COMPLETE ASYMPTOTICS FOR SHALLOW SHELLS

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Abstract. In this paper we study the asymptotics of the three-dimensional displacement field for clamped and free linear elastic shallow shells as the thickness tends to zero. As in the case of plates, the asymptotics contains regular terms and boundary layers. The two-dimensional generators of the regular parts are solutions of two-dimensional problems governed by an elliptic system in the sense of S. Agmon, A. Douglis and L. Nirenberg. This asymptotics is justified by optimal error estimates and improves the results obtained by S. Busse, P. G. Ciarlet and B. Miara.

Introduction

A thin shell can be defined as a three-dimensional object with a small thickness compared to the other sizes of the mean surface. The expression *shallow shell* means that the curvature of the mean surface is *also* small with respect to the sizes of the mean surface.

We show (see theorem 1.1) that if S is a surface embedded in $\mathbb{R}^3$ and if its principal curvatures are small with respect to the intrinsic diameter of S, the surface is given by a graph over a surface *immersed* in $\mathbb{R}^2$. Moreover, we show that the height of the graph is of the order of the magnitude of the curvature. In the following, we consider shells whose middle surfaces have curvature of the same order as the thickness. This characteristic common length is measured by the number $\varepsilon > 0$.

The previous result then leads to consider a shallow shell as a shell of thickness 2ε whose middle surface is an element of a family of surfaces S_ε represented by the application

$$\omega \ni (x_1, x_2) \mapsto \big(x_1, x_2, \varepsilon\,\theta(x_1, x_2)\big) \in \mathbb{R}^3,$$

where ω is a flat surface with regular boundary immersed in $\mathbb{R}^2$ and θ a smooth function on ω. If ω is *embedded* in $\mathbb{R}^2$, the previous application is a classical graph over a domain of $\mathbb{R}^2$. The three-dimensional shallow shell is then represented by the image $\hat{\Omega}^\varepsilon$ of the application $\boldsymbol{\Phi}^\varepsilon$ defined by

$$\boldsymbol{\Phi}^\varepsilon : \overline{\Omega}^\varepsilon = \overline{\omega} \times [-\varepsilon, \varepsilon] \ni (x_1, x_2, x_3^\varepsilon) \mapsto \big(x_1, x_2, \varepsilon\,\theta(x_1, x_2)\big) + x_3^\varepsilon \boldsymbol{a}_3^\varepsilon(x_1, x_2),$$

where $\boldsymbol{a}_3^\varepsilon(x_1, x_2)$ denotes the unit normal vector to the surface S_ε. This is the definition given for the first time by P. G. Ciarlet and J. C. Paumier in [7] and our result in theorem 1.1 justifies it.

We suppose that $\hat{\Omega}^\varepsilon$ is made with an homogeneous and isotropic material, and that the shell is subjected to the action of volume forces and to conditions on the

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lateral boundary (clamped or free). The diffeomorphism $\mathbf{\Phi}^\varepsilon$ induces on $\hat{\Omega}^\varepsilon$ the system of curvilinear coordinates $(x_1, x_2, x_3^\varepsilon) \in \Omega^\varepsilon$. Our aim is to study the asymptotic behaviour of the displacement $\boldsymbol{u}^\varepsilon$, solution of the three-dimensional linear elasticity equation written in curvilinear coordinates. The body forces are represented by a vector field $\boldsymbol{f}^\varepsilon$.

With this definition, a plate is a special case of a shallow shell: if the mean surface is described by the function $\theta = 0$ and if ω is embedded in $\mathbb{R}^2$, then $\hat{\Omega}^\varepsilon = \omega \times (-\varepsilon + \varepsilon)$ is a plate in the usual sense, with mean surface $S \equiv \omega \subset \mathbb{R}^2$ independent on ε . Thus, we expect that the behaviour of the displacement when ε tends to zero can be described in a same way as for plates, with some differences coming from the existence of a non-zero curvature.

For plates, after the scaling $x_3^\varepsilon = \varepsilon x_3$ and the change of unknown $\boldsymbol{u}(\varepsilon)(x_*, x_3) = (u_*^\varepsilon, \varepsilon u_3^\varepsilon)(x_*, x_3^\varepsilon)$, and by supposing that there exists a vector field $\boldsymbol{f}(x_*, x_3)$ such that $f^{\alpha,\varepsilon}(x_*, x_3^\varepsilon) = f^\alpha(x_*, x_3)$ and $f^{3,\varepsilon}(x_*, x_3^\varepsilon) = \varepsilon f^3(x_*, x_3)$, the displacement $\boldsymbol{u}(\varepsilon)$ tends to a Kirchhoff-Love displacement $\big(\zeta_*^0(x_*) - x_3 \nabla_* \zeta_3^0(x_*), \zeta_3^0(x_*)\big)$, where x_* , ζ_*^0 and $\nabla_* \zeta_3^0$ are condensed notations for (x_1, x_2) , (ζ_1^0, ζ_2^0) and $(\partial_1 \zeta_3^0, \partial_2 \zeta_3^0)$. The generator $\boldsymbol{\zeta}^0 = (\zeta_*, \zeta_3)$ is solution of a two-dimensional problem posed on ω , involving the decoupled operator

$$\begin{pmatrix} L^{\mathrm{m}} & 0 \\ 0 & L^{\mathrm{b}} \end{pmatrix},$$

where L^{m} acts on ζ_* and is the membrane operator for plates, and where L^{b} acts on ζ_3 and is the bending operator (see [8, 11]). Note that this operator is the expression of Koiter's operator (see [16]) in the case of plates after the scaling of the unknown.

For clamped shallow shells, using the method of extracting subsequences, Ciarlet, Miara [6] in Cartesian coordinates, and Busse, Ciarlet and Miara [4] in curvilinear coordinates showed that the scaled displacement $\boldsymbol{u}(\varepsilon)$ converges to a Kirchhoff-Love term, whose generator $\boldsymbol{\zeta}^0$ is solution of a two-dimensional equation on ω involving an operator $\boldsymbol{P}$ that couples the membrane and bending parts with terms depending on θ . Two "different" problems are obtained depending on whether the three-dimensional linear elasticity problem is written in Cartesian or curvilinear coordinates (see [6] and [4]). A comparison between these two models is made in Andreoiu [3], showing that both models describe the same limit after a change of coordinate system. In this paper, the displacement is studied in curvilinear coordinates, as in [4]. With the help of the operator $\boldsymbol{P}$ and the introduction of boundary layer terms, we show the existence of complete asymptotic expansion for the displacement.

As for thin elastic plates, lateral boundary conditions influence the asymptotics of the the three-dimensional displacement field. Two lateral boundary conditions are studied in this paper: shallow shells clamped over the whole lateral boundary and free shallow shells. In the case of plates the influence of lateral boundary conditions is described in Dauge, Gruais & Rössle [11]. Many results of [11] will be used in this paper.

Our aim is to construct an infinite asymptotic expansion of the displacement, validated by optimal error estimates in $\mathbf{H}^1$ norm. As for plates (see [9, 11, 8] and also [18, 17]), this asymptotics includes:

an outer part containing displacements depending on the in-plane variable x_* and of the scaled transverse variable x_3 and

an inner part containing boundary layer terms depending on two scaled variables: x_3 and $t = \varepsilon^{-1} r$ where r is the distance to the lateral boundary.

The outer part contains Kirchhoff-Love fields. Their generators verify two-dimensional problems governed by the elliptic operator $\boldsymbol{P}$. The traces (Dirichlet in the clamped case and Neumann in the free case) for these generators are determined by the conditions ensuring the exponential decay at infinity of the boundary layer terms in the inner part.

Our paper is organised as follows: In section 1, we discuss the concept of shallow shell and prove a theorem showing that a surface with *small* curvature is a graph over a flat immersed surface of $\mathbb{R}^2$.

In section 2 we introduce the studied problem. We consider the three-dimensional linear elasticity equations on the domain $\hat{\Omega}^\varepsilon$. We then make a change of coordinates, using the special geometry of the shallow shell, in order to write the equations in curvilinear coordinates on $\Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon)$. Using scalings, the problem is written on a fixed manifold defined as $\Omega := \omega \times (-1, +1)$.

In section 3, we state the results concerning the mixed Ansatz containing outer and inner expansions and we describe the two-dimensional elliptic operator governing the equations for the Kirchhoff-Love generators of the outer part. We also describe the first Kirchhoff-Love terms for each boundary condition considered.

In section 4, we expand the three-dimensional elasticity operator with respect to ε. We then study the solution of the equations without boundary condition in formal series algebra in ε. In this way, we give the algorithm for the construction of the outer part: the three-dimensional solution is determined by two-dimensional Kirchhoff-Love generators satisfying equations involving the operator $\boldsymbol{P}$. This analysis correspond to the formal series solution for shells (see [14]).

In section 5, we investigate the lateral boundary conditions. To this aim, we introduce an expansion constituted by terms depending on the variable $t = \varepsilon^{-1} r$, where r is the distance to the lateral boundary. This new scaling implies a change of variables in the operators in order to set the equations acting on boundary layer terms. We then see that the first terms in ε of the operators written with the variables (t, s, x_3) are the same as for plates (s denote the arc-length on the lateral boundary). Finally, we match the boundary layer terms to the outer terms constructed in section 3. This matching is only possible on the lateral boundary. We then recall the definition of the spaces in which boundary layer terms will be found, and we review the properties of the operators governing the equations that boundary layers have to verify.

In section 6 & 7, we outline the proofs of the final result in both the clamped and free cases. The method and arguments are identical to those used for plates (see [11, 8]). We show how the requirement for the boundary layer terms to be exponentially decreasing gives rise to two-dimensional boundary conditions for the Kirchhoff-Love generators of the outer part. These conditions lead to well defined generators, which ensure the existence and uniqueness of the expansion. The construction of asymptotic expansion of the three-dimensional displacement is then standard.

1. Shallow shells

A shell $\widehat{\Omega}$ is characterised by its mean surface S and its thickness d: $\widehat{\Omega}$ is the subset of $\mathbb{R}^3$ formed by the points $P + h\boldsymbol{n}(P) \in \mathbb{R}^3$, where $P \in S$, $\boldsymbol{n}(P)$ is the unit normal vector to S in P and $h \in (-\frac{d}{2}, \frac{d}{2})$.

Reciprocally, let S be a smooth surface with boundary, and let $K_{\max}$ be the maximum of the absolute values of the principal curvatures of S. If $d < 1/K_{\max}$, then the map

$$S \times (-\tfrac{d}{2}, \tfrac{d}{2}) \ni (P, h) \mapsto P + h\boldsymbol{n}(P) \in \mathbb{R}^3$$

is a $\mathscr{C}^\infty$-diffeomorphism with image $\widehat{\Omega}$, and in this situation, $\widehat{\Omega}$ is a shell. We say that $\widehat{\Omega}$ is a *shallow shell* if $K_{\max}$ satisfies an estimate of the type $K_{\max} \le Cd$, where C does not depend on d.

Moreover, we have the following theorem:

Theorem 1.1. *Let S be an orientable, compact and connected surface with boundary, embedded in $\mathbb{R}^3$. Let $d(P, Q)$ denote the geodesic distance between two points P and Q of S, and let $D := \max_{P,Q \in S} d(P, Q)$ be the intrinsic diameter of S. We denote by $K_{\max}$ the maximum of the absolute values of the principal curvatures of S. Then if $K_{\max} \le \frac{1}{2D}$, there exists a point $P_0 \in S$, such that the orthogonal projection of S on its tangent plan in P_0 allows the representation of S as a $\mathscr{C}^\infty$ graph in $\mathbb{R}^3$:*

$$\omega \ni (x, y) \mapsto \big(x, y, \Theta(x, y)\big) \in S \subset \mathbb{R}^3,$$

where ω is a flat surface with smooth boundary immersed in $T_{P_0}S$, the tangent plan at S in P_0, subset of $\mathbb{R}^3$, and where Θ is a function over this surface. Moreover, we have

$$|\Theta| \le CK_{\max} \quad \textit{and} \quad \|\nabla\Theta\| \le CK_{\max}, \tag{1.1}$$

with constants C depending only on D.

Proof. Let $N(P)$ denote the outer normal field on S, and let $P_0 \in S$. We can suppose that $N(P_0)$ is the point $(0, 0, 1)$ in a Euclidean coordinates system (x, y, z), where $(x, y) \in T_{P_0}S$. If X is a vector field on S, and if $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^3$, we have that, for all $P \in S$,

$$X\langle N(P), N(P_0)\rangle = \langle\nabla_X N(P), N(P_0)\rangle, \tag{1.2}$$

where ∇ is the standard connection in $\mathbb{R}^3$. By definition of the principal curvatures, we then have

$$\forall P \in S, \quad |X\langle N(P), N(P_0)\rangle| \le K_{\max}\|X\|,$$

where $\|X\|$ denote the norm of X in P with respect to the metric in S induced by the Euclidean metric in $\mathbb{R}^3$. Let $f(P)$ be the function $P \mapsto \langle N(P), N(P_0)\rangle$. Let $P \in S$, and $\alpha(s)$ be a minimising geodesic joining P_0 to P, where s is the arc length on α. Thus, the vector $\alpha'(s)$ is of length 1. The previous estimate shows that

$$\forall s, \quad \left|\frac{\mathrm{d}}{\mathrm{d}s} f\big(\alpha(s)\big)\right| \le K_{\max}\|\alpha'(s)\| = K_{\max}.$$

By integrating this inequality, we find that

$$|f(P_0) - f(P)| = \left| \int_0^{d(P_0,P)} \frac{\mathrm{d}}{\mathrm{d}s} f\big(\alpha(s)\big) \mathrm{d}s \right| \le K_{\max} d(P_0, P) \le K_{\max} D. \tag{1.3}$$

But $f(P_0) = 1$, hence, if $K_{\max} \le \frac{1}{2D}$, we have that for all $P \in S$, $1 \ge f(P) \ge 1/2$. Thus, the normals $N(P)$ take values in a compact set of the hemisphere $\{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 1, z > 0\}$.

