

# ANALYSIS OF THE MONTE-CARLO ERROR IN A HYBRID SEMI-LAGRANGIAN SCHEME

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ABSTRACT. We consider Monte-Carlo discretizations of partial differential equations based on a combination of semi-lagrangian schemes and probabilistic representations of the solutions. We study the Monte-Carlo error in a simple case, and show that under an anti-CFL condition on the time-step  $\delta t$  and on the mesh size  $\delta x$  and for  $N$  - the number of realizations - reasonably large, we control this error by a term of order  $\mathcal{O}(\sqrt{\delta t/N})$ . We also provide some numerical experiments to confirm the error estimate, and to expose some examples of equations which can be treated by the numerical method.

## 1. INTRODUCTION

The goal of this paper is to analyze and give some error estimates for numerical schemes combining the principles of semi-lagrangian and Monte-Carlo methods for some partial differential equations. Let us describe the method in a very general case by considering first a linear transport equation of the form

$$\partial_t u(t, x) = f(x) \cdot \nabla u(t, x), \quad x \in \mathbb{R}^d, \quad u(0, x) = u_0(x),$$

where  $u_0$  is a given function. Under some regularity assumptions and existence of the flow associated with the vector field  $f(x)$  in  $\mathbb{R}^d$ , the solution of this equation is given by the characteristics representation  $u(t, x) = u(0, \varphi_t(x))$ , where  $\varphi_t(x)$  is the flow associated with the ordinary differential equation  $\dot{y} = f(y)$  in  $\mathbb{R}^d$ . In this context, semi-Lagrangian schemes can be described as follows. Let us consider a set of grid nodes  $x_j$ ,  $j \in K$  in  $\mathbb{R}^d$  ( $K = \mathbb{N}$  or a finite set) and an interpolant operator  $\mathcal{I}$  mapping vectors of values at the nodes,  $(u_j) \in \mathbb{R}^K$  to a function  $(\mathcal{I}u)(x)$  defined over the whole domain. In this paper we will consider the case where  $x_j = j(\delta x)$ ,  $j \in \mathbb{Z}^d$ ,  $\delta x$  is the space mesh size, and  $\mathcal{I}$  a standard linear interpolation operator. Given approximations  $u_j^n$  of the exact solution  $u(t_n, x_j)$  at times  $t_n = n(\delta t)$  and points  $x_j$ , the previous formula gives an approximation scheme for  $u_j^{n+1}$  obtained by solving the ordinary differential equation  $\dot{y} = f(y)$  between  $t_n$  and  $t_{n+1}$ :  $u_j^{n+1} = (\mathcal{I}u^n)(\Phi_{\delta t}(x_j))$ , where  $\Phi_h$  is the numerical flow associated with a time integrator.

These methods are particularly interesting when the vector field  $f(x; u)$  depends on the solution  $u$  making the transport equation nonlinear, see for instance [12, 4, 11] and the references therein. This is the case when an advection term is present for instance, or for Vlasov equations (see for instance [2]). In these situations, standard semi-lagrangian schemes are based on solving equations of the form

$$\partial_t u(t, x) = f(x; u^n) \cdot \nabla u(t, x),$$

between  $t_n$  and  $t_{n+1}$ , where  $u^n$  denotes the solution at time  $t_n$ . In other words the vector field is frozen in  $u^n$  (in the language of geometric numerical integration, it is Crouch and Grossman method, see [1]). If moreover the vector field  $f(x; u)$  possesses some geometric structure for all functions  $u$ , the numerical integrator can be chosen to preserve this structure (for example symplectic integrator in the Vlasov case).

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In many situations, a diffusion term is present, and the equation can be written (in the linear case)

$$(1.1) \quad \partial_t u(t, x) = \frac{1}{2} \sigma(x) \sigma(x)^T \Delta u + f(x) \cdot \nabla u(t, x), \quad x \in \mathbb{R}^d, \quad u(0, x) = u_0(x),$$

where  $\sigma(x)$  is a  $d \times n$  matrix. In this case the solution admits the characteristic representation

$$u(t, x) = \mathbb{E} u_0(X_t^x),$$

where  $X_t^x$  is the stochastic process associated with the stochastic differential equation

$$(1.2) \quad dX_t^x = f(X_t^x) dt + \sigma(X_t^x) dB_t, \quad X_0^x = x,$$

where  $(B_t)_{t \geq 0}$  is a standard  $n$ -dimensional Brownian Motion.

In general, the law of the random variable  $X_t^x$  is not explicitly known, and we are not able to compute the expectation. The classical approximation procedures for such problems are Monte-Carlo methods: if we assume that we are able to compute  $N$  independent realizations  $(X_t^{x,m})_{1 \leq m \leq N}$  of the law of  $X_t^x$ , we can approach  $u(t, x)$  with

$$(1.3) \quad \frac{1}{N} \sum_{m=1}^N u_0(X_t^{x,m}).$$

In general, the variance of the random variables  $u_0(X_t^{x,m})$  is of size  $t$  and the law of large numbers ensures that the statistic error made is typically of order  $\mathcal{O}(\sqrt{T/N})$  for an integration over the interval  $[0, T]$ . To this error must be added the error in the approximation of the process  $X_t^x$  by numerical schemes of Euler type for instance. This error is of order  $\mathcal{O}(\delta t)$ , see for instance [8, 10, 13] and the reference therein for analysis of the weak-error in the numerical approximation of stochastic differential equations. If a global knowledge of the solution is required, the above operation must be repeated for different values of  $x_j$  on the grid.

The numerical method we study in this paper is based on the Markov property of the associated stochastic processes: we have for any  $x_j$  on the spacial grid and locally in time

$$(1.4) \quad u(t_{n+1}, x_j) = \mathbb{E} u(t_n, X_{\delta t}^{x_j}),$$

which is the formula we aim at discretizing. Using the Euler method to compute a numerical approximation of  $X_{\delta t}^{x_j}$ , we end up with the following numerical scheme

$$(1.5) \quad u_j^{n+1} := \frac{1}{N} \sum_{m=1}^N (\mathcal{I}u^n)(x_j + \delta t f(x_j) + \sqrt{\delta t} \mathcal{N}^{n,m,j}),$$

where the random variables  $(\mathcal{N}^{n,m,j})_{1 \leq m \leq M}$  are independent standard Gaussian random variables. Note that the main difference between the standard Monte-Carlo method is that the average is computed at every time step.

Such numerical method were already introduced in [3]. The principle of using (random) characteristic curves in (1.4) over a time interval of size  $\delta t$  and to use an interpolation procedure to get functions defined on the whole domain fits in the semi-lagrangian framework. The addition of the Monte-Carlo approximation then justifies the use of the hybrid terminology.

As in the deterministic case described above, it is clear the method can be adapted in situations where the drift term  $f(x)$  and the noise term  $\sigma(x)$  depend on the solution  $u$ . We will present in the end of the paper some numerical experiment in such nonlinear situations.

Another remark is that different kind of boundary conditions can be considered. If the presentation made above was concerning the situation where the equation is set on  $\mathbb{R}^d$ , representation formulae such as (1.4) hold true in the case of periodic boundary conditions, Dirichlet or Neumann condition on bounded domains. Again, we give some numerical examples in the end of the paper.

But the main aim of this paper is to perform the numerical analysis of the scheme (1.5) in the simplest situation, that is  $f = 0$ , in dimension  $d = 1$ , with  $\sigma(x) = 1$  and periodic initial conditions such that the transport equation is set on a domain  $D = (0, 1) \subset \mathbb{R}$  with periodic boundary conditions. Note that using splitting strategy, this is the term that is new in (1.5) in comparison with standard semi-Lagrangian methods.

In essence, the result stated in the next section shows that in this situation, the algorithm (1.5) approximates the exact solution up to an error that is of the order  $\mathcal{O}(\delta x + \sqrt{\delta t/N})$ . The first term comes from the interpolation, and the second show a variance reduction phenomenon. To obtain this bound, we require an anti-CLF condition as usual for semi-Lagrangian methods. We also assume that  $N$  is sufficiently large in some relatively weak sense (see a precise statement below).

The paper is organized as follows. In Section 2, we present the method, introduce various notations and state our main result. In Section 3 we give some properties of random matrices arising in a natural way in the definition of the scheme (1.5), and that are needed in the proof of our main estimate, given in full details Section 4. Possible extensions of the method, with for instance Dirichlet boundary conditions, or more complicated PDEs, are evoked in Section 5, together with a few numerical results. In particular, we present simulations for the two-dimensional Burgers equation.

## 2. SETTING AND MAIN RESULT

We consider the linear heat equation on  $(0, 1)$ , with a smooth initial condition and with periodic boundary conditions; more precisely, we want to approximate the unique periodic solution of the following partial differential equation:

$$(2.1) \quad \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2},$$

such that  $u(0, \cdot) = u_0 : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth periodic function of period 1, and such that for any  $t \geq 0$  the function  $u(t, \cdot)$  is periodic of period 1. For periodic functions, we denote by  $L^2$  and  $H^1$  the usual function spaces associated with the respective norm and semi-norm

$$\|f\|_{L^2}^2 = \int_0^1 |f(x)|^2 dx, \quad \text{and} \quad |f|_{H^1}^2 = \int_0^1 |f'(x)|^2 dx.$$

**2.1. Interpolation operator.** For a given integer  $M_S \geq 1$ , we discretize the space interval  $(0, 1)$  with the introduction of nodes  $x_j = j\delta x$  for  $j \in S := \{0, \dots, M_S - 1\}$ , with the condition  $x_{M_S} = M_S\delta x = 1$ . We set  $V = \{(u_j)_{j \in S}\} \subset \mathbb{R}^M$  the set of discrete functions defined on the discrete points of the grid.

We use linear interpolation to reconstruct functions on the whole interval from values at the nodes. We define an appropriate basis made of periodic and piecewise linear functions for  $k \in S = \{0, \dots, M_S - 1\}$ . We set for  $x \in [x_k - 1/2, x_k + 1/2]$ ,

$$\phi_k(x) = \hat{\phi}\left(\frac{x - x_k}{\delta x}\right), \quad \text{where} \quad \hat{\phi}(x) = \begin{cases} 0 & \text{if } |x| > 1, \\ 1 - |x| & \text{if } |x| \leq 1, \end{cases}$$

and extend the function  $\phi_k$  by periodicity on  $(0, 1)$ . Hence  $\phi_k$  is the unique piecewise linear periodic function and satisfying  $\phi_k(x_j) = \delta_{kj}$  the Kronecker symbol. Note that we have  $\sum_{k \in S} \phi_k(x) = 1$  for all  $x \in (0, 1)$ .

We define the following projection and interpolation operators

$$\mathcal{P} : \begin{cases} H^1 \rightarrow V \\ f \mapsto (f(x_j))_{j \in S} \end{cases} \quad \text{and} \quad \mathcal{I} : \begin{cases} V \rightarrow H^1 \\ u = (u_k)_{k \in S} \mapsto \sum_{k=0}^{M_S-1} u_k \phi_k(x), \end{cases}$$

where  $H^1$  denote the Sobolev space of periodic function on  $(0, 1)$ . Clearly,  $\mathcal{P} \circ \mathcal{I}$  is the identity on  $V$ ; nevertheless the distance between the identity and the composition of the operators  $\mathcal{I} \circ \mathcal{P}$

depends on the functional spaces and on the norms. Below, we give the estimates that are useful in our setting. We just notice that  $\mathcal{I} \circ \mathcal{P}(\mathbf{1}) = \mathbf{1}$ , as  $\sum_k \phi_k(x) = 1$  for all  $x$ .

