)^2\langle t\rangle^{s+1-2n}\int_{\frac{n}{2k}t}^t\frac{t}{\langle\sigma\rangle^{s+1-2k}}\,\mathrm{d}\sigma\leq C\langle t\rangle^{2n-2k+2}.$$

This term is uniformly bounded since $1\leq n\leq k-1$.

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