Now, if $P \in S$, this shows that $T_P S$ and $T_{P_0} S$ are not orthogonal, and thus in a neighbourhood of P , the surface S is given as a graph over a domain $\omega_P \subset T_{P_0} S$. The collection of domains $\{\omega_P\}_{P \in S}$ then defines the atlas of a surface ω immersed in $T_{P_0} S$.

Now, we prove the estimates (1.1). Let $P \in S$ and $\alpha(s)$ a minimising geodesic joining P_0 and P , where $s \in \big(0, d(P_0, P)\big)$ denotes the arc-length. Thus, we have that $\alpha''(s) = \kappa(s) N\big(\alpha(s)\big)$, where $\kappa(s)$ is the principal curvature of S in the point $\alpha(s)$ in the direction $\alpha'(s)$ (see [12]). Thus, if

$$z(s) = \langle \alpha(s), N(P_0) \rangle$$

denotes the level function on S in the z-direction, we have

$$|z''(s)| = |\langle \alpha''(s), N(P_0) \rangle| \le K_{\max} |\langle N(\alpha(s)), N(P_0) \rangle| \le K_{\max}.$$

Moreover we have $z'(0) = \langle \alpha'(0), N(P_0) \rangle = 0$ because $\alpha'(0)$ belongs to $T_{P_0} S$ and $z(0) = 0$ because P_0 is at the height $z = 0$. By integrating twice with respect to s , using the fact that the length of α is less than D , we find that $|z(s)| \le C K_{\max}$, where C depends only on D . But at the point P , the height function is just the value of Θ at the point in ω corresponding to P . This proves the result.

In order to establish the estimate for $\nabla\Theta$, we proceed as follows: Let P be a point of S and let (x_P, y_P) be the coordinates of P in ω . Let Y be a vector of $\mathbb{R}^2$ and also let $\big(x(t), y(t)\big)$ be a curve in the (x, y) , such that $\big(x(0), y(0)\big) = (x_P, y_P)$ and $\big(x'(0), y'(0)\big) = Y$. We denote by $\beta(t) = \big(x(t), y(t), \Theta(x(t), y(t))\big)$ the corresponding curve on S , hence we have $\beta(0) = P$ and $\beta'(0) = \big(Y, \nabla\Theta(x_P, y_P) \cdot Y\big)$. Thus, we have

$$\nabla\Theta(x_P, y_P) \cdot Y = \langle \beta'(0), N(P_0) \rangle. \tag{1.4}$$

Using the fact that $\beta'(0)$ is orthogonal to $N(P)$, we have

$$|\langle \beta'(0), N(P_0) \rangle| \le \|Y\| \, \langle N(P), N(P_0) \rangle^{-1} \sqrt{1 - \langle N(P), N(P_0) \rangle^2}.$$

Hence, we have

$$\|\nabla\Theta(x_P, y_P)\| \le \langle N(P), N(P_0) \rangle^{-1} \sqrt{1 - \langle N(P), N(P_0) \rangle^2}. \tag{1.5}$$

Recall that $f(P) = \langle N(P), N(P_0) \rangle$. Using the previous notations, and equations (1.2) and (1.4), we find that

$$\begin{aligned} |\beta'(0) f(P)| &= |\langle \nabla_{\beta'(0)} N(P), N(P_0) \rangle| \\ &\le K_{\max} |\langle \beta'(0), N(P_0) \rangle| \\ &\le K_{\max} \|Y\| \, \|\nabla\Theta(x_P, y_P)\| \, . \end{aligned}$$

This equation is valid for all curve β lying on S, with Y the horizontal part of β'. Thus, we can reproduce the proof of estimate (1.3), and we find

$$|f(P) - f(P_0)| \leq K_{\max} D \|\nabla\Theta\|\,,$$

where $\|\nabla\Theta\| = \sup_{x,y\in\omega} \|\nabla\Theta(x,y)\|$. But, under the condition $K_{\max} \leq 1/2D$, we have $f(P) \geq 1/2$. The previous equations show that

$$\forall\, P \in S, \quad \|\nabla\Theta(x_P, y_P)\|^2 \leq 4\big(2K_{\max} D\|\nabla\Theta\| - K_{\max}^2 D^2 \|\nabla\Theta\|^2\big).$$

Hence, we find that

$$\|\nabla\Theta\| \leq 8DK_{\max}.$$

This ends the proof of the theorem. □

Thus, if S is a surface satisfying the condition $K_{\max} \leq Cd$ for d sufficiently small, ($d \leq 1/2CD$), S satisfies the conditions of the theorem. Hence, there exists $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$, the image $\widehat{\Omega}$ of the application

$$(x,y,h) \mapsto \big((x,y,\Theta(x,y)) + h\boldsymbol{n}(x,y,\Theta(x,y))\big) \quad \text{for} \quad (x,y,h) \in \omega \times (-\frac{\rho}{2}, \frac{\rho}{2}),$$

is an embedded open set of $\mathbb{R}^3$.

In the following, we suppose that $\rho_0 > d$. In comparison with the estimate on Θ and $\nabla\Theta$ under the condition $K_{\max} \leq Cd$, we normalise the graph by setting $\Theta = \frac{d}{2}\theta$, where θ is a function on the manifold ω.

Using classical notations (see [5]), we thus consider a shallow shell as an element of the family of sets $\hat{\Omega}^\varepsilon$ indexed by ε, image of the application $\boldsymbol{\Phi}^\varepsilon$ defined by

$$\boldsymbol{\Phi}^\varepsilon : \overline{\Omega}^\varepsilon = \overline{\omega} \times [-\varepsilon, \varepsilon] \ni (x_1, x_2, x_3^\varepsilon) \mapsto \big(x_1, x_2, \varepsilon\,\theta(x_1,x_2)\big) + x_3^\varepsilon \boldsymbol{a}_3^\varepsilon(x_1,x_2), \tag{1.6}$$

where $\boldsymbol{a}_3^\varepsilon(x_1,x_2)$ denotes the unit normal vector to the middle surface. Here, ω is a flat surface with smooth boundary immersed in $\mathbb{R}^2$ and θ a function on ω. The lateral boundary of $\hat{\Omega}^\varepsilon$ is the image of $\partial\omega \times [-\varepsilon, \varepsilon]$ by the application $\boldsymbol{\Phi}^\varepsilon$.

Remark 1.2. Shallow shells are also used to describe a simplification of Koiter's two-dimensional equations: see [19, 20]. In [19, 20], ε is considered as fixed, and no relation are imposed between the curvature of the middle surface and the thickness. The obtained model is more simple in order to study two-dimensional boundary layers appearing in shells.

2. THREE-DIMENSIONAL LINEAR ELASTICITY PROBLEM FOR SHALLOW SHELLS

On the domain $\hat{\Omega}^\varepsilon$, there are two natural coordinate systems: a system $\{\hat{x}_i\}$ of Cartesian coordinates coming from the ambient space $\mathbb{R}^3$, and the system $\{x_*, x_3^\varepsilon\}$ called *normal coordinate system*, coming from the diffeomorphism (1.6).

Associated with the normal coordinate system, we denote by $\boldsymbol{g}_i^\varepsilon(x^\varepsilon) = \frac{\partial\boldsymbol{\Phi}^\varepsilon}{\partial x_i^\varepsilon}(x^\varepsilon)$, $i = 1,2,3$ the *covariant basis* in $\mathbb{R}^3$. The metric tensor in normal coordinates then is written $g_{ij}^\varepsilon(x^\varepsilon) = \boldsymbol{g}_i^\varepsilon(x^\varepsilon) \cdot \boldsymbol{g}_j^\varepsilon(x^\varepsilon)$, where $\cdot$ is the Euclidean scalar product in the ambient space $\mathbb{R}^3$. We also let $g^\varepsilon = \det(g_{ij}^\varepsilon)$.

To the covariant basis $(\boldsymbol{g}_i^\varepsilon(x^\varepsilon))$ we associate *the contravariant basis* $(\boldsymbol{g}^{j,\varepsilon}(x^\varepsilon))$ by the formula $\boldsymbol{g}_i^\varepsilon(x^\varepsilon) \cdot \boldsymbol{g}^{j,\varepsilon}(x^\varepsilon) = \delta_i^j$, where δ_i^j is the Kronecker symbol. The contravariant components of the metric in $\hat{\Omega}^\varepsilon$ are written $g^{ij,\varepsilon} = \boldsymbol{g}^{i,\varepsilon} \cdot \boldsymbol{g}^{j,\varepsilon}$, and the Christoffel symbols associated to this metric then are given by $\Gamma_{ij}^{k,\varepsilon} = \frac{\partial \boldsymbol{g}_j^\varepsilon}{\partial x_i^\varepsilon} \cdot \boldsymbol{g}^{k,\varepsilon}$.

We assume that the shells $\hat{\Omega}^\varepsilon$, for $0 < \varepsilon \leq \varepsilon_0$, are constituted by a homogeneous, isotropic material with Lamé coefficients $\lambda > 0$ and $\mu > 0$. We denote by Γ_0^ε the lateral boundary $\partial\omega \times [-\varepsilon, +\varepsilon]$ of $\overline{\Omega^\varepsilon}$ and by $\hat{\Gamma}_0^\varepsilon = \boldsymbol{\Phi}^\varepsilon(\Gamma_0^\varepsilon)$ the geometrical lateral boundary.

2.1. **The equations of elasticity.**

In the following, Greek indices take their values in $\{1, 2\}$ and Latin indices in $\{1, 2, 3\}$. Moreover, the summation over repeated indices and exponents is used.

The starting point is the linear elasticity problem that we first describe in Cartesian coordinates. Let $\{\hat{x}_i\}$ be a Cartesian coordinate system in $\mathbb{R}^3$. We denote by $\boldsymbol{V}(\hat{\Omega}^\varepsilon)$ the variational space depending on the boundary conditions:

$$\boldsymbol{V}(\hat{\Omega}^\varepsilon) = \{\hat{\boldsymbol{v}}^\varepsilon = (\hat{v}_i^\varepsilon) \in \mathbf{H}^1(\hat{\Omega}^\varepsilon) \; ; \; \hat{\boldsymbol{v}}^\varepsilon = 0 \quad \text{on} \quad \hat{\Gamma}_0^\varepsilon\}$$

for clamped shallow shells and

$$\boldsymbol{V}(\hat{\Omega}^\varepsilon) = \mathbf{H}^1(\hat{\Omega}^\varepsilon),$$

in the free case.

The shell is subjected to the action of body forces represented by the vector field $\hat{\boldsymbol{f}}^\varepsilon = (\hat{f}^{i,\varepsilon})$ on $\hat{\Omega}^\varepsilon$.

The considered problem consists in finding $\hat{\boldsymbol{u}}^\varepsilon \in \boldsymbol{V}(\hat{\Omega}^\varepsilon)$ such that

$$\int_{\hat{\Omega}^\varepsilon} \hat{A}^{ijk\ell} \hat{e}_{ij}(\hat{\boldsymbol{u}}^\varepsilon) \hat{e}_{k\ell}(\hat{\boldsymbol{v}}^\varepsilon) \, \mathrm{d}\hat{x}_1 \mathrm{d}\hat{x}_2 \mathrm{d}\hat{x}_3 = \int_{\hat{\Omega}^\varepsilon} \hat{f}^{i,\varepsilon} \hat{v}_i^\varepsilon \, \mathrm{d}\hat{x}_1 \mathrm{d}\hat{x}_2 \mathrm{d}\hat{x}_3, \quad \forall \, \hat{\boldsymbol{v}}^\varepsilon \in \boldsymbol{V}(\hat{\Omega}^\varepsilon), \tag{2.1}$$

where $\hat{A}^{ijk\ell} = \lambda \delta^{ij} \delta^{k\ell} + \mu(\delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk})$ is the rigidity matrix and $\hat{e}_{ij}(\hat{\boldsymbol{u}}) = \frac{1}{2}(\partial_{\hat{x}_i}\hat{u}_j + \partial_{\hat{x}_j}\hat{u}_i)$. This is the classical three-dimensional elasticity problem posed in Cartesian coordinates on the domain $\hat{\Omega}^\varepsilon$ of $\mathbb{R}^3$.

The linear elasticity problem in normal coordinates has as unknown the vector $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ of the coefficients of the displacement of the shallow shell $\hat{\boldsymbol{u}}^\varepsilon$ in the contravariant basis: $\hat{\boldsymbol{u}}^\varepsilon(\boldsymbol{\Phi}^\varepsilon(x^\varepsilon)) = (u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon})(x^\varepsilon)$, $x^\varepsilon \in \Omega^\varepsilon$.

We now make a change of coordinate system in order to set the equations in normal coordinates on the domain Ω^ε. We write $\boldsymbol{f}^\varepsilon = (f^{i,\varepsilon})$ the body forces vector field in normal coordinates. That means we have $\hat{\boldsymbol{f}}^\varepsilon(\boldsymbol{\Phi}^\varepsilon(x^\varepsilon)) = (f^{i,\varepsilon}\boldsymbol{g}_{i,\varepsilon})(x^\varepsilon)$ for $x^\varepsilon \in \Omega^\varepsilon$.