**2.2. Discrete norms.** For any  $u = (u_j)_{j \in S}$  we define the discrete  $L^2$  norm and  $H^1$  semi-norm as

$$\|u\|_{\ell^2}^2 = \delta x \sum_{j \in S} u_j^2 \quad \text{and} \quad |u|_{h^1}^2 = \delta x \sum_{j \in S} \frac{(u_{j+1} - u_j)^2}{\delta x^2},$$

where we use the extension by periodicity of the sequence  $(u_j)$  for the definition of the  $h^1$  semi-norm: we thus have  $u_{M_S} = u_0$ . We also define a norm with  $\|u\|_{h^1} = (\|u\|_{\ell^2}^2 + |u|_{h^1}^2)^{1/2}$ .

With these notations, we have the following approximation results:

**Proposition 2.1.** *There exists a constant  $c > 0$  such that for any mesh size  $\delta x = 1/M_S$ , and any sequence  $u = (u_j) \in V$  we have:*

$$|u|_{h^1} = |\mathcal{I}u|_{H^1}, \quad \text{and} \quad \|u\|_{\ell^2}^2 = \|\mathcal{I}u\|_{L^2}^2 + c\delta x^2 |u|_{h^1}^2.$$

Moreover, for any function  $f \in H^1$  we have

$$\|f - (\mathcal{I} \circ \mathcal{P})f\|_{L^2}^2 \leq c\delta x^2 (|f|_{H^1}^2 + |(\mathcal{I} \circ \mathcal{P})f|_{H^1}^2).$$

Proof. The first equality follows from a direct computation. The second one is proved expanding in the  $L^2$  scalar product  $\mathcal{I}u = \sum_k u_k \phi_k$ , and rewriting the sums in order to make the  $h^1$  semi-norm appear: we have

$$\|\mathcal{I}u\|_{L^2}^2 = \left\| \sum_{k \in S} u_k \phi_k \right\|_{L^2}^2 = \sum_{k, \ell \in S} u_k u_\ell \langle \phi_k, \phi_\ell \rangle_{L^2} = \frac{2\delta x}{3} \sum_{k \in S} u_k^2 + \frac{\delta x}{6} \sum_{k \in S} (u_k u_{k+1} + u_k u_{k-1}),$$

where we define  $u_{M_S} = u_0$  and  $u_{-1} = u_{M_S-1}$ , that is we extend  $u \in V$  by periodicity. We also used the fact that for all  $k$ ,

$$\langle \phi_k, \phi_\ell \rangle_{L^2} = 0 \text{ if } \ell \notin \{k-1, k, k+1\}, \quad \langle \phi_k, \phi_k \rangle_{L^2} = \frac{2\delta x}{3}, \quad \text{and} \quad \langle \phi_k, \phi_{k-1} \rangle_{L^2} = \frac{\delta x}{6}.$$

Now, the equality contains  $\|u\|_{h^1}^2$  which appears with natural integration by parts - using periodicity:

$$\begin{aligned} \|u\|_{\ell^2}^2 - \|\mathcal{I}u\|_{L^2}^2 &= \frac{\delta x}{6} \sum_{k \in S} (u_k(u_k - u_{k-1}) + u_k(u_k - u_{k+1})) \\ &= \frac{\delta x}{6} \sum_{k \in S} (u_{k+1}(u_{k+1} - u_k) + u_k(u_k - u_{k+1})) \\ &= \frac{\delta x}{6} \sum_{k \in S} (u_{k+1} - u_k)^2 = \frac{1}{6} \delta x^2 \|u\|_{h^1}^2. \end{aligned}$$

To prove the last estimate, we have for any  $j \in S = \{0, 1, \dots, M_S - 1\}$  and for any  $x \in [x_j, x_{j+1}]$ ,

$$\begin{aligned} |f(x) - (\mathcal{I} \circ \mathcal{P}f)(x)|^2 &\leq 2 \left( \int_{x_j}^x f'(t) dt \right)^2 + 2 \left( \int_{x_j}^x \frac{f(x_{j+1}) - f(x_j)}{\delta x} dt \right)^2 \\ &\leq 2(x - x_j) \int_{x_j}^{x_{j+1}} |f'(t)|^2 dt + 2\delta x^2 \frac{|f(x_{j+1}) - f(x_j)|^2}{\delta x^2}, \end{aligned}$$

using the Cauchy-Schwarz inequality. Now we integrate over  $[x_j, x_{j+1}]$ , and then it remains to take the sum over  $j \in S$  of the following quantities:

$$(2.2) \quad \int_{x_j}^{x_{j+1}} |f(x) - (\mathcal{I} \circ \mathcal{P}f)(x)|^2 dx \leq \delta x^2 \int_{x_j}^{x_{j+1}} |f'(x)|^2 dx + 2\delta x^2 \frac{|f(x_{j+1}) - f(x_j)|^2}{\delta x^2} \delta x.$$

The first term of the right-hand side is controlled with  $|f|_{H^1}^2$ , while the second term involves  $|\mathcal{P}f|_{h^1}^2 = |(\mathcal{I} \circ \mathcal{P})f|_{H^1}^2$ .  $\square$

**2.3. Definition of the numerical method.** We consider a final time  $T > 0$ , and an integer  $M_T$ , such that we divide the interval  $[0, T]$  into  $M_T$  intervals of size  $\delta t := \frac{T}{M_T}$ . We are thus interested in approximating the solution  $u(t, x)$  of (2.1) at times  $t_n = n\delta t$  and nodes  $x_j$  for  $n \leq M_T$ . In the simple case of Equation (2.1), for which we have the representation formula  $u(t, x) = \mathbb{E}u_0(x + B_t)$ , the numerical scheme (1.5) is the application  $u^n \mapsto u^{n+1}$  from  $V$  to itself written

$$(2.3) \quad u_j^{n+1} = \frac{1}{N} \sum_{m=1}^N \left( \sum_{k \in S} u_k^n \phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,m,j}) \right),$$

where the random variables  $\mathcal{N}^{n,m,j}$ , indexed by  $0 \leq n \leq M_T - 1$ ,  $1 \leq m \leq N$  and  $j \in S$  are independent standard normal variables. More precisely, to avoid an error term due to the approximation of Brownian Motion at discrete times, we require that

$$(2.4) \quad \sqrt{\delta t} \mathcal{N}^{n,m,j} = B_{(n+1)\delta t}^{(m,j)} - B_{n\delta t}^{(m,j)}$$

for some independent Brownian Motions  $(B^{(m,j)})$  for  $1 \leq m \leq N$  and  $0 \leq j \leq M_S - 1$ .

We start with an initial condition  $u^0 = (u_k^0 = u_0(x_k))$ , which contains the values of the initial condition at the nodes. To obtain simple expressions with products of matrices, we consider that vectors like  $u^0$  are column vectors.

We then define the important auxiliary sequence  $v^n \in V$  satisfying the following relations:

$$(2.5) \quad \begin{aligned} v_j^{n+1} &= \frac{1}{N} \sum_{m=1}^N \left( \sum_{k \in S} v_k^n \mathbb{E}[\phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,m,j})] \right) \\ &= \sum_{k \in S} v_k^n \mathbb{E}[\phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,m,j})], \end{aligned}$$

with the initial condition  $v^0 = u^0$ . Indeed, for any  $0 \leq n \leq M_T$  the vector  $v^n$  is the expected value - defined component-wise - of the random vector  $u^n$ .

**2.4. Main result.** With the previous notations, we have the following error estimate:

**Theorem 2.2.** *Assume that the initial condition  $u_0$  is of class  $\mathcal{C}^2$ . For any  $p \in \mathbb{N}$  and any final time  $T > 0$ , there exists a constant  $C_p > 0$ , such that for any  $\delta t > 0$ ,  $\delta x > 0$  and  $N \in \mathbb{N}^*$  we have*

$$(2.6) \quad \sup_{j \in S} |u(t_n, x_j) - v_j^n| \leq C \frac{\delta x^2}{\delta t} \sup_{x \in [0,1]} |u_0''(x)|$$

and

$$(2.7) \quad \mathbb{E} \|u^n - v^n\|_{\ell^2}^2 \leq C_p |u^0|_{h^1}^2 \left(1 + \frac{\delta x^2}{\delta t}\right) \left(1 + \frac{\delta x}{\delta t} + \frac{\delta x^2}{\delta t^2} (1 + |\log(\delta t)|)\right)^p \left(\frac{\delta t}{N} + \frac{1}{N^{p+1}}\right).$$

The control of the first part of the error is rather classical, while the estimate on the Monte-Carlo error given by (2.7) is more original and requires more attention in its analysis and in its proof.

First, we observe that the estimate is only interesting if a condition of anti-CFL type is satisfied: for some constant  $c > 0$  we require

$$\frac{\delta x}{\delta t} \max(1, \sqrt{\log(\delta t)}) < c.$$

We then identify in (2.7) a leading term of size  $\frac{\delta t}{N}$ , which corresponds to the statistical error in a Monte-Carlo method for random variables of variance  $\delta t$ , and a remaining term, which goes to 0 with arbitrary order of convergence with respect to the number of realizations  $N$ . This second term

is obtained via a bootstrap argument. Indeed it is easy to get the classical estimate with  $p = 0$ . The core of the proof is contained in the recursion which allows to increase the order from  $p$  to  $p + 1$ ; it heavily relies on the spatial structure of the noise and on the choice of the  $\ell^2$ -norm.

Thanks to (2.7) when  $p = 1$ , we see that interpreting Theorem 2.2 as a reduction of variance to a size  $\delta t$  is valid: we bound the error with

$$\frac{\delta t}{N} + \frac{1}{N^2} \leq \frac{\delta t^2}{2} + \frac{3}{2N^2},$$

which can be compared with a classical Monte-Carlo bound with the variance  $\delta t$ : we have for any sample  $(Y_1, \dots, Y_N)$  of a random variable  $Y$

$$\text{Var} \left( \frac{1}{N} \sum_{i=1}^N Y_i \right) = \frac{\text{Var}(Y)}{N} \leq \frac{\text{Var}(Y)^2}{2} + \frac{1}{2N^2}.$$

The control of the Monte-Carlo error in Theorem 2.2 relies on several arguments. Firstly, the first factor corresponds to the accumulation of the variances appearing at each time step - where two sources of error are identified: the random variables involve a stochastic diffusion process evaluated at time  $\delta t$ , and an error is introduced by the interpolation procedure. To obtain another factor, we observe that the independence of the random variables appearing for different nodes implies that only diagonal entries of some matrices appear - see (4.12). However, this independence property also complicates the proof: the solutions are badly controlled with respect to the  $h^1$  semi-norm. We then propose a decomposition of the error where the number of realizations  $N$  appears in the variance with the different orders 1 and 2: the first part is controlled by  $\delta t$  and  $\delta x$ , while the second one is only bounded. We finally use recursively this decomposition in order to improve the estimate, with a bootstrap argument.

### 3. RANDOM MATRICES

The definition of the numerical scheme (2.3) can be rewritten with matrix notations: for column vectors of size  $M_S$  such that  $(u^n)_j = u_j^n$ , we see that

$$(3.1) \quad u^{n+1} = P^{(n)} u^n,$$

where the entries of square matrix satisfy for any  $1 \leq j, k \leq M_S$

$$(3.2) \quad P_{j,k}^{(n)} = \frac{1}{N} \sum_{m=1}^N \phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,m,j}).$$

Moreover we decompose these matrices into  $N$  independent parts: for  $1 \leq m \leq N$

$$(3.3) \quad P^{(n)} = \frac{1}{N} \sum_{m=1}^N P^{(n,m)},$$

with the entries  $(P^{(n,m)})_{j,k} = \phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,m,j})$ .

We observe that the matrices  $P^{(n,m)}$  are independent; in each one, the rows are independent, however in a row indexed by  $j$  two different entries are never independent, since they depend on the same random variable  $\mathcal{N}^{n,m,j}$ ; moreover, the sum of coefficients in a row is 1.

All matrices  $P^{(n,m)}$  have the same law; we define a matrix  $Q = \mathbb{E}P^{(n,m)} = \mathbb{E}P^{(n)}$ , by taking the expectations of each entry: for any  $j, k \in S$

$$(3.4) \quad Q_{j,k} = \mathbb{E}[\phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,m,j})].$$

The right-hand side above does not depend on  $n, m$  since we take expectation. It only depends on  $j$  through the position  $x_j$ , not through the random variable  $\mathcal{N}^{n,m,j}$ . With these notations, the vectors  $v^n$  satisfy the relation  $v^{n+1} = Qv^n$  - see (2.5) - and we have for any  $n \geq 0$

$$(3.5) \quad u^n = \prod_{i=0}^{n-1} P^{(i)} u^0 = P^{(n-1)} \dots P^{(0)} u^0, \quad \text{and} \quad v^n = Q^n u^0.$$

We only present a few basic properties of the matrices  $P^{(n,m)}$ ,  $P^{(n)}$  and  $Q$ . First, we show that they are stochastic matrices. Second, we control their behavior with respect to the discrete norms and semi-norms. In order to prove the convergence result, we need other more technical properties which are developed during the proof.

**Proposition 3.1.** *For any  $0 \leq n \leq M_T - 1$  and for any  $1 \leq m \leq N$ , almost surely  $P^{(n,m)}$  is a stochastic matrix: for any indices  $j, k \in S$  we have  $P_{j,k}^{(n,m)} \geq 0$ , and for any  $j \in S$  we have  $\sum_{k \in S} P_{j,k}^{(n,m)} = 1$ .*

*For any  $0 \leq n \leq M_T - 1$ ,  $P^{(n)}$  is also a random stochastic matrix.*

*The matrix  $Q$  is stochastic and symmetric - and therefore is bistochastic.*

Proof. The stochasticity of the random matrices  $P^{(n,m)}$  is a simple consequence of their definition (3.2) and of the relations  $\phi_k(x) \geq 0$  and  $\sum_{k \in S} \phi_{kl}(x) = 1$ . Since  $P^{(n)}$  is a convex sum of the  $P^{(n,m)}$ , the property for those matrices also holds.

Finally, by taking expectation  $Q$  is obviously stochastic; symmetry is a consequence of (3.4), and of the property  $\phi_{k+1}(x) = \phi_k(x - \delta x)$ :

$$\begin{aligned} Q_{j,k} &= \mathbb{E}[\phi_k(x_j + \sqrt{\delta t} \mathcal{N}^{n,m,j})] = \mathbb{E}[\phi_0(x_j - x_k + \sqrt{\delta t} \mathcal{N}^{n,m,j})] \\ &= \mathbb{E}[\phi_0(x_k - x_j - \sqrt{\delta t} \mathcal{N}^{n,m,j})] = \mathbb{E}[\phi_0(x_k - x_j + \sqrt{\delta t} \mathcal{N}^{n,m,k})] = Q_{k,j}, \end{aligned}$$

since  $\phi_0$  is an even function, and since the law of  $\mathcal{N}^{n,m,j}$  is symmetric and does not depend on  $j$ . However this symmetry property is not satisfied by the  $P$ -matrices, because the trajectories of these random variables are different when  $j$  changes.  $\square$

Thanks to the chain of equalities in the proof above, we see that  $Q_{j,k}$  only depends on  $k - j$ , but we observe that no similar property holds for the matrices  $P^{(n,m)}$ .

We now focus on the behavior of the matrices with respect to the  $\ell^2$ -norm. The following proposition is a discrete counterpart of the decreasing of the  $L^2$ -norm of solutions of the heat equation.

**Proposition 3.2.** *For any  $0 \leq n \leq M_T - 1$  and for any  $1 \leq m \leq N$ , and for any  $u \in V$  we have*

$$\mathbb{E} \|P^{(n,m)} u\|_{\ell^2}^2 \leq \|u\|_{\ell^2}^2 \quad \text{and} \quad \mathbb{E} \|P^{(n)} u\|_{\ell^2}^2 \leq \|u\|_{\ell^2}^2.$$

Proof. According to the definitions above (3.1) and (3.3), we have for any index  $j$   $(P^{(n,m)} u)_j = \sum_{k \in S} P_{j,k}^{(n,m)} u_k$ . Thanks to the previous Proposition 3.1, we use the Jensen inequality to get

$$\begin{aligned} \mathbb{E} \|P^{(n,m)} u\|_{\ell^2}^2 &= \delta x \sum_{j \in S} \mathbb{E} |(P^{(n,m)} u)_j|^2 \\ &\leq \delta x \sum_{j \in S} \sum_{k \in S} \mathbb{E} P_{j,k}^{(n,m)} |u_k|^2 \leq \delta x \sum_{k \in S} \left( \sum_{j \in S} Q_{j,k} \right) |u_k|^2; \end{aligned}$$

now we use the properties of the matrix  $Q$  - it is a bistochastic matrix according to Proposition 3.1 - to conclude the proof, since  $\sum_{j \in S} Q_{j,k} = 1$ . The extension to the matrices  $P^{(n)}$  is straightforward.  $\square$

The matrix  $Q$  satisfies the same decreasing property in the  $\ell^2$ -norm; moreover we easily obtain a bound relative to the  $h^1$ -semi norm:

**Proposition 3.3.** *For any  $u \in V$ , we have  $\|Qu\|_{\ell^2} \leq \|u\|_{\ell^2}$  and  $|Qu|_{h^1} \leq |u|_{h^1}$ .*

Proof. The proof of the first inequality is similar to the previous situation for the random matrices. To get the second one, it suffices to define a sequence  $\tilde{u}$  such that for any  $0 \leq j \leq M_S - 1$  we have  $\tilde{u}_j = \frac{u_{j+1} - u_j}{\delta x}$  - with the convention  $u_{M_S} = u_0$ . Then thanks to the properties of  $Q$  we have  $\widetilde{Qu} = Q\tilde{u}$ : for any  $j \in S$

$$\begin{aligned} (\delta x)\widetilde{Qu}_j &= (Qu)_{j+1} - (Qu)_j = \sum_{k \in S} Q_{j+1,k}u_k - \sum_{k \in S} Q_{j,k}u_k \\ &= \sum_{k \in S} Q_{j,k-1}u_k - \sum_{k \in S} Q_{j,k}u_k = \sum_{k \in S} Q_{j,k}(u_{k+1} - u_k) = \delta x(Q\tilde{u})_j, \end{aligned}$$

using a translation of indices with periodic conditions, and the equality  $Q_{j+1,k} = Q_{j,k-1}$  as explained above. As a consequence, we have  $|Qu|_{h^1} = \|\widetilde{Qu}\|_{\ell^2} = \|Q\tilde{u}\|_{\ell^2} \leq \|\tilde{u}\|_{\ell^2} = |u|_{h^1}$ .  $\square$

It is worth noting that the previous argument can not be used to control  $\mathbb{E}|P^{(n,m)}u|_{h^1}$ : for a matrix  $P = P^{(n,m)}$ , the corresponding quantity  $\widetilde{Pu}$  can not be easily expressed with  $\tilde{u}$ . Indeed, given a deterministic  $u$ , then  $(P^{(n,m)}u)_j$  and  $(P^{(n,m)}u)_{j+1}$  are independent random variables - since they are defined respectively with  $\mathcal{N}^{(n,m,j)}$  and  $\mathcal{N}^{(n,m,j+1)}$ . The only result that can be proved is Proposition 3.4 below. However, its only role in the sequel is to explain why we can not obtain directly a good error bound; as a consequence, we do not give its proof.

**Proposition 3.4.** *There exists a constant  $C$ , such that for any discretization parameters  $N \geq 1$ ,  $\delta t = \frac{T}{M_T}$  and  $\delta x = \frac{1}{M_S}$ , we have for any vector  $u \in V$*

$$(3.6) \quad \mathbb{E}|P^{(0)}u|_{h^1}^2 \leq (1 + C \frac{\delta t + \delta x^2}{N\delta x^2})|u|_{h^1}^2.$$

Due to independence of matrices involved at different steps of the scheme, the previous inequalities can be used in chain.

We thus observe that the matrices  $P^{(k)}$  and  $Q$  are quite different, even if  $Q = \mathbb{E}P^{(k)}$ . On the one hand, the matrix  $Q$  is symmetric, and therefore respects the structure of the heat equation - the Laplace operator is also symmetric with respect to the  $L^2$ -scalar product. On the other hand, the structure of the noise destroys this symmetry for matrices  $P^{(k)}$ , while it introduces many other properties due to independence - in some sense noise is white in space and implies first that solutions are not regular, but that on the average a better estimate can be obtained.

#### 4. PROOF OF THEOREM 2.2

We begin with a detailed proof of (2.7). A proof of the other part of the error (2.6) is given in Section 4.4 below. Easy computations give the following expression for the part corresponding to the Monte-Carlo error: for any  $0 \leq n \leq M_T$

$$\delta x \sum_{j=0}^{M_S-1} \text{Var}(u_j^n) = \delta x \sum_{j=0}^{M_S-1} \mathbb{E}|u_j^n - v_j^n|^2 = \mathbb{E}\|u^n - v^n\|_{\ell^2}^2 = \delta x \mathbb{E}(u^n - v^n)^*(u^n - v^n),$$

where the superscript  $*$  denotes transposition of matrices.



Since the vectors  $u^n$  and  $v^n$  satisfy (3.5), with the same deterministic initial condition  $u^0$ , we have

$$\begin{aligned}\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 &= \mathbb{E}\|(P^{(n-1)} \dots P^{(0)} - Q^n)u^0\|_{\ell^2}^2 \\ &= \delta x(u^0)^* \mathbb{E} \left( (P^{(n-1)} \dots P^{(0)} - Q^n)^* (P^{(n-1)} \dots P^{(0)} - Q^n) \right) u^0 \\ &= \delta x(u^0)^* \mathbb{E} \left( (P^{(0)})^* \dots (P^{(n-1)})^* P^{(n-1)} \dots P^{(0)} - (Q^n)^* Q^n \right) u^0,\end{aligned}$$

where the last inequality is a consequence of the relation  $\mathbb{E}P^{(k)} = Q$  and of the independence of the matrices  $P^{(k)}$ .