After the change of coordinates, following the notations in [5], let $e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon)$ denote the components of the linearised deformation tensor associated with a displacement $\boldsymbol{v}^\varepsilon = v_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$. We find that

$$e_{i\|j}^\varepsilon(\boldsymbol{v}^\varepsilon) = \frac{1}{2}(\partial_i^\varepsilon v_j^\varepsilon + \partial_j^\varepsilon v_i^\varepsilon) - \Gamma_{ij}^{p,\varepsilon} v_p^\varepsilon, \tag{2.2}$$

where ∂_i^ε is the derivation with respect to the coordinates in the system x^ε. We denote by $A^\varepsilon = (A^{ijk\ell,\varepsilon})$ the rigidity matrix in normal coordinates, and we have

$$A^{ijk\ell,\varepsilon} = \lambda g^{ij,\varepsilon} g^{k\ell,\varepsilon} + \mu(g^{ik,\varepsilon} g^{j\ell,\varepsilon} + g^{i\ell,\varepsilon} g^{jk,\varepsilon}).$$

The variational formulation of the linear elasticity problem in curvilinear coordinates is, see [5]: find $\boldsymbol{u}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon)$ such that

$$\int_{\Omega^\varepsilon} A^{ijk\ell,\varepsilon} e^\varepsilon_{i\|j}(\boldsymbol{u}^\varepsilon) e^\varepsilon_{k\|\ell}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon} = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}, \quad \forall\, \boldsymbol{v}^\varepsilon \in \boldsymbol{V}(\Omega^\varepsilon), \tag{2.3}$$

where $\boldsymbol{V}(\Omega^\varepsilon)$ is the space

$$\boldsymbol{V}(\Omega^\varepsilon) = \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \mathbf{H}^1(\Omega^\varepsilon)\ ;\ \boldsymbol{v}^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon\},$$

in the case of clamped shallow shells and

$$\boldsymbol{V}(\Omega^\varepsilon) = \mathbf{H}^1(\Omega^\varepsilon),$$

in the free case.

2.2. **Scaling and hypothesis on the data.** To every point $x^\varepsilon = (x_*, x_3^\varepsilon) \in \Omega^\varepsilon$ we associate a point in the fixed open set $\Omega = \omega \times (-1,1)$ by the scaling

$$\Omega^\varepsilon \ni x^\varepsilon \mapsto (x_*, x_3 = \varepsilon^{-1} x_3^\varepsilon) =: x \in \Omega. \tag{2.4}$$

The corresponding scaling on the unknown $\boldsymbol{u}^\varepsilon$ yields a new unknown $\boldsymbol{u}(\varepsilon)$ given by

$$u_\alpha^\varepsilon(x^\varepsilon) = u_\alpha(\varepsilon)(x),\ \alpha = 1,2, \quad \text{and} \quad u_3^\varepsilon(x^\varepsilon) = \varepsilon^{-1} u_3(\varepsilon)(x). \tag{2.5}$$

Moreover, we make the following assumption on the forces: we suppose that there exists a vector field $\boldsymbol{f} = (f^\alpha, f^3)$ on Ω that is independent of ε such that

$$f^{\alpha,\varepsilon}(x^\varepsilon) = f^\alpha(x) \quad \text{and} \quad f^{3,\varepsilon}(x^\varepsilon) = \varepsilon f^3(x), \tag{2.6}$$

where x and x^ε are related by the scaling (2.4). We assume that $\boldsymbol{f} = (f^i) \in \mathscr{C}^\infty(\overline{\Omega})^3$ in order to get an asymptotic of arbitrary order.

Now we can write the problem (2.3) on the fixed domain Ω. The test function spaces become $\boldsymbol{V}(\Omega) = \{\boldsymbol{v} = (v_i) \in \mathbf{H}^1(\Omega)\ ;\ \boldsymbol{v} = 0 \text{ on } \Gamma_0\}$ for clamped shells, where $\Gamma_0 = \partial\omega \times (-1,1)$, and $\boldsymbol{V}(\Omega) = \mathbf{H}^1(\Omega)$ in the free case.

We also define the scaled geometrical data of the shell on Ω, i.e. we define $\boldsymbol{g}_i(\varepsilon)$, $\boldsymbol{g}^i(\varepsilon)$, $g_{ij}(\varepsilon)$, $g^{ij}(\varepsilon)$, $g(\varepsilon)$ and $\Gamma^k_{ij}(\varepsilon)$ such that

$$\begin{array}{lcllcl}
\boldsymbol{g}_i(\varepsilon)(x) & = & \boldsymbol{g}_i^\varepsilon(x^\varepsilon), & \boldsymbol{g}^i(\varepsilon)(x) & = & \boldsymbol{g}^{i,\varepsilon}(x^\varepsilon),\\
g_{ij}(\varepsilon)(x) & = & g_{ij}^\varepsilon(x^\varepsilon), & g^{ij}(\varepsilon)(x) & = & g^{ij,\varepsilon}(x^\varepsilon),\\
g(\varepsilon)(x) & = & g^\varepsilon(x^\varepsilon), & \Gamma^k_{ij}(\varepsilon)(x) & = & \Gamma^{k,\varepsilon}_{ij}(x^\varepsilon).
\end{array}$$

We obtain an equivalent problem, see [4], which consists in finding $\boldsymbol{u}(\varepsilon) \in \boldsymbol{V}(\Omega)$ such that

$$\int_\Omega A^{ijk\ell}(\varepsilon) e_{i\|j}(\varepsilon; \boldsymbol{u}(\varepsilon)) e_{k\|\ell}(\varepsilon; \boldsymbol{v})\sqrt{g(\varepsilon)} = \int_\Omega f^i v_i \sqrt{g(\varepsilon)}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}(\Omega), \tag{2.7}$$

where we set,

$$\begin{aligned} e_{\alpha\|\beta}(\varepsilon;\boldsymbol{v}) &= e_{\alpha\beta}(\boldsymbol{v}) - \Gamma^{\sigma}_{\alpha\beta}(\varepsilon)v_\sigma - \varepsilon^{-1}\Gamma^{3}_{\alpha\beta}(\varepsilon)v_3,\\ e_{\alpha\|3}(\varepsilon;\boldsymbol{v}) &= \varepsilon^{-1}e_{\alpha3}(\boldsymbol{v}) - \Gamma^{\sigma}_{\alpha3}(\varepsilon)v_\sigma,\\ e_{3\|3}(\varepsilon;\boldsymbol{v}) &= \varepsilon^{-2}e_{33}(\boldsymbol{v}), \end{aligned} \tag{2.8}$$

with $e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$, where ∂_i stands for the derivative with respect to x_i. Moreover, we have

$$A^{ijk\ell}(\varepsilon) = \lambda g^{ij}(\varepsilon)g^{k\ell}(\varepsilon) + \mu\big(g^{ik}(\varepsilon)g^{j\ell}(\varepsilon) + g^{i\ell}(\varepsilon)g^{jk}(\varepsilon)\big). \tag{2.9}$$

2.3. **Three-dimensional compatibility condition.** We denote by $\mathcal{R}(\varepsilon,\Omega)$ the space of *rigid motions* $\mathcal{R}(\varepsilon,\Omega) = \big\{\boldsymbol{v}\in\boldsymbol{V}(\Omega)\,;\, e_{i\|j}(\varepsilon,\boldsymbol{v}) = 0 \quad i,j\in\{1,2,3\}\big\}$.

For clamped shallow shells, $\mathcal{R}(\varepsilon,\Omega) = \{0\}$ and in the free case, $\mathcal{R}(\varepsilon,\Omega)$ is a six-dimensional space spanned by

$$\begin{pmatrix}1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\end{pmatrix},\ \begin{pmatrix}-\hat{x}_3\\0\\\hat{x}_1\end{pmatrix},\ \begin{pmatrix}0\\-\hat{x}_3\\\hat{x}_2\end{pmatrix},\ \begin{pmatrix}\hat{x}_2\\-\hat{x}_1\\0\end{pmatrix}, \tag{2.10}$$

in Cartesian coordinates.

Thanks to Korn's inequality in curvilinear coordinates (see [13, 5]), if the right-hand side in (2.7) verifies the *compatibility condition*

$$\forall \boldsymbol{v}\in\mathcal{R}(\varepsilon,\Omega),\qquad \int_\Omega f^i v_i\sqrt{g(\varepsilon)} = 0, \tag{2.11}$$

then there exists an unique solution $\boldsymbol{u}(\varepsilon)$ of (2.7) such that

$$\forall \boldsymbol{v}\in\mathcal{R}(\varepsilon,\Omega),\qquad \int_\Omega g^{ij}(\varepsilon)u_i(\varepsilon)v_j\sqrt{g(\varepsilon)} = 0. \tag{2.12}$$

2.4. **Local coordinates near the boundary.** We introduce in-plane coordinates (r,s) in a neighbourhood of the boundary $\partial\omega$. Let $\boldsymbol{n}$ be the inner unit normal to $\partial\omega$ and let $\boldsymbol{\tau}$ be the tangent unit vector such that the basis $(\boldsymbol{n},\boldsymbol{\tau})$ is direct at each point of $\partial\omega$. This definition, usual if ω is a domain of $\mathbb{R}^2$, also makes sense for an immersed surface in $\mathbb{R}^2$. Let s be the arc-length along $\partial\omega$ oriented according to $\boldsymbol{\tau}$ and $\mathbb{S}$ be the set of the values of s for which we can associate a point in $\partial\omega$.

For a point x_* in the neighbourhood of $\partial\omega$, let $r = r(x_*)$ be its signed distance to $\partial\omega$ oriented along $\boldsymbol{n}$, i.e. r is this distance if $x_*\in\omega$, and minus this distance if $x_*\notin\omega$. Then, if $\mid r\mid$ is small enough, there exists a unique point $x^0_*\in\partial\omega$ such that $\mid r \mid= d(x_*,x^0_*)$ and we define $s = s(x_*)$ as the curvilinear abscissa of x^0_*. Thus, we have a tubular neighbourhood of $\partial\omega$ diffeomorphic to $(-r^0,r^0)\times\mathbb{S}$ via the change of variables $x_*\to(r,s)$. We extend the vectors $\boldsymbol{n}$ and $\boldsymbol{\tau}$ from $\mathbb{S}$ to $(-r^0,r^0)\times\mathbb{S}$ by letting

$$\forall r\in(-r^0,r^0),\ \forall s\in\mathbb{S},\ \ \boldsymbol{n}(r,s) = \boldsymbol{n}(s)\quad\text{and}\quad \boldsymbol{\tau}(r,s) = \boldsymbol{\tau}(s).$$

The following relations are satisfied:

$$\begin{aligned} \partial_r\boldsymbol{n} &= 0 &&\text{and}\quad \partial_r\boldsymbol{\tau} = 0,\\ \partial_s\boldsymbol{n} &= -\kappa\boldsymbol{\tau} &&\text{and}\quad \partial_s\boldsymbol{\tau} = \kappa\boldsymbol{n}, \end{aligned}$$

where κ is the curvature of $\partial\omega$ with respect to s. If $R = R(s)$ denotes the curvature radius of $\partial\omega$ at s from inside ω, then $\kappa = 1/R$. In Cartesian coordinates, we have

$$\boldsymbol{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\tau} = \begin{pmatrix} n_2 \\ -n_1 \end{pmatrix}.$$

Thus we obtain, (obviously $\partial_n = \partial_r$):

$$\partial_r = n_1\partial_1 + n_2\partial_2 \quad \text{and} \quad \partial_s = (1-\kappa r)(n_2\partial_1 - n_1\partial_2).$$

In the tubular neighbourhood $(-r^0, r^0) \times \mathbb{S}$, we introduce the in-plane normal and tangential components of $\boldsymbol{u}(\varepsilon)$ defined by

$$u_n(\varepsilon) = n_1 u_1(\varepsilon) + n_2 u_2(\varepsilon) \quad \text{and} \quad u_s(\varepsilon) = (1-\kappa r)(n_2 u_1(\varepsilon) - n_1 u_2(\varepsilon)). \tag{2.13}$$

2.5. **The boundary value problem.** In the following, Γ_+ and Γ_- denote the upper and lower faces of $\Omega = \omega \times (-1, +1)$. After integration by parts, the problem (2.7) can be written as the boundary value problem

$$\begin{cases} \boldsymbol{L}(\varepsilon)\boldsymbol{u}(\varepsilon) & = & -\boldsymbol{f} & \text{in} & \Omega, \\ \boldsymbol{G}(\varepsilon)\boldsymbol{u}(\varepsilon) & = & 0 & \text{on} & \Gamma_+ \cup \Gamma_-, \\ \boldsymbol{u}(\varepsilon) & = & 0 & \text{on} & \Gamma_0 \quad \text{(clamped shallow shells)}, \\ \boldsymbol{T}(\varepsilon)\boldsymbol{u}(\varepsilon) & = & 0 & \text{on} & \Gamma_0 \quad \text{(free shallow shells)}, \end{cases} \tag{2.14}$$

where $\boldsymbol{L}(\varepsilon)$ is the interior operator on Ω whose components are, for $\boldsymbol{u}$ on Ω,

$$\begin{aligned} L_\alpha(\varepsilon)\boldsymbol{u} &= \partial_\beta\big(A^{\alpha\beta\sigma\tau} e_{\sigma\|\tau}(\varepsilon;\boldsymbol{u}) + A^{\alpha\beta33}(\varepsilon) e_{3\|3}(\varepsilon;\boldsymbol{u})\big) \\ &\quad + \big(A^{\alpha\beta\sigma\tau}(\varepsilon) e_{\sigma\|\tau}(\varepsilon;\boldsymbol{u}) + A^{\alpha\beta33}(\varepsilon) e_{3\|3}(\varepsilon;\boldsymbol{u})\big)\Gamma^\tau_{\beta\tau}(\varepsilon) \\ &\quad + \big(A^{\delta\beta\sigma\tau}(\varepsilon) e_{\sigma\|\tau}(\varepsilon;\boldsymbol{u}) + A^{\delta\beta33}(\varepsilon) e_{3\|3}(\varepsilon;\boldsymbol{u})\big)\Gamma^\alpha_{\delta\beta}(\varepsilon) \\ &\quad + 4A^{\delta3\sigma3}(\varepsilon) e_{\delta\|3}(\varepsilon;\boldsymbol{u})\Gamma^\alpha_{\sigma3}(\varepsilon) + 2A^{\alpha3\delta3}(\varepsilon) e_{\delta\|3}(\varepsilon;\boldsymbol{u})\Gamma^\sigma_{\sigma3}(\varepsilon) \\ &\quad + \varepsilon^{-1}2\partial_3\big(A^{\delta3\alpha3}(\varepsilon) e_{\delta\|3}(\varepsilon;\boldsymbol{u})\big), \\ L_3(\varepsilon)\boldsymbol{u} &= \varepsilon^{-1}\big(A^{\alpha\beta\sigma\tau}(\varepsilon) e_{\sigma\|\tau}(\varepsilon;\boldsymbol{u}) + A^{\alpha\beta33}(\varepsilon) e_{3\|3}(\varepsilon;\boldsymbol{u})\big)\Gamma^3_{\alpha\beta}(\varepsilon) \\ &\quad + \varepsilon^{-1}2\partial_\sigma\big(A^{\alpha3\sigma3}(\varepsilon) e_{\alpha\|3}(\varepsilon;\boldsymbol{u})\big) + \varepsilon^{-1}2A^{\alpha3\sigma3}(\varepsilon) e_{\alpha\|3}(\varepsilon;\boldsymbol{u})\Gamma^\tau_{\sigma\tau}(\varepsilon) \\ &\quad + \varepsilon^{-2}\partial_3\big(A^{\alpha\beta33}(\varepsilon) e_{\alpha\|\beta}(\varepsilon;\boldsymbol{u}) + A^{3333}(\varepsilon) e_{3\|3}(\varepsilon;\boldsymbol{u})\big) \\ &\quad + \varepsilon^{-1}\big(A^{\alpha\beta33}(\varepsilon) e_{\alpha\|\beta}(\varepsilon;\boldsymbol{u}) + A^{3333}(\varepsilon) e_{3\|3}(\varepsilon;\boldsymbol{u})\big)\Gamma^\sigma_{\sigma3}(\varepsilon). \end{aligned} \tag{2.15}$$

The operator $\boldsymbol{G}(\varepsilon)$ is the traction operator on the lower and upper faces of the shallow shell, and we have

$$\begin{aligned} G_\alpha(\varepsilon)\boldsymbol{u} &= \varepsilon^{-1}2A^{\beta3\alpha3}(\varepsilon) e_{\beta\|3}(\varepsilon;\boldsymbol{u}), \\ G_3(\varepsilon)\boldsymbol{u} &= \varepsilon^{-2}\big(A^{\alpha\beta33}(\varepsilon) e_{\alpha\|\beta}(\varepsilon;\boldsymbol{u}) + A^{3333}(\varepsilon) e_{3\|3}(\varepsilon;\boldsymbol{u})\big). \end{aligned} \tag{2.16}$$

Finally, $\boldsymbol{T}(\varepsilon)$ is the traction operator on the lateral boundary Γ_0 on the shell, and we have, with n_α the component of the normal $\boldsymbol{n}$ along the boundary,

$$\begin{aligned} T_\alpha(\varepsilon)\boldsymbol{u} &= \big(A^{\alpha\beta\sigma\tau}(\varepsilon) e_{\sigma\|\tau}(\varepsilon;\boldsymbol{u}) + A^{\alpha\beta33}(\varepsilon) e_{3\|3}(\varepsilon;\boldsymbol{u})\big) n_\beta, \\ T_3(\varepsilon)\boldsymbol{u} &= \varepsilon^{-1}2A^{\alpha3\sigma3}(\varepsilon) e_{\sigma\|3}(\varepsilon;\boldsymbol{u}) n_\alpha. \end{aligned} \tag{2.17}$$

In the coordinate system (r,s,x_3) near the lateral boundary Γ_0, the traction operator on Γ_0 writes $(T_r(\varepsilon),T_s(\varepsilon),T_3(\varepsilon))$, where

$$T_r(\varepsilon)=n_1T_1(\varepsilon)+n_2T_2(\varepsilon) \quad \text{and} \quad T_s(\varepsilon)=(1-\kappa r)^{-1}\left(n_2T_1(\varepsilon)-n_1T_2(\varepsilon)\right).$$

3. Description of results

In this section, we give and describe the main result of this paper. The structure of the different terms of the asymptotic is also given.

As in the case of plates, the asymptotics of $\boldsymbol{u}(\varepsilon)$ contains three kinds of terms (for $k\geq 0$):

$\boldsymbol{u}^k_{\mathrm{KL}}(x_*,x_3)$: Kirchhoff-Love displacements depending on two-dimensional $\mathscr{C}^\infty$ generators $\boldsymbol{\zeta}^k(x_*)=\left(\zeta^k_*(x_*),\zeta^k_3(x_*)\right)$ such that

$$\boldsymbol{u}^k_{\mathrm{KL}}(x)=\left(\zeta^k_*(x_*)-x_3\nabla_*\zeta^k_3(x_*),\zeta^k_3(x_*)\right),$$

$\boldsymbol{v}^k(x_*,x_3)$: three-dimensional $\mathcal{C}^\infty$ displacements with zero mean value, i.e.

$$\forall\, x_*\in\omega, \qquad \int_{-1}^{1}\boldsymbol{v}^k(x_*,x_3)\,\mathrm{d}x_3=0.$$

$\boldsymbol{w}^k(t,s,x_3)$: boundary layer term, uniformly exponentially decreasing as $t\to\infty$, concentrating the singularities due to the edge of the shallow shell.

The main result is the following:

Theorem 3.1. *Let* $\boldsymbol{u}(\varepsilon)$ *be the unique solution of the problem* (2.7) *satisfying the orthogonality condition* (2.12). *Then, for all* $k\geq 0$, *there exist Kirchhoff-Love fields* $\boldsymbol{u}^k_{\mathrm{KL}}$, *displacements* $\boldsymbol{v}^k$ *with zero mean value and boundary layer terms* $\boldsymbol{w}^k$, *such that if* $\chi(r)$ *is a cut-off function equal to* 1 *in a neighbourhood of* $\partial\omega$, *and if we define for all* $k\geq 0$ *the displacement*

$$\boldsymbol{u}^k(x,\varepsilon^{-1}r)=\boldsymbol{u}^k_{\mathrm{KL}}(x)+\boldsymbol{v}^k(x)+\chi(r)\boldsymbol{w}^k(\varepsilon^{-1}r,s,x_3), \tag{3.1}$$

then for all $k\geq 0$, $\boldsymbol{u}^k\in\boldsymbol{V}(\Omega)$ *and moreover, we have for all* $N\geq 0$

$$\|\boldsymbol{u}(\varepsilon)(x)-\sum_{k=0}^{N}\varepsilon^k\boldsymbol{u}^k(x,\varepsilon^{-1}r)\|_{\mathbf{H}^1(\Omega)}\leq C\varepsilon^{N+1/2}, \tag{3.2}$$

where C *is some constant. Moreover,* $\boldsymbol{v}^0=0$, $\boldsymbol{w}^0=0$, $\boldsymbol{v}^1=0$ *and* $w^1_3=0$. *In particular,*

$$\boldsymbol{u}^0=\boldsymbol{u}^0_{\mathrm{KL}} \quad \textit{and} \quad \boldsymbol{u}^1=\boldsymbol{u}^1_{\mathrm{KL}}(x)+\chi(r)\boldsymbol{w}^1(\varepsilon^{-1}r,s,x_3).$$

We deduce from this theorem that

$$\|\boldsymbol{u}(\varepsilon)(x)-\boldsymbol{u}^0_{\mathrm{KL}}(x)\|_{\mathbf{H}^1(\Omega)}\leq C\varepsilon^{1/2}. \tag{3.3}$$

3.1. **The Kirchhoff-Love generators.** The generators $\boldsymbol{\zeta}^k = (\zeta_*^k, \zeta_3^k)$ of the above Kirchhoff-Love fields are solutions of a two-dimensional problem on ω with boundary conditions on $\partial\omega$.

Let $\tilde{\lambda}$ be the homogenized Lamé coefficient

$$\tilde{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu}, \tag{3.4}$$

and let $b^{\alpha\beta\sigma\tau}$ denote the contravariant components of the two-dimensional elasticity tensor,

$$b^{\alpha\beta\sigma\tau} = \tilde{\lambda}\delta^{\alpha\beta}\delta^{\sigma\tau} + \mu(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\sigma}). \tag{3.5}$$

We also let

$$\tilde{e}_{\alpha\beta}(\boldsymbol{\zeta}) = e_{\alpha\beta}(\boldsymbol{\zeta}) - (\partial_{\alpha\beta}\theta)\zeta_3. \tag{3.6}$$

The two-dimensional problem is governed by a bilinear mapping on $\boldsymbol{V}(\omega) \times \boldsymbol{V}(\omega)$, where $\boldsymbol{V}(\omega) = \mathrm{H}_0^1(\omega) \times \mathrm{H}_0^1(\omega) \times \mathrm{H}_0^2(\omega)$ in the clamped case, and $\boldsymbol{V}(\omega) = \mathrm{H}^1(\omega) \times \mathrm{H}^1(\omega) \times \mathrm{H}^2(\omega)$ in the free case:

$$a(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \int_\omega b^{\alpha\beta\sigma\tau}\left(\tilde{e}_{\alpha\beta}(\boldsymbol{\zeta})\tilde{e}_{\sigma\tau}(\boldsymbol{\eta}) + \frac{1}{3}(\partial_{\alpha\beta}\zeta_3)(\partial_{\sigma\tau}\eta_3)\right) \mathrm{d}\omega. \tag{3.7}$$

In the case of plates ($\theta = 0$) this bilinear form is just the sum of a membrane bilinear form acting on ζ_* and a bending bilinear form acting on ζ_3. In the general case, both operators are coupled by lower order terms.

In the coordinates system (r, s, x_3), the Dirichlet operator is given by:

$$\boldsymbol{\zeta} \mapsto (\zeta_n, \zeta_s, \zeta_3, \partial_n\zeta_3)\big|_{\partial\omega},$$

while the Neumann operator is:

$$\boldsymbol{\zeta} \mapsto \big(B_s(\boldsymbol{\zeta}), B_n(\boldsymbol{\zeta}), M_n(\zeta_3), N_n(\zeta_3)\big)\big|_{\partial\omega},$$

where

$$\begin{aligned} B_n(\boldsymbol{\zeta}) &= \tilde{\lambda}\mathrm{div}_*\zeta_* + 2\mu\partial_n\zeta_n - \zeta_3\big((\tilde{\lambda} + 2\mu)\partial_{nn}\theta + \tilde{\lambda}(\partial_{ss}\theta - \kappa\partial_n\theta)\big), \\ B_s(\boldsymbol{\zeta}) &= \mu\left(\partial_n\zeta_s + \partial_s\zeta_n + 2\kappa\zeta_s - 2\zeta_3(\partial_{ns}\theta + \kappa\partial_s\theta)\right), \\ M_n(\zeta_3) &= \frac{1}{3}\left(\tilde{\lambda}\Delta_*\zeta_3 + 2\mu\partial_{nn}\zeta_3\right), \\ N_n(\zeta_3) &= -\frac{1}{3}\left((\tilde{\lambda} + 2\mu)\partial_n(\Delta_*\zeta_3) + 2\mu\partial_s(\partial_r + \kappa)\partial_s\zeta_3\right), \end{aligned} \tag{3.8}$$

where $\mathrm{div}_*\boldsymbol{\zeta}_* = \partial_1\zeta_1 + \partial_2\zeta_2$ and $\Delta_* = \partial_{11} + \partial_{22}$ with respect to (x_1, x_2).

Let $\boldsymbol{P} = (P_i)$ be the two-dimensional operator associated with the bilinear form $a(\cdot,\cdot)$. By integration by parts, we obtain that:

$$\begin{aligned} P_\sigma(\boldsymbol{\zeta}) &= -\tilde{\lambda}\partial_\sigma\tilde{e}_{\alpha\alpha}(\boldsymbol{\zeta}) - 2\mu\partial_\alpha\tilde{e}_{\alpha\sigma}(\boldsymbol{\zeta}), \\ P_3(\boldsymbol{\zeta}) &= \frac{1}{3}(\tilde{\lambda} + 2\mu)\Delta^2\zeta_3 - \tilde{\lambda}(\Delta\theta)\tilde{e}_{\alpha\alpha}(\boldsymbol{\zeta}) - 2\mu(\partial_{\alpha\beta}\theta)\tilde{e}_{\alpha\beta}(\boldsymbol{\zeta}). \end{aligned} \tag{3.9}$$

Note that the degrees of the operator $\boldsymbol{P}$ can be represented as

$$\deg\boldsymbol{P} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix},$$

and moreover the terms depending on θ are of order at most one. Thus the principal symbol of the operator is the same as for plates. The same remark holds for the Neuman operator, and thus we have the following proposition, which gives the regularity properties of the operator $\boldsymbol{P}$:

Proposition 3.2. *The operator $\boldsymbol{P}$ is a self adjoint operator, strongly elliptic in the sense of Agmon, Douglis and Nirenberg (see* [1]*), with indices of equations $t_1 = t_2 = 1$, $t_3 = 2$ and indices of unknown $s_1 = s_2 = 1$ and $s_3 = 2$. Moreover, the Dirichlet and Neumann boundary conditions satisfy the complementing boundary condition.*

Let $\mathcal{K}(\omega) = \{\boldsymbol{\zeta} \in \boldsymbol{V}(\omega) \ ; \ a(\boldsymbol{\zeta}, \boldsymbol{\zeta}) = 0\}$ denote the space of two-dimensional rigid displacements. We then have:

$$\mathcal{K}(\omega) = \{\boldsymbol{\zeta} \in \boldsymbol{V}(\omega) \ ; \ \tilde{e}_{\alpha\beta}(\boldsymbol{\zeta}) = 0 \quad \text{and} \quad \partial_{\alpha\beta}\zeta_3 = 0\}.$$

It easily seen that $\mathcal{K}(\omega) = 0$ in the case of boundary conditions of clamping and that, in the free case, $\mathcal{K}(\omega)$ is six-dimensional and spanned by the terms

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \partial_1\theta \\ \partial_2\theta \\ 1 \end{pmatrix}, \begin{pmatrix} x_2\partial_1\theta \\ x_2\partial_2\theta - \theta \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1\partial_1\theta - \theta \\ x_1\partial_2\theta \\ x_1 \end{pmatrix}. \tag{3.10}$$

According to the boundary conditions considered, the Kirchhoff-Love generators $\boldsymbol{\zeta}^k$ are solutions to two kinds of two-dimensional boundary value problems: In the case of clamped shallow shells, the generators satisfy, for all $k \geq 0$, equations of the type:

$$\left\{ \begin{array}{lcll} \boldsymbol{P}(\boldsymbol{\zeta}^k) & = & \boldsymbol{r}^k & \text{in} \quad \omega, \\ (\zeta_r^k, \zeta_s^k, \zeta_3^k, \partial_n\zeta_3^k)\,\big|_{\partial\omega} & = & \boldsymbol{h}^k & \text{on} \quad \partial\omega, \end{array} \right. \tag{3.11}$$

with regular right-hand sides $\boldsymbol{r}^k \in (\mathscr{C}^\infty(\omega))^3$ and $\boldsymbol{h}^k = (h_r^k, h_s^k, h_3^k, h_n^k) \in (\mathscr{C}^\infty(\partial\omega))^4$.