Therefore we need to study the matrix  $S_n = \mathbb{E} \left( (P^{(0)})^* \dots (P^{(n-1)})^* P^{(n-1)} \dots P^{(0)} - (Q^n)^* Q^n \right)$  given by the expression above, such that

$$\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 = \delta x(u^0)^* S_n u^0.$$

**4.1. Decompositions of the error.** We propose two decompositions of  $S_n$  into sums of  $n$  terms, involving products of matrices  $P^{(k)}$ , of  $Q$  and of the difference between two matrices  $P^{(k)}$  and  $Q$ , which corresponds to a one-step error:

$$(4.1) \quad P^{(n-1)} \dots P^{(0)} - Q^n = \sum_{k=0}^{n-1} P^{(n-1)} \dots P^{(k+1)} (P^{(k)} - Q) Q^k,$$

and

$$(4.2) \quad P^{(n-1)} \dots P^{(0)} - Q^n = \sum_{k=0}^{n-1} Q^{n-1-k} (P^{(k)} - Q) P^{(k-1)} \dots P^{(0)}.$$

These decompositions lead to the following expressions for  $S_n$  - where we use the independence of the matrices  $P^{(k)}$  for different values of  $k$ :

$$\begin{aligned}S_n &= \mathbb{E} \sum_{k=0}^{n-1} (Q^k)^* (P^{(k)} - Q)^* (P^{(k+1)})^* \dots (P^{(n-1)})^* P^{(n-1)} \dots P^{(k+1)} (P^{(k)} - Q) Q^k \\ &= \mathbb{E} \sum_{k=0}^{n-1} (P^{(0)})^* \dots (P^{(k-1)})^* (P^{(k)} - Q)^* (Q^{n-1-k})^* Q^{n-1-k} (P^{(k)} - Q) P^{(k-1)} \dots P^{(0)}\end{aligned}$$

Therefore we obtain the following expressions for the error:

$$(4.3) \quad \begin{aligned}\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 &= \delta x(u^0)^* S_n u^0 \\ &= \sum_{k=0}^{n-1} \mathbb{E}\|P^{(n-1)} \dots P^{(k+1)} (P^{(k)} - Q) Q^k u^0\|_{\ell^2}^2 \\ &= \sum_{k=0}^{n-1} \mathbb{E}\|Q^{n-1-k} (P^{(k)} - Q) P^{(k-1)} \dots P^{(0)} u^0\|_{\ell^2}^2.\end{aligned}$$

Before we show how each decomposition is used to obtain a convergence result, we focus on the variance induced by one step of the scheme. In fact, only the second one gives the improved estimate of Theorem 2.2. Nevertheless, we also get a useful error bound thanks to the first one.

**4.2. One-step variance.** In the previous Section, we have introduced decompositions of the error, and we observed that we need a bound on the error made after each time-step. The following Proposition states that the variance after one step of the scheme is of size  $\delta t$  if we consider the  $\ell^2$  norm, and that a residual term of size  $\delta x^2$  appears due to the interpolation procedure. If we consider  $N$  independent realizations, Corollary 4.2 below states that the variance is divided by  $1/N$  if we look at the full matrix of the scheme.

**Proposition 4.1.** *There exists a constant  $C$ , such that for any discretization parameters  $\delta t = \frac{T}{M_T}$  and  $\delta x = \frac{1}{M_S}$ , and for any  $1 \leq m \leq N$  and  $0 \leq n \leq M_T - 1$ , we have for any vector  $u \in \mathbb{R}^{M_S}$*

$$(4.4) \quad \mathbb{E}\|(P^{(n,m)} - Q)u\|_{\ell^2}^2 \leq C(\delta t + \delta x^2)|u|_{h^1}^2.$$

**Corollary 4.2.** *For any  $0 \leq n \leq M_T - 1$  and for any vector  $u \in \mathbb{R}^{M_S}$ , we have*

$$\mathbb{E}\|(P^{(n)} - Q)u\|_{\ell^2}^2 \leq C \frac{(\delta t + \delta x^2)}{N} |u|_{h^1}^2.$$

The proof of the corollary is straightforward, since  $P^{(n)} = \frac{1}{N} \sum_{m=1}^N P^{(n,m)}$  with independent and identically distributed matrices  $P^{(n,m)}$ . However, the proof of Proposition 4.1 is very technical.

One difficulty of the proof is the dependence of the noise on the position  $j$ : for different indices  $j_1$  and  $j_2$ , the random variables  $(P^{(n,m)}u)_{j_1}$  and  $(P^{(n,m)}u)_{j_2}$  are independent. To deal with this problem, for each  $j$  we introduce an appropriate auxiliary function and we analyze the error on each interval  $[x_j, x_{j+1}]$  separately. We also need to take care of some regularity properties of the functions - they are  $H^1$  functions, piecewise linear, but they are not in general of class  $C^1$  - in order to obtain bounds involving the  $h^1$  and  $H^1$  semi-norms.

Proof of Proposition 4.1. To simplify the notations, we assume that  $n = 0$  and that  $m = 1$  so that we only work with one matrix  $P$  with entries

$$P_{j,k} = \phi_k(x_j + B_{\delta t}^j),$$

where the  $B^j$  are independent Brownian Motions.

We define the following auxiliary periodic functions: for any  $x \in \mathbb{R}$

$$(4.5) \quad V(x) = \mathbb{E}\mathcal{I}u(x + B_{\delta t}^j),$$

and for any index  $0 \leq j \leq M_S - 1$

$$(4.6) \quad U^{(j)}(x) = \mathcal{I}u(x + B_{\delta t}^j).$$

We observe that since we take expectation in (4.5) the index  $j$  plays no role there. Moreover we have the following relations for any  $j \in S$ :

$$V(x_j) = (Qu)_j \quad \text{and} \quad U^{(j)}(x_j) = (Pu)_j, \quad \text{but} \quad U^{(j)}(x_{j+1}) \neq (Pu)_{j+1}.$$

The last relation is the reason why we need to introduce different auxiliary functions  $U^{(j)}$  for each index  $j$ .

We finally introduce the following function depending on two variables: for any  $0 \leq t \leq \delta t$  and  $x \in \mathbb{R}$ ,

$$(4.7) \quad \mathcal{V}(t, x) = \mathbb{E}\mathcal{I}u(x + B_t),$$

for some standard Brownian Motion  $B$ . This function is solution of the backward Kolmogorov equation associated with the Brownian Motion, with the initial condition  $\mathcal{V}(0, \cdot) = \mathcal{I}u$ , and for  $t > 0$

$$\partial_t \mathcal{V} = \frac{1}{2} \partial_{xx}^2 \mathcal{V}.$$

Moreover we have  $\mathcal{V}(\delta t, \cdot) = V$ .

We have the following expression for the mean-square error, integrated over an interval  $[x_j, x_{j+1}]$ : for any index  $j \in S$

$$(4.8) \quad \int_{x_j}^{x_{j+1}} \mathbb{E}|U^{(j)}(x) - V(x)|^2 dx = \int_0^{\delta t} \int_{x_j}^{x_{j+1}} \mathbb{E}|\partial_x \mathcal{V}(\delta t - s, x + B_s^j)|^2 dx ds.$$

The proof of this identity is as follows. First, thanks to smoothing properties of the heat semi-group, for any  $t > 0$  the function  $\mathcal{V}(t, \cdot)$  is smooth. Using Itô formula, with the Brownian Motion  $B^{(j)}$  corresponding to the function  $U^{(j)}$ ,

$$d\mathcal{V}(\delta t - s, x + B_s^j) = \partial_x \mathcal{V}(\delta t - s, x + B_s^j) dB_s^j,$$

for  $0 \leq s \leq \delta t - \epsilon$  and for any  $\epsilon \in (0, \delta t)$ , and the isometry property implies

$$\mathbb{E}|\mathcal{V}(\delta t, x) - \mathcal{V}(\epsilon, x + B_{\delta t - \epsilon}^j)|^2 = \int_0^{\delta t - \epsilon} |\partial_x \mathcal{V}(\delta t - s, x + B_s^j)|^2 ds.$$

We integrate over  $x \in [x_j, x_{j+1}]$ , and we then pass to the limit  $\epsilon \rightarrow 0$ , since  $\mathcal{V}(0, \cdot) = \mathcal{I}u$  is a piecewise linear function. Moreover, we use the identity  $\mathcal{V}(\delta t, \cdot) = V$ . We observe that in the right-hand side of the last equality we take expectation, so that we replace  $B^j$  with the Brownian Motion  $B$ , which does not depend on  $j$ .

Summing over indices  $j \in S$ , we then get, thanks to an affine change of variables  $y = x + B_s$

$$\begin{aligned} \sum_{j \in S} \int_{x_j}^{x_{j+1}} \mathbb{E}|U^{(j)}(x) - V(x)|^2 dx &= \int_0^{\delta t} \int_0^1 \mathbb{E}|\partial_x \mathcal{V}(\delta t - s, x + B_s)|^2 dx ds \\ &= \int_0^{\delta t} \int_0^1 |\partial_x \mathcal{V}(\delta t - s, x)|^2 dx ds = \int_0^{\delta t} |\mathcal{V}(\delta t - s, \cdot)|_{H^1}^2 ds \\ &\leq \int_0^{\delta t} |\mathcal{V}(0, \cdot)|_{H^1}^2 ds = \delta t |\mathcal{I}u|_{H^1}^2 = \delta t |u|_{h^1}^2. \end{aligned}$$

The inequality (4.4) is then a consequence of the two following estimates: first,

$$(4.9) \quad \sum_{j \in S} \int_{x_j}^{x_{j+1}} \mathbb{E} \left| U^{(j)}(x) - V(x) - \mathcal{I} \circ \mathcal{P}(U^{(j)} - V)(x) \right|^2 dx \leq C \delta x^2 |u|_{h^1}^2,$$

and second we show that

$$(4.10) \quad \left| \mathbb{E} \|Pu - Qu\|_{\ell^2}^2 - \sum_{j \in S} \int_{x_j}^{x_{j+1}} \mathbb{E} |\mathcal{I} \circ \mathcal{P}(U^{(j)} - V)(x)|^2 dx \right| \leq C \delta x^2 |u|_{h^1}^2.$$

To get (4.9), we use the inequality (2.2) on each interval  $[x_j, x_{j+1}]$ , for a fixed realization of  $B_{\delta t}^j$ :

$$\begin{aligned} \int_{x_j}^{x_{j+1}} |U^{(j)}(x) - V(x) - \mathcal{I} \circ \mathcal{P}(U^{(j)} - V)(x)|^2 dx &\leq C \delta x^2 \int_{x_j}^{x_{j+1}} |\partial_x (U^{(j)} - V)(x)|^2 dx \\ &\quad + C \delta x \delta x^2 \frac{|[U^{(j)}(x_{j+1}) - V(x_{j+1})] - [U^{(j)}(x_j) - V(x_j)]|^2}{\delta x^2}. \end{aligned}$$

Taking the sum over indices  $j \in S$  and expectation, we see that

$$\begin{aligned} \sum_{j \in S} \int_{x_j}^{x_{j+1}} \mathbb{E} |\partial_x (U^{(j)} - V)(x)|^2 dx &\leq 2 \sum_{j \in S} \int_{x_j}^{x_{j+1}} \mathbb{E} |\partial_x (\mathcal{I}u)(x + B_{\delta t}^j)|^2 dx + \sum_{j \in S} \int_{x_j}^{x_{j+1}} |\partial_x V(x)|^2 dx \\ &\leq 2(|\mathcal{I}u|_{H^1}^2 + |\mathcal{V}(\delta t, \cdot)|_{H^1}^2) \leq 4|\mathcal{I}u|_{H^1}^2 = 4|u|_{h^1}^2, \end{aligned}$$

since  $V = \mathcal{V}(\delta t, \cdot)$ . Indeed, taking expectation allows to consider a single Brownian Motion  $B$ , without  $j$ -dependence.