In the case of free shallow shells, the two-dimensional boundary value problems are, for all $k \geq 0$, of the type:

$$\left\{ \begin{array}{lcll} \boldsymbol{P}(\boldsymbol{\zeta}^k) & = & \boldsymbol{r}^k & \text{in} \quad \omega, \\ \big(B_n(\boldsymbol{\zeta}^k), B_s(\boldsymbol{\zeta}^k), N_n(\zeta_3^k), M_n(\zeta_3^k)\big)\,\big|_{\partial\omega} & = & \boldsymbol{g}^k & \text{on} \quad \partial\omega, \end{array} \right. \tag{3.12}$$

with $\boldsymbol{g}^k = (g_n^k, g_s^k, g_3^k, g_m^k) \in (\mathscr{C}^\infty(\partial\omega))^4$.

Let us define on $\partial\omega$:

$$g_1^k = n_1 g_n^k + n_2 g_s^k \quad \text{and} \quad g_2^k = n_2 g_n^k - n_1 g_s^k, \quad \text{for each } k \geq 0. \tag{3.13}$$

In order to have a solution, the boundary value problem (3.12) must satisfy the following compatibility condition:

$$\int_\omega r_i^k \eta_i \,\mathrm{d}\omega + \int_\gamma g_i^k \eta_i + \int_\gamma g_m^k \partial_n \eta_3 = 0, \quad \forall \boldsymbol{\eta} \in \mathcal{K}(\omega) \tag{3.14}$$

and in this case, the unique solution $\boldsymbol{\zeta}^k$ satisfies:

$$\int_\omega \zeta_i^k \eta_i = 0, \quad \forall \boldsymbol{\eta} \in \mathcal{K}(\omega). \tag{3.15}$$

3.2. **The first Kirchhoff-Love generators $\boldsymbol{\zeta}^0$.** Recall that the first term of the expansion is a Kirchhoff-Love field $u^0_{\mathrm{KL}} = (\zeta^0_* - x_3\nabla_*\zeta^0_3, \zeta^0_3)$. We give here the equations satisfied by $\boldsymbol{\zeta}^0$ in both cases of boundary conditions.

Let $p^i(x_*)$ and $q^i(x_*)$ be the following functions on ω, constructed from the three-dimensional vector field $\boldsymbol{f} = (f^i)$ (see (2.6)):

$$p^i(x_*) = \int_{-1}^{1} f^i(x_*, x_3)\,\mathrm{d}x_3 \quad \text{and} \quad q^i(x_*) = \int_{-1}^{1} x_3 f^i(x_*, x_3)\,\mathrm{d}x_3. \tag{3.16}$$

In the case of clamped shallow shells, the generator $\boldsymbol{\zeta}^0$ is the unique solution to the two-dimensional problem:

$$\begin{cases} \boldsymbol{P}(\boldsymbol{\zeta}^0) & = & \frac{1}{2}(p^1, p^2, \partial_\alpha q^\alpha + p^3) & \text{in} & \omega, \\ (\zeta^0_r, \zeta^0_s, \zeta^0_3, \partial_n\zeta^0_3)\big|_{\partial\omega} & = & (0,0,0,0) & \text{on} & \partial\omega. \end{cases} \tag{3.17}$$

S. Busse, P. G. Ciarlet & B. Miara found in [4] the same problem for $\boldsymbol{\zeta}^0$. With (3.3), we improve their result by giving an estimate for the convergence.

In the free case, $\boldsymbol{\zeta}^0$ is the unique solution satisfying condition (3.15) to the Neumann problem:

$$\begin{cases} \boldsymbol{P}(\boldsymbol{\zeta}^0) & = & \dfrac{1}{2}(p^1, p^2, \partial_\alpha q^\alpha + p^3) & \text{in} & \omega, \\ \big(B_n(\boldsymbol{\zeta}^0), B_s(\boldsymbol{\zeta}^0), N_n(\zeta^0_3), M_n(\zeta^0_3)\big)\big|_{\partial\omega} & = & (0, 0, -\frac{1}{2}n_\alpha q^\alpha, 0) & \text{on} & \partial\omega. \end{cases} \tag{3.18}$$

4. OUTER EXPANSION

In this section, we study the solution in *formal series* of the three-dimensional elasticity equations without boundary conditions on the lateral boundary. Thus, we search for a formal series in powers of ε:

$$\underline{\boldsymbol{u}}(\varepsilon)(x) = \underline{\boldsymbol{u}}^0(x) + \varepsilon\underline{\boldsymbol{u}}^1(x) + \varepsilon^2\underline{\boldsymbol{u}}^2(x) + \cdots, \tag{4.1}$$

(with coefficients $\underline{\boldsymbol{u}}^k(x)$ displacement fields on Ω), solution of the two first equations in (2.14) in the sense of formal series.

To this aim, we first expand the operators $\boldsymbol{L}(\varepsilon)$ and $\boldsymbol{G}(\varepsilon)$ with respect to ε, and define the formal series problem. We will see that this problem can be solved and that the coefficients $\underline{\boldsymbol{u}}^k(x)$ are determined by generators $\boldsymbol{\zeta}^k$ solutions of problems of type

$$\boldsymbol{P}(\boldsymbol{\zeta}^k) = \boldsymbol{r}^k \quad \text{in } \omega.$$

Hence, we do not have the uniqueness of such an expansion, because no traces on $\partial\omega$ are imposed on the $\boldsymbol{\zeta}^k$ terms. We will show in the next section that the generators $\boldsymbol{\zeta}^k$ are fully determined after the introduction of boundary layer terms. The analysis of this section is similar to that in ([14]) for shells.

4.1. Asymptotic expansion of the elasticity operator.

Definition 4.1. We say that a function $f(\varepsilon)$ depending on ε is $\mathcal{O}(\varepsilon^k)$ if $f(\varepsilon)/\varepsilon^k$ is bounded when ε approaches zero. If for every $N \in \mathbb{N}$, we have $f(\varepsilon) - \sum_{k\geq 0}^{N} \varepsilon^k f_k = \mathcal{O}(\varepsilon^{N+1})$, we write

$$f(\varepsilon) \sim \sum_{k\geq 0} \varepsilon^k f_k$$

and we can write $f(\varepsilon) = \sum_{k\geq 0} \varepsilon^k f_k$ in the sense of asymptotic expansions.

The particular form of the middle surface of the shell yields asymptotic expansions of the geometrical data such as $\Gamma_{ij}^k(\varepsilon)$, $g^{ij}(\varepsilon)$ (and hence $A^{ijk\ell}(\varepsilon)$).

In the following, we denote $s^\theta = (\partial_1\theta)^2 + (\partial_2\theta)^2$.

Proposition 4.2. *The geometrical data admit the following expansions:*

$$\boldsymbol{g}_\alpha(\varepsilon) \sim \sum_{j\geq 0} \varepsilon^j \boldsymbol{g}_{\alpha;j}, \quad \boldsymbol{g}^\alpha(\varepsilon) \sim \sum_{j\geq 0} \varepsilon^j \boldsymbol{g}^{\alpha;j} \quad \text{and} \quad \boldsymbol{g}_3(\varepsilon) = \boldsymbol{g}^3(\varepsilon) \sim \sum_{j\geq 0} \varepsilon^j \boldsymbol{g}_{3;j}, \tag{4.2}$$

with

$$\boldsymbol{g}_{\alpha;0} = \begin{pmatrix} \delta_{\alpha 1} \\ \delta_{\alpha 2} \\ 0 \end{pmatrix}, \quad \boldsymbol{g}_{\alpha;1} = \begin{pmatrix} 0 \\ 0 \\ \partial_\alpha\theta \end{pmatrix}, \quad \boldsymbol{g}_{\alpha;2} = \begin{pmatrix} -x_3\partial_{\alpha 1}\theta \\ -x_3\partial_{\alpha 2}\theta \\ 0 \end{pmatrix},$$

$$\boldsymbol{g}^{\alpha;0} = \begin{pmatrix} \delta^{\alpha 1} \\ \delta^{\alpha 2} \\ 0 \end{pmatrix}, \quad \boldsymbol{g}^{\alpha;1} = \begin{pmatrix} 0 \\ 0 \\ \partial_\alpha\theta \end{pmatrix}, \quad \boldsymbol{g}^{\alpha;2} = \begin{pmatrix} x_3\partial_{\alpha 1}\theta - (\partial_\alpha\theta)\partial_1\theta \\ x_3\partial_{\alpha 2}\theta - (\partial_\alpha\theta)\partial_2\theta \\ 0 \end{pmatrix},$$

and moreover, the first two components of $\boldsymbol{g}_{\alpha;2j+1}$ *and* $\boldsymbol{g}^{\alpha;2j+1}$ *and the last component of* $\boldsymbol{g}_{\alpha;2j}$ *and* $\boldsymbol{g}^{\alpha;2j}$ *are zero for all* j. *For the third vector, we have*

$$\boldsymbol{g}_{3;0} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{g}_{3;1} = \begin{pmatrix} -\partial_1\theta \\ -\partial_2\theta \\ 0 \end{pmatrix}, \quad \boldsymbol{g}_{3;2} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2}s^\theta \end{pmatrix},$$

and the first two components of $\boldsymbol{g}_{3;2j}$ *and the last component of* $\boldsymbol{g}_{3;2j+1}$ *are zero for all* j. *For the metric tensors we have:*

$$g_{\alpha\beta}(\varepsilon) \sim \sum_{j\geq 0} \varepsilon^{2j} g_{\alpha\beta;2j} \quad \text{and} \quad g^{\alpha\beta}(\varepsilon) \sim \sum_{j\geq 0} \varepsilon^{2j} g^{\alpha\beta;2j}, \tag{4.3}$$

with

$$g_{\alpha\beta;0} = \delta_{\alpha\beta}, \quad g_{\alpha\beta;2} = (\partial_\alpha\theta)\partial_\beta\theta - 2x_3\partial_{\alpha\beta}\theta, \quad g^{\alpha\beta;0} = \delta^{\alpha\beta} \quad \text{and} \quad g^{\alpha\beta;2} = -g_{\alpha\beta;2}.$$

Moreover we have:

$$g_{\alpha 3}(\varepsilon) = 0, \quad g^{\alpha 3}(\varepsilon) = 0 \quad \text{and} \quad g_{33}(\varepsilon) = g^{33}(\varepsilon) = 1. \tag{4.4}$$

The Christoffel symbols admit the expansions:

$$\begin{aligned} \Gamma_{\alpha\beta}^\sigma(\varepsilon) &\sim \textstyle\sum_{j\geq 1} \varepsilon^{2j}\Gamma_{\alpha\beta}^{\sigma;2j}, \\ \Gamma_{\alpha\beta}^3(\varepsilon) &\sim \textstyle\sum_{j\geq 0} \varepsilon^{2j+1}\Gamma_{\alpha\beta}^{3;2j+1}, \\ \Gamma_{\alpha 3}^\sigma(\varepsilon) &\sim \textstyle\sum_{j\geq 0} \varepsilon^{2j+1}\Gamma_{\alpha 3}^{\sigma;2j+1}, \end{aligned} \tag{4.5}$$

with

$$\Gamma^{\sigma;2}_{\alpha\beta} = (\partial_\sigma\theta)\partial_{\alpha\beta}\theta - x_3\partial_{\alpha\beta\sigma}\theta, \quad \Gamma^{3;1}_{\alpha\beta} = \partial_{\alpha\beta}\theta \quad \textit{and} \quad \Gamma^{\sigma;1}_{\alpha 3} = -\partial_{\alpha\sigma}\theta.$$

Finally, we have:

$$\sqrt{g(\varepsilon)} \sim \sum_{j\geq 0} \varepsilon^{2j} g^{1/2;2j}, \tag{4.6}$$

with

$$g^{1/2;0} = 1 \quad \textit{and} \quad g^{1/2;2} = 1/2(s^\theta - 2x_3\Delta\theta).$$

Proof. These computations are obtained by using the special form of the surface and Taylor expansions. The fact that $\Gamma^3_{\alpha\beta}(\varepsilon) = \varepsilon\partial_{\alpha\beta}\theta + \cdots$ means that the second fundamental form of the surface S_ε is of order ε. □

We deduce from this proposition that the operators $\boldsymbol{L}(\varepsilon)$, $\boldsymbol{G}(\varepsilon)$ and $\boldsymbol{T}(\varepsilon)$ also admit expansions with respect to ε: this is due to fact that their coefficients only depend on the geometrical data expanded in Proposition 4.2. Thus we have:

$$L_\alpha(\varepsilon)\boldsymbol{u} \sim \sum_{k\geq 0} \varepsilon^{2k-2} L_{\alpha;2k}\boldsymbol{u} \quad \text{and} \quad L_3(\varepsilon)\boldsymbol{u} \sim \sum_{k\geq 0} \varepsilon^{2k-4} L_{3;2k}\boldsymbol{u}, \tag{4.7}$$

where

$$\left\{ \begin{array}{lcl} L_{\alpha;0}\boldsymbol{u} &=& 2\mu\partial_3 e_{\alpha 3}(\boldsymbol{u}) + \lambda\partial_\alpha e_{33}(\boldsymbol{u}), \\ L_{3;0}\boldsymbol{u} &=& (\lambda+2\mu)\partial_3 e_{33}(\boldsymbol{u}). \end{array} \right. \tag{4.8}$$

Note that the normal component starts with ε^{-4} and that the horizontal components start with ε^{-2}.