We now decompose the remaining term as follows:

$$\begin{aligned} \frac{|[U^{(j)}(x_{j+1}) - V(x_{j+1})] - [U^{(j)}(x_j) - V(x_j)]|^2}{\delta x^2} &\leq 2 \frac{|U^{(j)}(x_{j+1}) - U^{(j)}(x_j)|^2}{\delta x^2} \\ &\quad + 2 \frac{|V(x_{j+1}) - V(x_j)|^2}{\delta x^2}. \end{aligned}$$

With the second part, using Proposition 3.3 we see that

$$\delta x \sum_{j \in S} \frac{|V(x_{j+1}) - V(x_j)|^2}{\delta x^2} = \delta x \sum_{j \in S} \frac{|(Qu)_{j+1} - (Qu)_j|^2}{\delta x^2} = |Qu|_{h^1}^2 \leq |u|_{h^1}^2.$$

To treat the first part, we make the fundamental observation that for a fixed  $j \in S$ , the same noise process  $B^j$  is used to compute all values  $U^{(j)}(x)$  when  $x$  varies. As a consequence, we can use a pathwise, almost sure version of the argument leading to the proof of Proposition 3.3 which concerns the behavior of  $Q$  with respect to the  $h^1$  semi norm.

$$\begin{aligned} U^{(j)}(x_{j+1}) - U^{(j)}(x_j) &= \sum_{k \in S} u_k [\phi_k(x_{j+1} + B_{\delta t}^j) - \phi_k(x_j + B_{\delta t}^j)] \\ &= \sum_{k \in S} u_k [\phi_{k-1}(x_j + B_{\delta t}^j) - \phi_k(x_j + B_{\delta t}^j)] \\ &= \sum_{k \in S} [u_{k+1} - u_k] \phi_k(x_j + B_{\delta t}^j), \end{aligned}$$

using the relation  $\phi_{k+1}(x) = \phi_k(x - \delta x)$  and an integration by parts.

Now summing over indices  $j \in S$  and using the Jensen inequality - thanks to Proposition 3.1 - we obtain

$$\begin{aligned} \delta x \sum_{j \in S} \mathbb{E} \frac{|U^{(j)}(x_{j+1}) - U^{(j)}(x_j)|^2}{\delta x^2} &\leq \delta x \sum_{k \in S, j \in S} \mathbb{E} \phi_k(x_j + B_{\delta t}^j) \frac{|u_{k+1} - u_k|^2}{\delta x^2} \\ &\leq \delta x \sum_{k \in S} \frac{|u_{k+1} - u_k|^2}{\delta x^2} = |u|_{h^1}^2. \end{aligned}$$

Having proved (4.9), we now focus on (4.10). We have, since  $\mathcal{P}(U^{(j)} - V)_j = [(P - Q)u]_j$

$$\begin{aligned} & \left| \sum_{j \in S} \int_{x_j}^{x_{j+1}} |\mathcal{I} \circ \mathcal{P}(U^{(j)} - V)(x)|^2 dx - \delta x \sum_j |[(P - Q)u]_j|^2 \right| \\ & \leq C \delta x^2 \delta x \sum_{j \in S} \frac{|(U^{(j)} - V)(x_{j+1}) - (U^{(j)} - V)(x_j)|^2}{\delta x^2}. \end{aligned}$$

It remains to take expectation and to conclude like for (4.9). □

**4.3. Proof of Theorem 2.2.** As we have explained in the introduction, we consider that  $\delta x$  is controlled by  $\delta t$  thanks to a anti-CFL condition. Roughly, from Proposition 4.1 we thus see that the variance obtained after one step of the scheme is of size  $\delta t$ , and that the error depends on the solution through the  $h^1$  semi-norm. Moreover, from Propositions 3.3 and 3.4 we remark that the behaviors of the matrices  $Q$  and  $P^{(n)}$  with respect to this semi-norm are quite different.

Using the first decomposition of the error in (4.3), we use in chain the bounds given above in Propositions 3.2, 3.3 and 4.1 and Corollary 4.2:

$$\begin{aligned}\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 &= \sum_{k=0}^{n-1} \mathbb{E}\|P^{(n-1)} \dots P^{(k+1)}(P^{(k)} - Q)Q^k u^0\|_{\ell^2}^2 \leq \sum_{k=0}^{n-1} \mathbb{E}\|(P^{(k)} - Q)Q^k u^0\|_{\ell^2}^2 \\ &\leq \sum_{k=0}^{n-1} C \frac{(\delta t + \delta x^2)}{N} |Q^k u^0|_{h^1}^2 \leq \sum_{k=0}^{n-1} C \frac{(\delta t + \delta x^2)}{N} |u^0|_{h^1}^2 \\ &\leq C \frac{1 + \delta x^2/\delta t}{N} |u^0|_{h^1}^2.\end{aligned}$$

If the continuous problem is initialized with the function  $u_0$ , which is periodic and of class  $\mathcal{C}^1$ , then  $u^0 = \mathcal{P}u_0$  satisfies  $|u^0|_{h^1} \leq \sup_{x \in [0,1]} |u_0'(x)|$ . Moreover we assume that an anti-CFL condition is satisfied, so that the term  $\delta x^2/\delta t$  is bounded. As a consequence, we find a classical Monte-Carlo estimate, where the error does not decrease when  $\delta t$  goes to 0 and is only controlled with the number of realizations:

$$(4.11) \quad \mathbb{E}\|u^n - v^n\|_{\ell^2}^2 \leq C \frac{1 + \delta x^2/\delta t}{N} |u^0|_{h^1}^2.$$

In fact, (4.11) shows that the variances obtained at each time step can be summed to obtain some control of the variance at the final time. To get an improved bound, we thus need other arguments.

The main observation is that using independence of rows in the  $P$ -matrices, we only need to focus on diagonal terms

$$\sup_{j \in S} \left( (Q^\ell)^* Q^\ell \right)_{jj} = \sup_{j \in S} \left( Q^{2\ell} \right)_{jj},$$

for indices  $0 \leq \ell = n - k - 1 \leq n - 1$ . We recall that indeed  $Q$  is a symmetric matrix, so that  $(Q^\ell)^* Q^\ell = Q^{2\ell}$ .

More precisely, the error can be written

$$\begin{aligned}\mathbb{E}\|u^n - v^n\|_{\ell^2}^2 &= \delta x (u^0)^* S_n u^0 = \delta x \sum_{i,j \in S} u_i^0 (S_n)_{i,j} u_j^0 \\ &= \delta x \sum_{k=0}^{n-1} \sum_{i,j \in S} u_i^0 \mathbb{E} \left[ (A_k)^* (P^{(k)} - Q) Q^{2(n-1-k)} (P^{(k)} - Q) A_k \right]_{i,j} u_j^0,\end{aligned}$$

where for simplicity we use the notation  $A_k := P^{(k-1)} \dots P^{(0)}$ . We compute for any  $i, j \in S$ , using the independence properties at different steps

$$\begin{aligned}\mathbb{E} \left[ (A_k)^* (P^{(k)} - Q) Q^{2(n-1-k)} (P^{(k)} - Q) A_k \right]_{i,j} &= \sum_{k_1, k_2, k_3, k_4 \in S} \mathbb{E} \left[ (A_k)_{k_1, i} (P^{(k)} - Q)_{k_2, k_1} (Q^{2(n-1-k)})_{k_2, k_3} (P^{(k)} - Q)_{k_3, k_4} (A_k)_{k_4, j} \right] \\ &= \sum_{k_1, k_2, k_3, k_4 \in S} \mathbb{E} \left[ (A_k)_{k_1, i} (A_k)_{k_4, j} \right] \mathbb{E} \left[ (P^{(k)} - Q)_{k_2, k_1} (P^{(k)} - Q)_{k_3, k_4} \right] (Q^{2(n-1-k)})_{k_2, k_3}.\end{aligned}$$

The observation is now that if  $k_2 \neq k_3$ , then the independence of the random variables for different nodes implies that

$$(4.12) \quad \mathbb{E} \left[ (P^{(k)} - Q)_{k_2, k_1} (P^{(k)} - Q)_{k_3, k_4} \right] = 0,$$

since it is the covariance of two independent random variables - see (3.2). Moreover, when  $k_2 = k_3$  we see that  $((Q^{(n-1-k)})^* Q^{(n-1-k)})_{k_2, k_3}$  only depends on  $n - k - 1$ , due to invariance properties of

the equation. Therefore we rewrite the former expansion in the following way:

$$\begin{aligned}
(4.13) \quad \mathbb{E}\|u^n - v^n\|_{\ell^2}^2 &= \delta x \sum_{k=0}^{n-1} \sum_{i,j \in S} u_i^0 \mathbb{E}((A_k)^*(P^{(k)} - Q)Q^{2(n-1-k)}(P^{(k)} - Q)A_k)_{i,j} u_j^0 \\
&= \delta x \sum_{k=0}^{n-1} \sum_{i,j \in S} u_i^0 u_j^0 (Q^{2(n-1-k)})_{1,1} \sum_{k_2 \in S} \mathbb{E} \left[ \left( (P^{(k)} - Q)A_k \right)_{k_2,i} \left( (P^{(k)} - Q)A_k \right)_{k_2,j} \right] \\
&= \sum_{k=0}^{n-1} \left( Q^{2(n-1-k)} \right)_{1,1} \mathbb{E}\|(P^{(k)} - Q)P^{(k-1)} \dots P^{(0)}u^0\|_{\ell^2}^2.
\end{aligned}$$

We thus have to control  $(Q^{2\ell})_{1,1} = (Q^{2\ell})_{j,j}$  for any  $j \in S$ . The following Lemma 4.3 gives a control of this expression. The first estimate means that the coefficients  $Q_{j_1, j_2}^{2\ell}$  are approximations of the solution of the PDE at time  $2\ell\delta t$ , at position  $j_2$ , starting from the initial condition  $\phi_{j_1}$ , with an error due to interpolation. The second estimate is fundamental in the proof of the Theorem, since it allows to introduce an additional factor  $\delta x$ ; however, we need to treat carefully the denominator.