For the traction operators on $\Gamma_\pm$, we have (see (2.16)):

$$G_\alpha(\varepsilon)\boldsymbol{u} \sim \sum_{k\geq 0} \varepsilon^{2k-2} G_{\alpha;2k}\boldsymbol{u} \quad \text{and} \quad G_3(\varepsilon)\boldsymbol{u} \sim \sum_{k\geq 0} \varepsilon^{2k-4} G_{3;2k}\boldsymbol{u} \tag{4.9}$$

with

$$\left\{ \begin{array}{lcl} G_{\alpha;0}\boldsymbol{u} &=& 2\mu e_{\alpha 3}(\boldsymbol{u}), \\ G_{3;0}\boldsymbol{u} &=& (\lambda+2\mu) e_{33}(\boldsymbol{u}). \end{array} \right. \tag{4.10}$$

Note that the normal component starts with ε^{-4} and that the horizontal components start with ε^{-2}.

Remark 4.3. The first operators $L_{i;0}$ and $G_{i;0}$ do not depend on θ and are thus, the same as the first terms for plates, see [11]. This is the reason of the fact that the structure of the outer expansion, involving Kirchhoff-Love terms, is the same as for plates.

Finally, using Proposition 4.2, we show that the traction operator on the lateral boundary admits the following expansions in ε:

$$\left\{ \begin{array}{lcl} T_n(\varepsilon)\boldsymbol{u} &\sim& \sum_{k\geq 0} \varepsilon^{2k-2} T_{n;2k}\boldsymbol{u}, \\ T_s(\varepsilon)\boldsymbol{u} &\sim& \sum_{k\geq 0} \varepsilon^{2k} T_{s;2k}\boldsymbol{u}, \\ T_3(\varepsilon)\boldsymbol{u} &\sim& \sum_{k\geq 0} \varepsilon^{2k-2} T_{3;2k}\boldsymbol{u}, \end{array} \right. \tag{4.11}$$

and we compute that:

$$
\left\{
\begin{array}{rcl}
T_{n;0}\boldsymbol{u} & = & \lambda\partial_3 u_3, \\
T_{s;0}\boldsymbol{u} & = & \mu\big(\partial_n u_s + \partial_s u_n + 2\kappa u_s - 2u_3(\partial_{rs}\theta + \kappa\partial_s\theta)\big) \\
 & & +\lambda\partial_3 u_3\big(2x_3(\partial_{rs}\theta + \kappa\partial_s\theta) - (\partial_r\theta)\partial_s\theta\big), \\
T_{3;0}\boldsymbol{u} & = & \mu\big(\partial_3 u_r + \partial_r u_3\big).
\end{array}
\right.
\tag{4.12}
$$

We also need the following expressions:

$$
\left\{
\begin{array}{rcl}
T_{n;2}\boldsymbol{u} & = & \lambda\mathrm{div}_*\boldsymbol{u}_* + 2\mu\partial_n u_n \\
 & & -u_3\big(\lambda(\partial_{rr}\theta + \partial_{ss}\theta - \kappa\partial_r\theta) + 2\mu\partial_{rr}\theta\big) \\
 & & +\lambda\partial_3 u_3\big(2x_3\partial_{rr}\theta - (\partial_r\theta)^2\big), \\
T_{3;2}\boldsymbol{u} & = & 2\mu\big(\partial_{rr}\theta u_r + (\partial_{rs}\theta + \kappa\partial_s\theta)u_s\big) \\
 & & +\mu\big[\big(2x_3\partial_{rr}\theta - (\partial_r\theta)^2\big)(\partial_r u_3 + \partial_3 u_r) \\
 & & +\big(2x_3(\partial_{rs}\theta + \kappa\partial_s\theta) - (\partial_r\theta)\partial_s\theta\big)(\partial_s u_3 + \partial_3 u_s)\big],
\end{array}
\right.
\tag{4.13}
$$

Remark 4.4. For the three-dimensional traction on the lateral boundary, the first operators $T_{n;0}$ and $T_{3;0}$ are the same as those for plates, see [11]. But the tangential component contains the function θ from the first order, which is not the case for plates defined with $\theta = 0$.

4.2. **The outer expansion.** The previous expansions associate to each operator a formal series in power of ε. Our aim is to construct a formal series:

$$
\underline{\boldsymbol{u}}(\varepsilon)(x) = \sum_{k=0} \varepsilon^k \underline{\boldsymbol{u}}^k(x),
\tag{4.14}
$$

solution of the problem (in the sense of formal series)

$$
\left\{
\begin{array}{rcll}
\boldsymbol{L}(\varepsilon)\underline{\boldsymbol{u}}(\varepsilon) & = & -\boldsymbol{f} & \text{in} \quad \Omega, \\
\boldsymbol{G}(\varepsilon)\underline{\boldsymbol{u}}(\varepsilon) & = & 0 & \text{on} \quad \Gamma_+ \cup \Gamma_-,
\end{array}
\right.
\tag{4.15}
$$

which represents the following set of equations, obtained by taking the product of the formal series and identifying the terms in powers of ε:

$$
\forall\, k \geq 0, \quad
\left\{
\begin{array}{rclll}
\sum_{\ell=0}^k L_{\alpha;\ell}\underline{\boldsymbol{u}}^{k-\ell} & = & f^\alpha\delta_2^k & \text{in} \quad \Omega, & \alpha = 1,2, \\
\sum_{\ell=0}^k L_{3;\ell}\underline{\boldsymbol{u}}^{k-\ell} & = & f^3\delta_4^k & \text{in} \quad \Omega, & \alpha = 1,2, \\
\sum_{\ell=0}^k G_{i;\ell}\underline{\boldsymbol{u}}^{k-\ell} & = & 0 & \text{on} \quad \Gamma_+ \cup \Gamma_-, & i = 1,2,3,
\end{array}
\right.
$$

where δ_ℓ^k is the Kronecker symbol, and where we set $L_{i;2\ell+1} = 0$ and $G_{i;2\ell+1} = 0$ for all $\ell \geq 0$, and $\underline{\boldsymbol{u}}^\ell = 0$ for $\ell < 0$.

Using the expansions (4.7), (4.9) and the explicit forms of the operators $L_{3;0}$ and $G_{3;0}$, the transverse components of the former equations are written:

(4.16)

$$
\begin{array}{rcll}
(\lambda + 2\mu)\partial_{33}\underline{u}_3^k & = & -L_{3;2}\underline{\boldsymbol{u}}^{k-2} - L_{3;4}\underline{\boldsymbol{u}}^{k-4} - \sum_{m=3}^{[k/2]} L_{3;2m}\underline{\boldsymbol{u}}^{k-2m} - f^3\delta_4^k & \text{in} \quad \Omega, \\
(\lambda + 2\mu)\partial_3\underline{u}_3^k & = & -G_{3;2}\underline{\boldsymbol{u}}^{k-2} - G_{3;4}\underline{\boldsymbol{u}}^{k-4} - \sum_{m=3}^{[k/2]} G_{3;2m}\underline{\boldsymbol{u}}^{k-2m} & \text{on} \quad \Gamma_\pm,
\end{array}
$$

where $[a]$ denotes the integer part of a real number a.

For fixed k, problem (4.16) is a Neumann problem whose solvability relies on the compatibility condition:

$$(\lambda+2\mu)\int_{-1}^{1}\partial_{33}\underline{u}_3^k\,\mathrm{d}x_3=(\lambda+2\mu)\,\left[\partial_3\underline{u}_3^k\right]_{-1}^{+1}.$$

This condition is equivalently expressed as:

$$\begin{aligned}\int_{-1}^{1}\Big(L_{3;2}\underline{\boldsymbol{u}}^{k-2}+L_{3;4}\underline{\boldsymbol{u}}^{k-4}+\sum_{m=3}^{[k/2]}L_{3;2m}\underline{\boldsymbol{u}}^{k-2m}+f^3\delta_4^k\Big)\,\mathrm{d}x_3&\\ =\Big[G_{3;2}\underline{\boldsymbol{u}}^{k-2}+G_{3;4}\underline{\boldsymbol{u}}^{k-4}+\sum_{m=3}^{[k/2]}G_{3;2m}\underline{\boldsymbol{u}}^{k-2m}\Big]_{-1}^{+1}&,\end{aligned}$$

relation which involves $\underline{\boldsymbol{u}}^\ell$ for $\ell<k$.

Using Proposition 4.2, we find that:

$$\left\{\begin{array}{lcl}L_{3;2}\boldsymbol{v}&=&\lambda\partial_3\left(e_{\alpha\alpha}(\boldsymbol{v})-(\Delta\theta)v_3\right)-2\mu(\Delta\theta)e_{33}(\boldsymbol{v})+2\mu\partial_\sigma e_{\sigma3}(\boldsymbol{v}),\\ G_{3;2}\boldsymbol{v}&=&\lambda\left(e_{\alpha\alpha}(\boldsymbol{v})-(\Delta\theta)v_3\right),\end{array}\right.\tag{4.17}$$

and

$$\left\{\begin{array}{lcl}L_{3;4}\boldsymbol{v}&=&\lambda\partial_3\big((\Gamma^{3;3}_{\alpha\alpha}-g^{\alpha\beta;2}\partial_{\alpha\beta}\theta)v_3\big)+\lambda\partial_3\left(g^{\alpha\beta;2}e_{\alpha\beta}(\boldsymbol{v})-\Gamma^{\sigma;2}_{\alpha\alpha}v_\sigma\right)\\ &&-\big((\lambda+2\mu)\Gamma^{\sigma;3}_{\sigma3}+\lambda(\Gamma^{3;3}_{\alpha\alpha}-g^{\alpha\beta;2}\partial_{\alpha\beta}\theta)\big)\,e_{33}(\boldsymbol{v})\\ &&+2\mu\partial_{\alpha\beta}\theta\left(e_{\alpha\beta}(\boldsymbol{v})-(\partial_{\alpha\beta}\theta)v_3\right)+2\mu\partial_\sigma\left(g^{\alpha\sigma;2}e_{\alpha3}(\boldsymbol{v})+(\partial_{\alpha\sigma}\theta)v_\alpha\right)\\ &&+2\mu\Gamma^{\sigma;2}_{\alpha\sigma}e_{\alpha3}(\boldsymbol{v}),\\ G_{3;4}\boldsymbol{v}&=&\lambda(\Gamma^{3;3}_{\alpha\alpha}-g^{\alpha\beta;2}\partial_{\alpha\beta}\theta)v_3+\lambda\left(g^{\alpha\beta;2}e_{\alpha\beta}(\boldsymbol{v})-\Gamma^{\sigma;2}_{\alpha\alpha}v_\sigma\right).\end{array}\right.\tag{4.18}$$

The compatibility condition then becomes:

$$\begin{aligned}&2\mu\Delta\theta\int_{-1}^{+1}\partial_3\underline{u_3}^{k-2}\,\mathrm{d}x_3-2\mu\partial_\sigma\Big(\int_{-1}^{+1}e_{\sigma3}(\underline{\boldsymbol{u}}^{k-2})\,\mathrm{d}x_3\Big)\\ &+\int_{-1}^{+1}\big((\lambda+2\mu)\Gamma^{\sigma;3}_{\sigma3}+\lambda\big(\Gamma^{3;3}_{\alpha\alpha}-g^{\alpha\beta;2}\partial_{\alpha\beta}\theta\big)\big)\,\partial_3\underline{u_3}^{k-4}\,\mathrm{d}x_3\\ &-2\mu\partial_{\alpha\beta}\theta\int_{-1}^{+1}\big(e_{\alpha\beta}(\underline{\boldsymbol{u}}^{k-4})-(\partial_{\alpha\beta}\theta)\underline{u_3}^{k-4}\big)\,\mathrm{d}x_3\\ &-2\mu\partial_\sigma\left(\int_{-1}^{+1}\big(g^{\alpha\sigma;2}e_{\alpha3}(\underline{\boldsymbol{u}}^{k-4})+(\partial_{\alpha\sigma}\theta)\underline{u_\alpha}^{k-4}\big)\,\mathrm{d}x_3\right)\\ &-2\mu\int_{-1}^{+1}\Gamma^{\sigma;2}_{\sigma\alpha}e_{\alpha3}(\underline{\boldsymbol{u}}^{k-4})\,\mathrm{d}x_3-\int_{-1}^{+1}f^3\delta_4^k\,\mathrm{d}x_3\\ &=\sum_{m=3}^{[k/2]}\int_{-1}^{+1}L_{3;2m}\underline{\boldsymbol{u}}^{k-2m}\,\mathrm{d}x_3-\sum_{m=3}^{[k/2]}G_{3;2m}\underline{\boldsymbol{u}}^{k-2m}(1)+\sum_{m=3}^{[k/2]}G_{3;2m}\underline{\boldsymbol{u}}^{k-2m}(-1).\end{aligned}\tag{4.19}$$

Using the expansions (4.7), (4.9) and the explicit forms of the operators $L_{\sigma;0}$ and $G_{\sigma;0}$, we get for the horizontal components ($\sigma = 1, 2$) and for $k \geq 0$,

(4.20)
$$\begin{aligned} 2\mu\partial_3 e_{\gamma 3}(\underline{\boldsymbol{u}}^k) + \lambda\partial_\gamma(\partial_3 \underline{u}_3^k) &= -L_{\gamma;2}\underline{\boldsymbol{u}}^{k-2} - \textstyle\sum_{m=2}^{[k/2]} L_{\gamma;2m}\underline{\boldsymbol{u}}^{k-2m} - f^\gamma\delta_2^k \quad \text{in} \quad \Omega, \\ 2\mu e_{\gamma 3}(\underline{\boldsymbol{u}}^k) &= -G_{\gamma;2}\underline{\boldsymbol{u}}^{k-2} - \textstyle\sum_{m=2}^{[k/2]} G_{\gamma;2m}\underline{\boldsymbol{u}}^{k-2m} \qquad \text{on} \quad \Gamma_\pm. \end{aligned}$$

For each $k \geq 0$, we must thus have

$$2\mu \int_{-1}^{1} \partial_3 e_{\gamma 3}(\underline{\boldsymbol{u}}^k)\,\mathrm{d}x_3 = 2\mu\left[e_{\gamma 3}(\underline{\boldsymbol{u}}^k)\right]_{-1}^{+1},$$

and this relation is written as:

$$\begin{aligned} &\int_{-1}^{1}\Big(\lambda\partial_{\gamma 3}\underline{u}_3^k + L_{\gamma;2}\underline{\boldsymbol{u}}^{k-2} + \sum_{m=2}^{[k/2]} L_{\gamma;2m}\underline{\boldsymbol{u}}^{k-2m} + f^\gamma\delta_2^k\Big)\,\mathrm{d}x_3 \\ &\qquad\qquad\qquad = \Big[G_{\gamma;2}\underline{\boldsymbol{u}}^{k-2} + \sum_{m=2}^{[k/2]} G_{\gamma;2m}\underline{\boldsymbol{u}}^{k-2m}\Big]_{-1}^{+1}. \end{aligned}$$

We find that:

$$\text{(4.21)} \quad \left\{ \begin{aligned} L_{\gamma;2}\boldsymbol{v} &= 2\mu\partial_3\left(g^{\alpha\gamma;2}e_{\alpha 3}(\boldsymbol{v}) + (\partial_{\gamma\sigma}\theta)v_\sigma\right) + \lambda\partial_\gamma\left(e_{\alpha\alpha}(\boldsymbol{v}) - (\Delta\theta)v_3\right) \\ &\quad + 2\mu\partial_\sigma\left(e_{\sigma\gamma}(\boldsymbol{v}) - (\partial_{\sigma\gamma}\theta)v_3\right) + \lambda\partial_\sigma\left(g^{\gamma\sigma;2}e_{33}(\boldsymbol{v})\right) \\ &\quad + \lambda(\Gamma^{\gamma;2}_{\alpha\alpha} + \Gamma^{\tau;2}_{\gamma\tau})e_{33}(\boldsymbol{v}) - 2\mu(\Delta\theta)e_{\gamma 3}(\boldsymbol{v}) - 4\mu(\partial_{\alpha\gamma}\theta)e_{\alpha 3}(\boldsymbol{v}), \\ G_{\gamma;2}\boldsymbol{v} &= 2\mu\left(g^{\alpha\gamma;2}e_{\alpha 3}(\boldsymbol{v}) + \partial_{\gamma\sigma}\theta v_\sigma\right). \end{aligned} \right.$$

Hence we have the following condition:

$$\text{(4.22)} \quad \begin{aligned} &\lambda\partial_\gamma\left(\int_{-1}^{+1}\partial_3\underline{u}_3^k\,\mathrm{d}x_3\right) + \lambda\partial_\gamma\left(\int_{-1}^{+1}\left(e_{\alpha\alpha}(\underline{\boldsymbol{u}}^{k-2}) - (\Delta\theta)\underline{u}_3^{k-2}\right)\mathrm{d}x_3\right) \\ &+ 2\mu\partial_\sigma\left(\int_{-1}^{+1}\left(e_{\sigma\gamma}(\underline{\boldsymbol{u}}^{k-2}) - (\partial_{\sigma\gamma}\theta)\underline{u}_3^{k-2}\right)\mathrm{d}x_3\right) \\ &+ \lambda\partial_\sigma\left(\int_{-1}^{+1} g^{\gamma\sigma;2}e_{33}(\underline{\boldsymbol{u}}^{k-2})\,\mathrm{d}x_3\right) + \lambda\int_{-1}^{+1}\left(\Gamma^{\gamma;2}_{\alpha\alpha} + \Gamma^{\tau;2}_{\gamma\tau}\right)e_{33}(\underline{\boldsymbol{u}}^{k-2})\,\mathrm{d}x_3 \\ &- 2\mu\Delta\theta\int_{-1}^{+1} e_{\gamma 3}(\underline{\boldsymbol{u}}^{k-2})\,\mathrm{d}x_3 - 4\mu\partial_{\alpha\gamma}\theta\int_{-1}^{+1} e_{\alpha 3}(\underline{\boldsymbol{u}}^{k-2})\,\mathrm{d}x_3 + \int_{-1}^{+1} f^\gamma\delta_2^k\,\mathrm{d}x_3 \\ &= -\int_{-1}^{+1}\sum_{m=2}^{[k/2]} L_{\gamma;2m}\underline{\boldsymbol{u}}^{k-2m}\,\mathrm{d}x_3 - \sum_{m=2}^{[k/2]} G_{\gamma;2m}\underline{\boldsymbol{u}}^{k-2m}(1) + \sum_{m=2}^{[k/2]} G_{\gamma;2m}\underline{\boldsymbol{u}}^{k-2m}(-1). \end{aligned}$$

Now we will study the solution of the equations (4.16) and (4.20) for all $k \geq 0$. For $k = 0$, the equations are:

(4.23)
$$\left\{ \begin{aligned} (\lambda + 2\mu)\partial_{33}\underline{u}_3^0 &= 0 \quad \text{in} \quad \Omega, \\ (\lambda + 2\mu)\partial_3\underline{u}_3^0 &= 0 \quad \text{on} \quad \Gamma_\pm, \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} 2\mu\partial_3 e_{\gamma 3}(\underline{\boldsymbol{u}}^0) + \lambda\partial_{\gamma 3}\underline{u}_3^0 &= 0 \quad \text{in} \quad \Omega, \\ 2\mu e_{\gamma 3}(\underline{\boldsymbol{u}}^0) &= 0 \quad \text{on} \quad \Gamma_\pm, \end{aligned} \right.$$

and the compatibility conditions are trivially satisfied. We deduce that there exists $\boldsymbol{\zeta}^0(x_*)$ independent on x_3 such that $\underline{u}^0$ is the Kirchhoff-Love displacement associated to $\boldsymbol{\zeta}^0$: we have $\underline{\boldsymbol{u}}^0 = \big(\zeta^0_*(x_*) - \nabla_*\zeta^0_3, \zeta^0_3\big)$. The same result hold for $\underline{\boldsymbol{u}}^1$ with a generator denoted $\boldsymbol{\zeta}^1$.

The generators $\boldsymbol{\zeta}^0$ and $\boldsymbol{\zeta}^1$ are not determined yet. Roughly speaking, the compatibility condition in the next steps will give equations involving $\boldsymbol{\zeta}^0$ and $\boldsymbol{\zeta}^1$. These equations will take the form $\boldsymbol{P}(\boldsymbol{\zeta}^0) = \boldsymbol{r}^0$ and $\boldsymbol{P}(\boldsymbol{\zeta}^1) = \boldsymbol{r}^1$, where $\boldsymbol{P}$ is the operator described in the former section, where $\boldsymbol{r}^0$ is given in (3.17) and (3.18), and where $\boldsymbol{r}^1 = 0$.

At each step, we get the same first equations (4.23) involving $\underline{\boldsymbol{u}}^k$, with non-vanishing rights hand sides depending on the fields $\underline{\boldsymbol{u}}^\ell$ for $\ell < k$. Thus, each term $\underline{\boldsymbol{u}}^k$ only depends on the previous terms $\underline{\boldsymbol{u}}^\ell$ for $\ell < k$ and on a Kirchhoff-Love term associated to an undetermined generator $\boldsymbol{\zeta}^k$. This is due to the fact that the kernel of the operators (4.23) consists of Kirchhoff-Love terms. The generators $\boldsymbol{\zeta}^k$ are then determined by the compatibility conditions in the next steps by relations $\boldsymbol{P}(\boldsymbol{\zeta}^k) = \boldsymbol{r}^k$, where $\boldsymbol{r}^k$ depends on $\boldsymbol{f}$, and on the generators $\boldsymbol{\zeta}^\ell$ for $\ell < k$. The following theorem gives the structure of the terms $\underline{\boldsymbol{u}}^k$:

Theorem 4.5. *For any $k \geq 0$ there exist a Kirchhoff-Love displacement $\boldsymbol{u}^k_{\mathrm{KL}}$, whose generator is denoted by $\boldsymbol{\zeta}^k$ and a displacement field $\boldsymbol{v}^k$ with zero mean value:*

$$\forall x_* \in \overline{\omega}, \qquad \int_{-1}^{1} \boldsymbol{v}^k(x_*, x_3)\,\mathrm{d}x_3 = 0,$$

such that

$$\underline{\boldsymbol{u}}^k = \boldsymbol{u}^k_{\mathrm{KL}} + \boldsymbol{v}^k \tag{4.24}$$

is solution of (4.16) *and* (4.20).

Moreover, for $k \geq 0$, $\boldsymbol{\zeta}^k$ is solution of an equation governed by the operator $\boldsymbol{P}$, defined in (3.9)*:*

$$\boldsymbol{P}(\boldsymbol{\zeta}^k) = \boldsymbol{r}^k \quad \textit{in} \quad \omega, \tag{4.25}$$

where for each k, $\boldsymbol{r}^k$ is determined by the previous functions $\boldsymbol{\zeta}^{k-2m}$, $1 \leq m \leq [k/2]$ and by $\boldsymbol{f}$. Moreover, $\boldsymbol{r}^0 = 1/2(p^1, p^2, \partial_\alpha q^\alpha + p^3)$ and $\boldsymbol{r}^1 = (0,0,0)$.

We also have $\boldsymbol{v}^0 = \boldsymbol{v}^1 = 0$ and for $k \geq 2$, $\boldsymbol{v}^k$ depends only on $\boldsymbol{f}$ and on the functions $\boldsymbol{\zeta}^{k-2m}$ for $1 \leq m \leq [k/2]$.

Proof. Let us formulate our induction hypothesis for any $\ell \in \mathbb{N}$ and let us denote it $(\mathcal{F}^\ell)$:

for every $k \leq \ell - 4$, $\underline{\boldsymbol{u}}^k$ is determined and (4.16), (4.20), (4.19) and (4.22) are satisfied;

the function $\underline{\boldsymbol{u}}^{\ell-2}$ is determined and (4.16), (4.19), (4.20), (4.22) are solved for $k = \ell - 2$;

there exist $\boldsymbol{v}^\ell$ such that $\displaystyle\int_{-1}^{1} \boldsymbol{v}^\ell\,\mathrm{d}x_3 = 0$ and (4.16), (4.20) and conditions (4.19), (4.22) are satisfied at $k = \ell$ for $\underline{\boldsymbol{u}}^\ell = \boldsymbol{v}^\ell$;

the compatibility condition (4.19) is satisfied for $k = \ell + 2$ by $\underline{\boldsymbol{u}}^\ell = \boldsymbol{v}^\ell$.

We see that $(\mathcal{F}^0)$ and $(\mathcal{F}^1)$ hold with $\boldsymbol{v}^0 = 0$, $\boldsymbol{v}^1 = 0$.

Let us assume that $(\mathcal{F}^\ell)$ holds. We prove that $(\mathcal{F}^{\ell+2})$ also holds. Since v_3^ℓ is solution of (4.16) for $k = \ell$, we see that for any function ζ_3^ℓ not depending on x_3, $v_3^\ell + \zeta_3^\ell$ is still solution of (4.16).

Since v_γ^ℓ is solution of (4.20) for $k = \ell$, for any function ζ_γ^ℓ independent on x_3, we get another solution for this equation, namely $v_\gamma^\ell + \zeta_\gamma^\ell - x_3 \partial_\gamma \zeta_3^\ell$, for $\underline{u}_3^\ell = v_3^\ell + \zeta_3^\ell$.

Therefore, for any Kirchhoff-Love field $\boldsymbol{u}_{\mathrm{KL}}^\ell$, $\underline{\boldsymbol{u}}^\ell = \boldsymbol{v}^\ell + \boldsymbol{u}_{\mathrm{KL}}^\ell$ is solution of (4.16) and (4.20) for $k = \ell$. The condition (4.19) is satisfied for $k = \ell + 2$ and $\underline{\boldsymbol{u}}^\ell = \boldsymbol{v}^\ell$ and still holds for $\underline{\boldsymbol{u}}^\ell = \boldsymbol{v}^\ell + \boldsymbol{u}_{\mathrm{KL}}^\ell$, where $\boldsymbol{u}_{\mathrm{KL}}^\ell$ is an arbitrary Kirchhoff-Love field. This allows to denote by $v_3^{\ell+2}$ the solution of (4.16) with $\displaystyle\int_{-1}^{1} v_3^{\ell+2}\,\mathrm{d}x_3 = 0$ for $k = \ell + 2$ and $\underline{\boldsymbol{u}}^\ell = \boldsymbol{v}^\ell + \boldsymbol{u}_{\mathrm{KL}}^\ell$.

Now we investigate the condition (4.22) for $k = \ell + 2$ and $\underline{\boldsymbol{u}}^\ell = \boldsymbol{v}^\ell + \boldsymbol{u}_{\mathrm{KL}}^\ell$. Integrating (4.16) from -1 to x_3 for $k = \ell + 2$, we obtain the expression of $\partial_3 v_3^\ell$ with respect to $\boldsymbol{\zeta}^\ell$, $\boldsymbol{v}^\ell$, $\underline{\boldsymbol{u}}^{\ell-2}, \dots \underline{\boldsymbol{u}}^0$. Let P_γ be defined by (3.9). By separating the contributions of $\boldsymbol{v}^\ell$ and $\boldsymbol{u}_{\mathrm{KL}}^\ell$ in the investigated condition, we compute that equation (4.22) takes the form

$$P_\gamma(\boldsymbol{\zeta}^\ell) = r_\gamma^\ell \quad \text{in} \quad \omega, \tag{4.26}$$

for r_γ^ℓ depending on $\boldsymbol{v}^\ell$, $\underline{\boldsymbol{u}}^{\ell-2}, \dots \underline{\boldsymbol{u}}^0$. For $\ell = 0, 1$, we have $r_\gamma^0 = \frac{1}{2} p^\gamma$ and $r_\gamma^1 = 0$.