**Lemma 4.3.** *There exists a constant  $C$  such that for any discretization parameters  $\delta t = \frac{T}{M_T}$  and  $\delta x = \frac{1}{M_S}$ , we have for any  $1 \leq \ell \leq M_T - 1$  and for any  $0 \leq j_1, j_2 \leq M_S - 1$*

$$(4.14) \quad |Q_{j_1, j_2}^{2\ell} - \mathbb{E}\phi_{j_1}(x_{j_2} + B_{2\ell\delta t})| \leq C \frac{\delta x^2}{\delta t} (1 + |\log(\delta t)|).$$

Moreover, for any  $j \in S$ , we have for any  $1 \leq \ell \leq M_T$

$$(4.15) \quad \mathbb{E}\phi_j(x_j + B_{2\ell\delta t}) \leq C \frac{\delta x}{\sqrt{2\ell\delta t}}.$$

**Remark 4.4.** *The singularities when  $\delta t \rightarrow 0$  with a fixed  $\delta x$  come from the use of regularization of the heat semi-group - when we consider the  $\phi_j$ 's as initial conditions.*

For the second estimate (4.15), we make two important remarks. First, the constant  $C$  depends on the final time  $T$ , and we cannot directly let  $\ell$  tend to  $+\infty$ : we have

$$\lim_{\ell \rightarrow +\infty} \mathbb{E}\phi_j(x_j + B_{2\ell\delta t}) = \int_{x_j - 1/2}^{x_j + 1/2} \phi_j(x) dx = \delta x \neq 0.$$

Second, from (4.15) we get for any  $\ell > 0$  and for any fixed  $\delta t$

$$\lim_{\delta x \rightarrow 0} \mathbb{E}\phi_j(x_j + B_{2\ell\delta t}) = 0,$$

while we know that for a fixed  $\delta x > 0$  and a fixed  $\ell$ , we have

$$\lim_{\delta t \rightarrow 0} \mathbb{E}\phi_j(x_j + B_{2\ell\delta t}) = \phi_j(x_j) = 1.$$

These two behaviors are different and from (4.15) we see the kind of relations that the parameters  $\delta x$  and  $\delta t$  must satisfy for obtaining one convergence or the other.  $\square$

Proof of Lemma 4.3. For any  $0 \leq \ell \leq 2M_T$ , we define

$$M_\ell = \sup_{i,j \in S} |(Q^\ell)_{i,j} - \mathbb{E}\phi_j(x_i + B_{\ell\delta t})|,$$

where  $(B_t)_{t \geq 0}$  is a standard Brownian Motion.

We have  $M_0 = 0$ , and by definition of  $Q$  we also have  $M_1 = 0$ .

We define some auxiliary functions  $W_j$ , for any index  $j \in S$ : for any  $x \in \mathbb{R}$  and any  $t \geq 0$

$$W_j(t, x) = \mathbb{E}\phi_j(x + B_t).$$

$W_j$  is solution of the heat equation, with periodic boundary conditions and initial condition  $\phi_j$ . For any  $t > 0$ ,  $W_j(t, \cdot)$  is therefore a smooth function - thanks to regularization properties of the heat semi-group - and since  $\phi_j$  is bounded by 1 we easily see that we have the following estimates, for some constant  $C$ :

$$(4.16) \quad \|\partial_x W_j(t, \cdot)\|_\infty \leq \frac{C}{\sqrt{t}} \quad \text{and} \quad \|\partial_{xx}^2 W_j(t, \cdot)\|_\infty \leq \frac{C}{t}.$$

We now prove the following estimate on the sequence  $(M_\ell)$ : for any  $1 \leq \ell \leq M_T - 1$

$$(4.17) \quad M_{\ell+1} \leq M_\ell + C \frac{\delta x^2}{\ell \delta t}.$$

The error comes from the interpolation procedure which is made at each time step.

For any  $i, j \in S$ , Markov property implies that

$$\begin{aligned} (Q^{\ell+1})_{i,j} - \mathbb{E}\phi_j(x_i + B_{(\ell+1)\delta t}) &= \sum_{k \in S} Q_{i,k}(Q^\ell)_{k,j} - \mathbb{E}W_j(\ell\delta t, x_i + B_{\delta t}) \\ &= \sum_{k \in S} Q_{i,k}(Q^\ell)_{k,j} - \mathbb{E}\mathcal{I} \circ \mathcal{P}(W_j(\ell\delta t, \cdot))(x_i + B_{\delta t}) \\ &\quad + \mathbb{E}[\mathcal{I} \circ \mathcal{P}(W_j(\ell\delta t, \cdot)) - W_j(\ell\delta t, \cdot)](x_i + B_{\delta t}). \end{aligned}$$

For the first term, we remark that it is bounded by  $M_\ell$ ; indeed we see that

$$\begin{aligned} \mathbb{E}\mathcal{I} \circ \mathcal{P}(W_j(\ell\delta t, \cdot))(x_i + B_{\delta t}) &= \sum_{k \in S} W_j(\ell\delta t, x_k) \mathbb{E}\phi_k(x_i + B_{\delta t}) \\ &= \sum_{k \in S} Q_{i,k} \mathbb{E}\phi_j(x_k + B_{\ell\delta t}). \end{aligned}$$

To conclude, it remains to use the stochasticity of the matrix  $Q$ : entries are positive, and their sum over each line is equal to 1.

The second term is bounded using the following argument:

$$\|\mathcal{I} \circ \mathcal{P}(W_j(\ell\delta t, \cdot)) - W_j(\ell\delta t, \cdot)\|_\infty \leq C\delta x^2 \|\partial_{xx}^2 W_j(\ell\delta t, \cdot)\|_\infty \leq C \frac{\delta x^2}{\ell \delta t},$$

according to well-known interpolation estimates and to (4.16).

From (4.17), using  $M_1 = 0$  we obtain for any  $1 \leq \ell \leq M_T$

$$M_\ell \leq C \frac{\delta x^2}{\delta t} \sum_{k=1}^{M_T-1} \frac{1}{k} \leq C \frac{\delta x^2}{\delta t} (|\log(T)| + |\log(\delta t)|).$$

which gives the result, with a constant depending on  $T$ .

Now we prove the second estimate of the Lemma. Thanks to the relation  $\phi_{k+1}(x) = \phi_k(x - \delta x)$  we see that the left-hand side does not depend on  $j \in S$ ; moreover we expand the calculation of the expectation using the periodicity of the function  $\phi_j$  and relation definition of  $\hat{\phi}$ , the description of

its support as  $x_j + \bigcup_{k \in \mathbb{Z}} [k - \delta x, k + \delta x]$ : we get for  $1 \leq \ell \leq M_T$

$$\begin{aligned}
\mathbb{E} \phi_j(x_j + B_{2\ell\delta t}) &= \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{2\pi\ell\delta t}} \int_{k-\delta x}^{k+\delta x} \hat{\phi}\left(\frac{z-k}{\delta x}\right) e^{-|z|^2/(2\ell\delta t)} dz \\
&\leq \frac{1}{\sqrt{2\pi\ell\delta t}} \sum_{k \in \mathbb{Z}} \int_{k-\delta x}^{k+\delta x} e^{-|z|^2/(2\ell\delta t)} dz \\
&\leq \frac{1}{\sqrt{2\pi\ell\delta t}} \sum_{k \in \mathbb{Z}} \int_{k-\delta x}^{k+\delta x} e^{-|z|^2/(2T)} dz \\
&\leq \frac{1}{\sqrt{2\pi\ell\delta t}} \sum_{k \in \mathbb{Z}} C\delta x e^{-k^2/(2T)} \\
&\leq C \frac{\delta x}{\sqrt{2\ell\delta t}}.
\end{aligned}$$

□

The estimate of Lemma 4.3 is now used in (4.13), and we obtain:

$$\begin{aligned}
(4.18) \quad \sum_{k=0}^{n-1} \left( Q^{2(n-1-k)} \right)_{1,1} &\leq C \sum_{k=0}^{n-2} \left( \frac{\delta x^2}{\delta t} (1 + |\log(\delta t)|) + \frac{\delta x}{\sqrt{(n-1-k)\delta t}} \right) + 1 \\
&\leq C \left( \frac{\delta x^2}{\delta t^2} (1 + |\log(\delta t)|) + \frac{\delta x}{\delta t} + 1 \right) =: CA.
\end{aligned}$$

To conclude one more argument is necessary: we need to apply Proposition 4.1 in order to sum the variances. However this involves the quantity  $\mathbb{E}|P^{(k-1)} \dots P^{(0)} u^0|_{h^1}^2$ , which is badly controlled according to Proposition 3.4: for example, when  $\delta t = \delta x$  the accumulation only implies that

$$\mathbb{E}|P^{(k-1)} \dots P^{(0)} u|_{h^1}^2 \leq (1 + C \frac{\delta t}{N\delta x^2})^k |u|_{h^1}^2 \leq e^{\frac{CT}{N\delta x^2}} |u|_{h^1}^2,$$

for any  $k \leq \frac{T}{\delta t}$ . We recall that this bad behavior of the matrices  $P^{(n)}$  with respect to the  $h^1$ -semi norm is a consequence of the independence of the random variables for different nodes, whereas this independence property is essential to get the improved estimate, since it allows to use the second estimate of Lemma 4.3.

**Remark 4.5.** *Instead of considering Gaussian random variables which are independent with respect to the spatial index  $j$ , we could more generally introduce - like in [7] - a correlation matrix  $K$ , and try to minimize the variance with respect to the choice of  $K$ . Here we have chosen  $K$  as the identity matrix, so that the noise is white in space; the error bound (2.7) we obtain is a nontrivial consequence of an averaging effect due to this choice - see (4.12). A natural question - which is not answered here - would be to analyze the situation for general  $K$ : do we still improve the variance, and can we get more regular solutions?*

The solution we propose relies on the following idea: if above we could replace  $P^{(k-1)} \dots P^{(0)}$  with  $Q^k$ , we could easily conclude. Another error term appears, which is controlled by  $1/N$  instead of  $1/\sqrt{N}$ . More precisely, independence properties yield for  $k \geq 1$

$$\begin{aligned}
(4.19) \quad \mathbb{E} \|(P^{(k)} - Q)P^{(k-1)} \dots P^{(0)} u^0\|_{\ell^2}^2 &= \mathbb{E} \|(P^{(k)} - Q)Q^k u^0\|_{\ell^2}^2 \\
&\quad + \mathbb{E} \|(P^{(k)} - Q) \left( P^{(k-1)} \dots P^{(0)} - Q^k \right) u^0\|_{\ell^2}^2.
\end{aligned}$$

The roles of the different terms are as follows. On the one hand, the first term gives the part of size  $\frac{\delta t}{N}$ , thanks to Lemma 4.3: according to Corollary 4.2 and to Proposition 3.3, we have for any  $k \geq 1$



with  $k\delta t \leq T$

$$(4.20) \quad \mathbb{E}\|(P^{(k)} - Q)Q^k u^0\|_{\ell^2}^2 \leq C \frac{\delta t + \delta x^2}{N} |Q^k u^0|_{h^1}^2 \leq C \frac{\delta t + \delta x^2}{N} |u^0|_{h^1}^2.$$

On the other hand, the second term is now used to improve recursively the error estimate, since we have

$$(4.21) \quad \mathbb{E}\|(P^{(k)} - Q) \left( P^{(k-1)} \dots P^{(0)} - Q^k \right) u^0\|_{\ell^2}^2 \leq \frac{C}{N} \mathbb{E}\| \left( P^{(k-1)} \dots P^{(0)} - Q^k \right) u^0\|_{\ell^2}^2.$$

The independence of realizations at step  $k$  gives the factor  $\frac{1}{N}$ ; we remark that we cannot use the estimation of the one-step variance given by Corollary 4.2: otherwise we would need to control  $\mathbb{E}\| \left( P^{(k-1)} \dots P^{(0)} - Q^k \right) u^0\|_{h^1}^2$ .

Using also (4.18) and (4.19) into (4.13), we see that

$$(4.22) \quad \sup_{n \in \mathbb{N}, n\delta t \leq T} \mathbb{E}\|u^n - v^n\|_{\ell^2}^2 \leq C\delta t \mathcal{A} \frac{1 + \frac{\delta x^2}{\delta t}}{N} |u^0|_{h^1}^2 + C \frac{\mathcal{A}}{N} \sup_{n \in \mathbb{N}, n\delta t \leq T} \mathbb{E}\|u^n - v^n\|_{\ell^2}^2.$$

The proof of the Theorem now reduces to the study of the following recursive inequalities, for  $p \geq 0$

$$E^{(p+1)} \leq C\delta t \mathcal{A} \frac{1 + \frac{\delta x^2}{\delta t}}{N} |u^0|_{h^1}^2 + \frac{C\mathcal{A}}{N} E^{(p)},$$

with an initialization  $E^{(0)} = C \frac{\mathcal{B}}{N}$ , according to (4.11), with the notation  $\mathcal{B} := (1 + \frac{\delta x^2}{\delta t}) |u^0|_{h^1}^2$ . We remark that the control of the matrices  $P^{(k)}$  and  $Q$  with respect to the  $l^2$ -norm leads to another possibility for the initialization:  $E^{(0)} = 2\|u^0\|_{\ell^2}^2$ ; we observe that the recursion then yields the same kind of estimate.

We finally easily prove that for any  $p \geq 0$  there exists a constant  $C_p \geq 1$  such that

$$(4.23) \quad \sup_{n \in \mathbb{N}, n\delta t \leq T} \mathbb{E}\|u^n - v^n\|_{\ell^2}^2 \leq C_p \left( \frac{\mathcal{A}^p \mathcal{B}}{N^{p+1}} + \mathcal{A} \mathcal{B} \frac{\delta t}{N} \right),$$

and the proof of Theorem 2.2 is finished.

**Remark 4.6.** *If we consider the equation  $\frac{\partial u}{\partial t} = \frac{\nu}{2} \frac{\partial^2 u}{\partial x^2}$  with a viscosity parameter  $\nu > 0$ , the quantities  $\mathcal{A}$  and  $\mathcal{B}$  appearing in the proof are transformed into*

$$\mathcal{A}_\nu = \left( 1 + \frac{\delta x}{\sqrt{\nu} \delta t} + \frac{\delta x^2}{\nu \delta t^2} (1 + |\log(\delta t)|) \right) \quad \text{and} \quad \mathcal{B}_\nu = \left( \nu + \frac{\delta x^2}{\delta t} \right) |u^0|_{h^1}^2.$$

where the constant  $C$  does not depend on  $\nu$ .

The first change in the proof concerns the analysis of the one-step variance: in (4.4), the right-hand side is replaced by  $C(\nu\delta t + \delta x^2)$ . We observe that the error due to interpolation remains the same.

The second change concerns Lemma 4.3, where we use some regularization properties thanks to gaussian noise: when  $\nu$  goes to 0 the estimates degenerates.

As a consequence, we may observe that the estimate (2.7) gives a bound valid for a fixed value of  $\nu$ , while (4.11) becomes more interesting when  $\nu$  is small compared with the discretization parameters.

**4.4. Accumulation of the interpolation error.** To obtain Theorem 2.2, it remains to control the deterministic part of the error of the scheme, without the discretization of the expectation with the Monte-Carlo method. We thus need to prove (2.6):

for any  $n \in \mathbb{N}$  such that  $n\delta t \leq T$ , and for any  $j \in \mathbb{N}$  with  $0 \leq x_j = j\delta x < 1$ , we have

$$(4.24) \quad |u(n\delta t, x_j) - v_j^n| \leq C \frac{\delta x^2}{\delta t} \sup_{x \in [0,1]} |u''(x)|,$$

where  $u$  is the exact solution and where  $v^n$  is defined by (2.5).

Since  $\|u(n\delta t, x) - v^n\|_{\ell^2} \leq \sup_j |u(n\delta t, x_j) - v_j^n|$ , we easily obtain an estimate in the  $l^2$ -norm. Therefore, the conditions imposed on  $\delta x$  and  $\delta t$  by (2.7) are not restrictive, and can be seen as consequences of the semi-lagrangian framework.

The proof of (4.24) in our context is as follows: using the exact representation formula and its discrete counterpart (2.5), we have

$$\begin{aligned} u((n+1)\delta t, x_j) - v_j^{n+1} &= \mathbb{E}u(n\delta t, x_j + B_{\delta t}) - \mathbb{E} \sum_{k \in S} v_k^n \phi_k(x_j + B_{\delta t}) \\ &= \sum_{k \in S} (u(n\delta t, x_k) - v_k^n) \mathbb{E} \phi_k(x_j + B_{\delta t}) \\ &\quad + \mathbb{E} \left( u(n\delta t, x_j + B_{\delta t}) - \sum_{k \in S} u(n\delta t, x_k) \phi_k(x_j + B_{\delta t}) \right), \end{aligned}$$

where  $B_{\delta t}$  is a Brownian Motion at time  $\delta t$ .

It is easy to see that

$$\left| \sum_{k \in S} (u(n\delta t, x_k) - v_k^n) \mathbb{E} \phi_k(x_j + B_{\delta t}) \right| \leq \sup_{k \in S} |u(n\delta t, x_k) - v_k^n|,$$

and we see that the other term depends on the interpolation error:

$$\begin{aligned} \left| \mathbb{E} \left[ u(n\delta t, x_j + B_{\delta t}) - \sum_{k \in S} u(n\delta t, x_k) \phi_k(x_j + B_{\delta t}) \right] \right| &\leq \sup_{x \in [0,1]} |u(n\delta t, x) - \mathcal{I} \circ \mathcal{P}u(n\delta t, \cdot)(x)| \\ &\leq C\delta x^2 \sup_{x \in [0,1]} \left| \frac{\partial^2 u}{\partial x^2}(n\delta t, x) \right| \leq C\delta x^2 \sup_{x \in [0,1]} \left| \frac{\partial^2 u}{\partial x^2}(0, x) \right|. \end{aligned}$$

To conclude, we remark that for  $n = 0$  we have  $u(0, x_j) = v_j^0$ .

## 5. NUMERICAL RESULTS AND EXTENSIONS

**5.1. Illustration of Theorem 2.2.** The first numerical example we consider is a simulation of the solution of the heat equation in the spatial domain  $(0, 1)$  in periodic setting. We introduce the viscosity parameter  $\nu$  so that the problem is

$$(5.1) \quad \frac{\partial u(t, x)}{\partial t} = \nu \frac{\partial^2 u(t, x)}{\partial x^2}, \quad \text{for } t > 0, x \in (0, 1), \quad u(0, x) = u_0(x) \text{ for } x \in (0, 1),$$

with the boundary condition  $u(t, 1) = u(t, 0)$  for  $t \geq 0$ . For the numerical simulation of Figure 1, we choose  $\nu = 0.01$ , and  $u_0(x) = \sin(2\pi x)$ . The exact solution satisfies  $u(t, x) = \exp(-4\pi^2\nu t) \sin(2\pi x)$ . The discretization parameters are  $\delta t = \delta x = 0.01$  and  $N = 100$ .

The bound of Theorem 2.2 is illustrated with Figure 2, where we represent the error in logarithmic scales for different values of the parameters.

We study the convergence of the scheme, with a numerical simulation which confirms the order of convergence with respect to the parameters  $\delta t = \delta x$  of the Monte-Carlo error. The final time is  $T = 0.1$ , the viscosity is  $\nu = 0.1$  and the initial condition is  $u_0(x) = \cos(2\pi x)$ . We compare the numerical solution  $u^n$  with the exact solution; we only observe the Monte-Carlo error, which is dominant with respect to the deterministic part of the error according to Theorem 2.2. The mean-square error in the  $\ell^2$  norm is estimated with a sample of size 20.

The error in Figure 2 is represented in logarithmic scales. The parameters  $\delta t$  and  $\delta x$  are equal and satisfy  $\delta t = \delta x = \frac{1}{n}$  for the following values  $n = 50, 100, 200, 400, 800, 1600, 3200$ . Each line is obtained when we draw the logarithm of the Error as a function of  $\log_{10}(n)$ , for a fixed value of  $N \in \{10, 20, 40, 80\}$ . The dot-line represents a straight-line with slope  $-1/2$ .

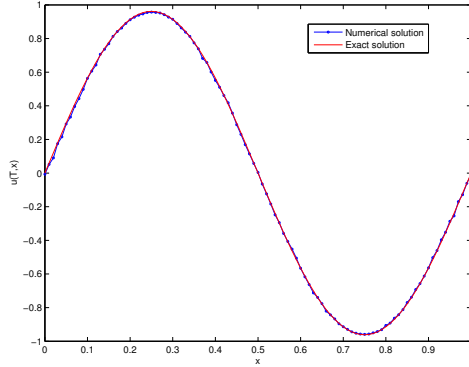


FIGURE 1. Solution at time  $T = 0.1$  with  $\delta t = \delta x = 1/N = 0.01$

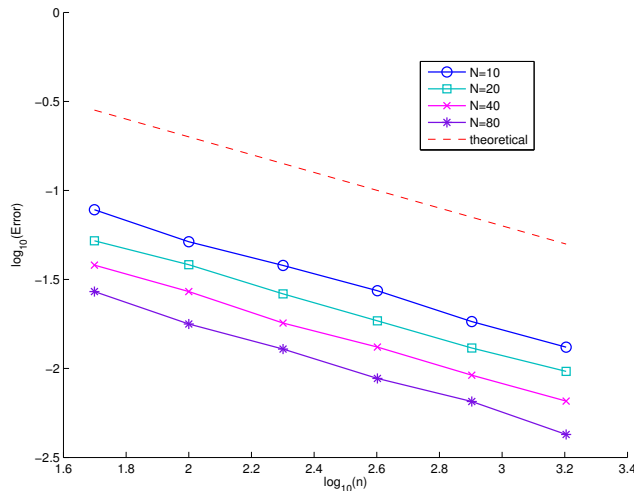


FIGURE 2. Error for periodic boundary conditions when  $\delta t = \delta x = 1/n$ , in logarithmic scales.

This experiment confirms that the Monte-Carlo error is of order  $1/2$  with respect to the parameters when  $\delta t = \delta x$ , as (2.7) claims. Indeed, the shift between the lines when  $N$  varies also corresponds to the size  $1/\sqrt{N}$  of the Monte-Carlo error.

**5.2. The method for Dirichlet boundary conditions.** We would like now show how it is possible to adapt our method in the case of Dirichlet boundary conditions. Let us consider the equation (5.1), but with boundary conditions  $u(t, x) = 0$  for  $t > 0$  and  $x \in \partial D = \{0, 1\}$ . The representation formula then involves the family of the first-exit times of the process  $X_t^x = x + \sqrt{\nu}B_t$  starting from the different points of the domain: If we define  $\tau^x = \inf \{t > 0; X_t^x \in D^c\}$ , then the solution satisfies

$$(5.2) \quad u(t, x) = \mathbb{E} [u_0(X_t^x) \mathbb{1}_{t \leq \tau^x}];$$

the stochastic process is killed when it reaches the boundary. Note that this formula extends to more general PDE of the form (1.1) with the associated process (1.2).

The numerical approximation becomes more complicated, since we also need an accurate approximation of the stopping times. This problem is well-known, and solutions have been proposed in [6]

and [9] for the computation of (5.2) at a given point  $x$  using time discretizations of the stochastic process  $X_t^x$ .

In our case, we take advantage of the semi-lagrangian context to do a refinement near the boundary: for a discretization between the times  $t_n$  and  $t_{n+1}$ , we introduce a decomposition of the domain into an "interior" zone and a "boundary" zone, with different treatments. In the boundary zone, we refine in time and use a subdivision of  $[n\delta t, (n+1)\delta t]$  of mesh size  $\tau \leq \delta t$  and we use a possibly different value  $N_b$  for the number of Monte-Carlo realizations. Moreover, following [6] and [9], we introduce an exit test in the boundary zone, based on the knowledge of the law of exit of the diffusion process.

In the interior part, less care is necessary and we can take  $\tau = \delta t$  and  $N_i < N_b$  for the size of the sample.

We give in Figure 3 the result of investigations on the convergence of the method when Dirichlet boundary conditions are applied. We draw in logarithmic scales the error in terms of  $n = 1/\delta t = 1/\delta x$ , with  $n = 50, 100, 200, 400, 800$ , with different values of the Monte-Carlo parameter  $N_i = 10, 20, 40, 80$ . We have chosen on the interval  $(-1, 1)$  the initial function  $u_0(x) = \sin(\pi \frac{x+1}{2})$ , with the viscosity  $\nu = 0.1$ . The boundary zone is made of the intervals  $(-1, -0.9)$  and  $(0.9, 1)$ , where we take  $\tau = \delta t/10$  and  $N_b = 10N_i$ . The solutions are computed until time  $T = 0.1$ . Like in the case of periodic boundary conditions, the statistical error is dominant with respect to the other error terms; we compare with the exact solution, and to estimate the variance we use a sample of size 100.

The observation of Figure 3 shows that the Monte-Carlo error depends on the parameter  $\delta t = \delta x$ ; the comparison with the "theoretical" line with slope  $-1/2$  indicates a conjecture that the error is also of order  $1/2$ , like for the periodic case. The shift between the curves for different values of  $N$  corresponds in the error to a factor  $1/\sqrt{N}$ .

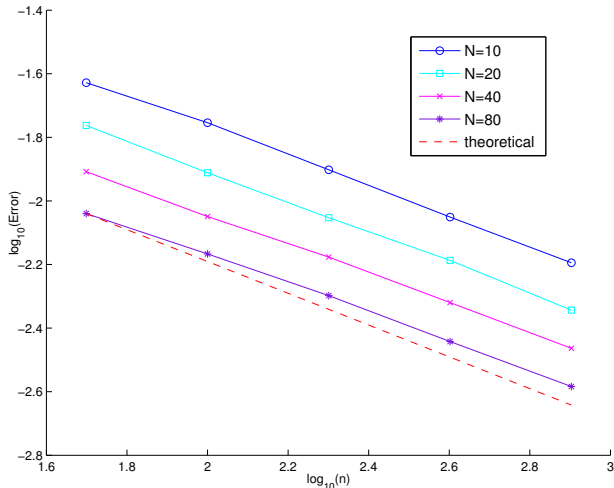


FIGURE 3. Error for Dirichlet boundary conditions when  $\delta t = \delta x = 1/n$ , in logarithmic scales.

**5.3. The method for some non-linear PDEs.** We present a simple method to obtain approximations of the solution of the viscous Burgers equation in dimension  $d = 2$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \Delta u + f.$$

It is defined on the domain  $(-1, 1)^2$ , with homogeneous Dirichlet boundary conditions - periodic ones would also have been possible. Compared with the situations described so far, we add a forcing term  $f$ , which may depend on time  $t$ , position  $x$  and the solution  $u$ .

As explained in the Introduction, we construct approximations  $u^n$  of the solution at discrete times  $n\delta t$ , introducing functions  $v^n$  such that for any  $n \geq 0$  with the following semi-implicit scheme:

$$(5.3) \quad \frac{\partial v^{n+1}}{\partial t} + (u^n \cdot \nabla)v^{n+1} = \nu \Delta v^{n+1} + f^n,$$

for any time  $n\delta t \leq t \leq (n+1)\delta t$  and  $x \in D$ . The initial condition is  $v^{n+1}(n\delta t, \cdot) = u^n = v^n(n\delta t, \cdot)$ . The discrete-time approximation then satisfies  $u^0 = u_0$  and  $u^n = v^n(n\delta t, \cdot)$ . The forcing term here satisfies  $f^n(t, x) = f(n\delta t, x, u^n(x))$ .

On each subinterval  $[n\delta t, (n+1)\delta t]$ , we have

$$v^{n+1}(t, x) = \mathbb{E}[v^{n+1}(n\delta t, X_t^x) \mathbf{1}_{t < \tau^x} + \int_{n\delta t}^{t \wedge \tau^x} f^n(X_s^x) ds],$$

where the diffusion process  $X$  satisfies

$$dX_t^x = -u^n(X_t^x)dt + \sqrt{2\nu}dB_t, X_{n\delta t}^x = x.$$

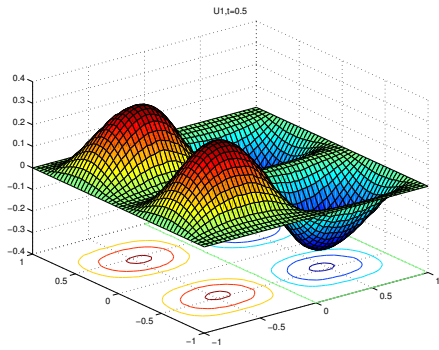
The stopping times  $\tau^x$  represent the first exit time of the process in the time interval  $[n\delta t, (n+1)\delta t]$ . Since  $v^{n+1}(n\delta t, \cdot) = u^n$ , the scheme only requires the knowledge of the approximations  $u^n$ .

For the numerical simulations, we take the initial condition to be 0, and the forcing is  $f(t, x) = (-\sin(\pi t) \sin(\pi x) \sin(\pi y)^2, -\sin(\pi t) \sin(\pi x)^2 \sin(\pi y))$ . The viscosity parameter is  $\nu = 0.001$ . The time step satisfies  $\delta t = 0.02$ , and the spatial mesh size is  $\delta x = 0.04$ . The "interior" zone is  $(-0.8, +0.8)^2$ , where  $N_i = 10$ ; on the "boundary" zone, we have  $N_b = 100 = 10N_i$ , and  $\tau = 0.002 = \delta t/10$ .

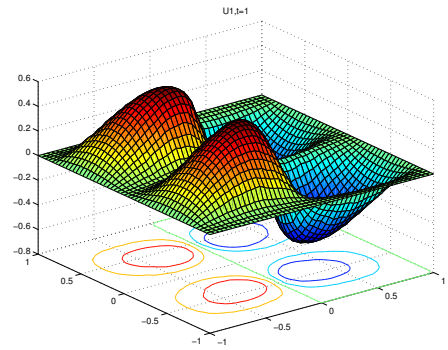
Both components of the velocity field  $u$  are represented in Figures 4 and 5 below at different times  $t = 0.5, 1, 1.5, 2$ .

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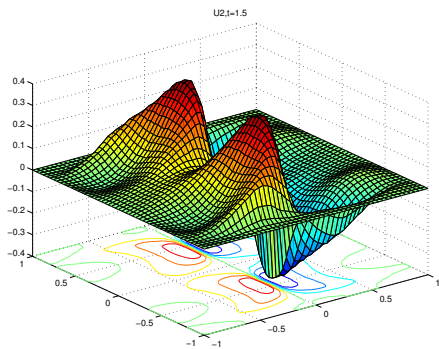
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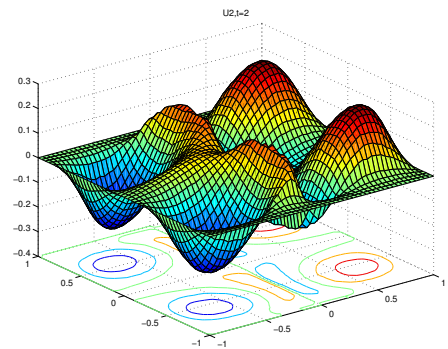
(a)  $u_1, t = 0.5$



(b)  $u_1, t = 1$



(c)  $u_1, t = 1.5$



(d)  $u_1, t = 2$

FIGURE 4. Solution of the 2D Burgers equation at different times - first component

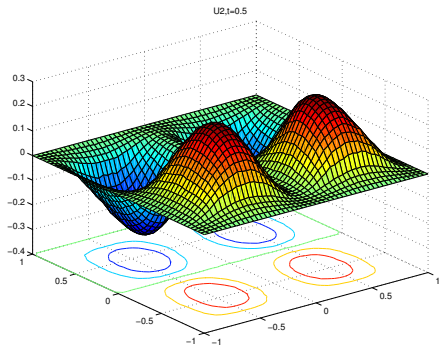
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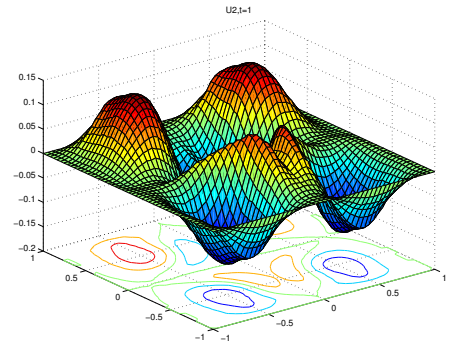
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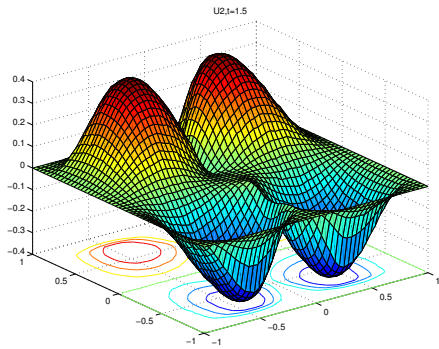
*E-mail address:* `Erwan.Faou@inria.fr`



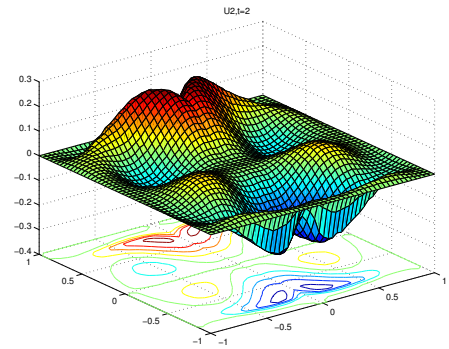
(a)  $u_2, t = 0.5$



(b)  $u_2, t = 1$



(c)  $u_2, t = 1.5$



(d)  $u_2, t = 2$

FIGURE 5. Solution of the 2D Burgers equation at different times - second component