For a function $\boldsymbol{\zeta}^\ell$ verifying (4.26), let $v_\gamma^{\ell+2}$ be the solution of (4.20) for $k = \ell + 2$ and for $\underline{u}_3^{\ell+2} = v_3^{\ell+2}$, having zero mean value over $(-1, 1)$.

In order to investigate the compatibility condition (4.19) for $k = \ell + 4$ and $\underline{\boldsymbol{u}}^{\ell+2} = \boldsymbol{v}^{\ell+2}$, we integrate (4.20) from -1 to x_3. Using the first equation (4.26), we get that this compatibility condition takes the form

$$P_3(\boldsymbol{\zeta}^\ell) = r_3^\ell \quad \text{in} \quad \omega, \tag{4.27}$$

with a left side depending on $\boldsymbol{v}^\ell$, $\underline{\boldsymbol{u}}^{\ell-2}, \dots \underline{\boldsymbol{u}}^0$. For $\ell = 0, 1$, we have $r_3^0 = 1/2(\partial_\gamma q^\gamma + p^3)$ and $r_3^1 = 0$.

We also calculate that:

$$v_3^2(x_*, x_3) = \frac{\lambda}{\lambda + 2\mu} \left(\frac{x_3^2}{2} - \frac{1}{6} \right) \Delta \zeta_3^0 - \frac{\lambda}{\lambda + 2\mu} x_3 (\mathrm{div}_* \zeta_*^0 - (\Delta\theta) \zeta_3^0). \tag{4.28}$$

Therefore, if we take $\boldsymbol{\zeta}^\ell$ verifying (4.26), (4.27) and we define $\underline{\boldsymbol{u}}^\ell = \boldsymbol{v}^\ell + u_{\mathrm{KL}}^\ell$, then the induction condition $(\mathcal{F}^{\ell+2})$ is established. $\square$

5. Construction of the inner expansion

In the previous section, starting from solutions $\boldsymbol{\zeta}^k$ of equations (4.25), we constructed formal series (4.14) satisfying equations (4.15). However, we can show that for all solutions $\boldsymbol{\zeta}^k$, the equations on the lateral boundary Γ_0 are usually not satisfied.

As in the case of plates, we introduce scaled boundary layer terms $\boldsymbol{w}(\varepsilon^{-1}r, s, x_3)$ exponentially decreasing in $t = \varepsilon^{-1}r$. Thus, we have to make the change of variable $(r, s, x_3) \mapsto (t, s, x_3)$ in order to pose the equations, and as this change of variable depends on ε , it has an influence on the underlying formal series.

5.1. **Interior equations and horizontal boundary conditions.** In the following, (r, s, x_3) denotes the coordinate system described in section 2.4, in a neighbourhood of Γ_0 , and $t = \varepsilon^{-1}r$ is a scaled coordinate. The system (t, s, x_3) lies in $\Sigma^+ \times \mathbb{S}$, with $(t, x_3) \in \Sigma^+ := \mathbb{R}^+ \times (-1, +1)$. Thus, Σ^+ is a half strip with two corners $(t = 0, x_3 = \pm 1)$, and whose boundary consists of a lateral boundary $\gamma_0 = \{0\} \times (-1, +1)$ and of upper and lower edges $\gamma_\pm = \mathbb{R}_+ \times \{\pm 1\}$.

In order to define the operators acting on boundary layer terms, we introduce the following scaling operator: let $\mathcal{D}(\varepsilon)$ be defined as

$$\mathcal{D}(\varepsilon)\boldsymbol{\varphi} = (\boldsymbol{\varphi}_*, \varepsilon\, \varphi_3),$$

for all triple $\boldsymbol{\varphi} = (\boldsymbol{\varphi}_*, \varphi_3)$.

Recall that in coordinates (r, s, x_3) , the components of the operator $\boldsymbol{L}(\varepsilon)$ are given by:

$$\begin{aligned}
L_r(\varepsilon)(r, s, x_3; \partial_r, \partial_s, \partial_3) &= \big(n_1 L_1(\varepsilon) + n_2 L_2\big)(x; \partial_x),\\
L_s(\varepsilon)(r, s, x_3; \partial_r, \partial_s, \partial_3) &= (1 - \kappa r)^{-1}\big(n_2 L_1(\varepsilon) - n_1 L_2(\varepsilon)\big)(x; \partial_x),\\
L_3(\varepsilon)(r, s, x_3; \partial_r, \partial_s, \partial_3) &= L_3(\varepsilon)(x; \partial_x),
\end{aligned}$$

where $x = (x_1, x_2, x_3)$ and $\partial_x = (\partial_1, \partial_2, \partial_3)$, and that the same holds for the operators $\boldsymbol{G}(\varepsilon)$ and $\boldsymbol{T}(\varepsilon)$.

In order to obtain an operator of order 0 in ε , by using $\mathcal{D}(\varepsilon)$ we define the following operator:

$$\left\{\begin{aligned}
\mathcal{L}_t(\varepsilon)(t, s, x_3; \partial_t, \partial_s, \partial_3) &:= \varepsilon^2 L_r(\varepsilon)(\varepsilon\, t, s, x_3; \varepsilon^{-1}\partial_t, \partial_s, \partial_3) \circ \mathcal{D}(\varepsilon),\\
\mathcal{L}_s(\varepsilon)(t, s, x_3; \partial_t, \partial_s, \partial_3) &:= \varepsilon^2 L_s(\varepsilon)(\varepsilon\, t, s, x_3; \varepsilon^{-1}\partial_t, \partial_s, \partial_3) \circ \mathcal{D}(\varepsilon),\\
\mathcal{L}_3(\varepsilon)(t, s, x_3; \partial_t, \partial_s, \partial_3) &:= \varepsilon^3 L_3(\varepsilon)(\varepsilon\, t, s, x_3; \varepsilon^{-1}\partial_t, \partial_s, \partial_3) \circ \mathcal{D}(\varepsilon).
\end{aligned}\right.$$

Using Taylor expansions, we see that we can associate to this operator a formal series:

$$\boldsymbol{\mathcal{L}}(\varepsilon) = \boldsymbol{\mathcal{L}}^0 + \varepsilon\, \boldsymbol{\mathcal{L}}^1 + \sum_{k \geq 2} \varepsilon^k \boldsymbol{\mathcal{L}}^k,$$

where the $\boldsymbol{\mathcal{L}}^k$ are operators of degree 2 on $\Sigma^+ \times \mathbb{S}$ that are polynomial in x_3 and t . Moreover, we identify:

$$\left\{\begin{aligned}
\mathcal{L}_t^0\boldsymbol{\varphi} &= \mu(\partial_{tt}\varphi_t + \partial_{33}\varphi_t) + (\lambda + \mu)\partial_t(\partial_t\varphi_t + \partial_3\varphi_3),\\
\mathcal{L}_s^0\boldsymbol{\varphi} &= \mu(\partial_{tt}\varphi_s + \partial_{33}\varphi_s),\\
\mathcal{L}_3^0\boldsymbol{\varphi} &= \mu(\partial_{tt}\varphi_3 + \partial_{33}\varphi_3) + (\lambda + \mu)\partial_3(\partial_t\varphi_t + \partial_3\varphi_3),
\end{aligned}\right. \tag{5.1}$$

and we see that this operator is the same as that for plates (see [11, 8]).

Similarly, we define the following traction operator on $\Sigma^+\times\mathbb{S}$:

$$\left\{\begin{array}{lcl} \mathcal{G}_t(\varepsilon)(t,s,x_3;\partial_t,\partial_s,\partial_3) & := & \varepsilon^2 G_r(\varepsilon)(\varepsilon\, t,s,x_3;\varepsilon^{-1}\partial_t,\partial_s,\partial_3)\circ\mathcal{D}(\varepsilon),\\ \mathcal{G}_s(\varepsilon)(t,s,x_3;\partial_t,\partial_s,\partial_3) & := & \varepsilon^2 G_s(\varepsilon)(\varepsilon\, t,s,x_3;\varepsilon^{-1}\partial_t,\partial_s,\partial_3)\circ\mathcal{D}(\varepsilon),\\ \mathcal{G}_3(\varepsilon)(t,s,x_3;\partial_t,\partial_s,\partial_3) & := & \varepsilon^3 G_3(\varepsilon)(\varepsilon\, t,s,x_3;\varepsilon^{-1}\partial_t,\partial_s,\partial_3)\circ\mathcal{D}(\varepsilon). \end{array}\right.$$

Using Taylor expansions, we see that we can associate to this operator a formal series:

$$\boldsymbol{\mathcal{G}}(\varepsilon)=\boldsymbol{\mathcal{G}}^0+\varepsilon\,\boldsymbol{\mathcal{G}}^1+\sum_{k\geq 2}\varepsilon^k\boldsymbol{\mathcal{G}}^k,$$

where the operators $\boldsymbol{\mathcal{G}}^k$ are of degree one on $\Sigma^+\times\mathbb{S}$ and are polynomials in x_3 and t. Moreover, we see that the first term is given by:

$$\left\{\begin{array}{lcl} \mathcal{G}^0_t\boldsymbol{\varphi} & = & \mu(\partial_t\varphi_3+\partial_3\varphi_t),\\ \mathcal{G}^0_s\boldsymbol{\varphi} & = & \mu\partial_3\varphi_s,\\ \mathcal{G}^0_3\boldsymbol{\varphi} & = & (\lambda+2\mu)\partial_3\varphi_3+\lambda\partial_t\varphi_t. \end{array}\right. \tag{5.2}$$

Thus we see that this operator is the same as that for plates, see [11, 8].

Hence, we are searching for formal series $\boldsymbol{\varphi}(\varepsilon)=\sum_{k\geq 0}\varepsilon^k\boldsymbol{\varphi}^k$ solution of the equations (in formal series):

$$\boldsymbol{\mathcal{L}}(\varepsilon)\boldsymbol{\varphi}(\varepsilon)=0,$$

$$\boldsymbol{\mathcal{G}}(\varepsilon)\boldsymbol{\varphi}(\varepsilon)=0.$$

Before studying these equations, we perform the same change of variables for the operators on the lateral boundary, taking into account the previous result concerning the outer expansion. Hence, we obtain boundary equations in order to get the matching of the terms of this outer part.

5.2. **Conditions on the lateral boundary.**

5.2.1. *Lateral Dirichlet boundary conditions.* Let $\underline{\boldsymbol{u}}(\varepsilon)=\sum_{k\geq 0}\varepsilon^k\underline{\boldsymbol{u}}^k$ be a formal series constructed in the former section. Recall that, according to Theorem 4.5, we have:

$$\begin{array}{lcl} \underline{u}^k_n & = & \zeta^k_n-x_3\partial_n\zeta^k_3+v^k_n,\\ \underline{u}^k_s & = & \zeta^k_s-x_3\partial_s\zeta^k_3+v^k_s,\\ \underline{u}^k_3 & = & \zeta^k_3+v^k_3, \end{array} \tag{5.3}$$

where the $\boldsymbol{\zeta}^k$ are two-dimensional generators. Due to the scaling operator $\mathcal{D}(\varepsilon)$, we are looking for a boundary layer formal series of the type $\boldsymbol{w}(\varepsilon)=\mathcal{D}(\varepsilon)\circ\boldsymbol{\varphi}(\varepsilon)$. Thus, we want to find a formal series $\boldsymbol{\varphi}(\varepsilon)$ satisfying the relation:

$$\sum_{k\geq 0}\varepsilon^k\underline{\boldsymbol{u}}^k\,\big|_{\Gamma_0}+\sum_{k\geq 0}\varepsilon^k(\boldsymbol{\varphi}^k_*,\varepsilon\varphi^k_3)\,\big|_{t=0}=0. \tag{5.4}$$

In 5.4 we identify the coefficients of ε and we get:

$$\left\{\begin{array}{lclcl} \varphi^0_t\,\big|_{t=0} & + & \underline{u}^0_n\,\big|_{\partial\omega} & = & 0,\\ \varphi^0_s\,\big|_{t=0} & + & \underline{u}^0_s\,\big|_{\partial\omega} & = & 0,\\ & & \underline{u}^0_3\,\big|_{\partial\omega} & = & 0, \end{array}\right. \quad\text{and}\quad \varphi^0_3\,\big|_{t=0}+\underline{u}^1_3\,\big|_{\partial\omega}=0,$$

and, for $k \geq 1$,

$$\left\{ \begin{array}{lclcl} \varphi_t^k \big|_{t=0} & + & \underline{u}_n^k \big|_{\partial\omega} & = & 0, \\ \varphi_s^k \big|_{t=0} & + & \underline{u}_s^k \big|_{\partial\omega} & = & 0, \\ \varphi_3^k \big|_{t=0} & + & \underline{u}_3^{k+1} \big|_{\partial\omega} & = & 0. \end{array} \right. \tag{5.5}$$

5.2.2. *Lateral Neumann boundary conditions.* As before, we define the following operator:

$$\left\{ \begin{array}{lcl} \mathcal{T}_t(\varepsilon)(t,s,x_3;\partial_t,\partial_s,\partial_3) & := & \varepsilon T_r(\varepsilon)(\varepsilon\, t,s,x_3;\varepsilon^{-1}\partial_t,\partial_s,\partial_3) \circ \mathcal{D}(\varepsilon), \\ \mathcal{T}_s(\varepsilon)(t,s,x_3;\partial_t,\partial_s,\partial_3) & := & \varepsilon T_s(\varepsilon)(\varepsilon\, t,s,x_3;\varepsilon^{-1}\partial_t,\partial_s,\partial_3) \circ \mathcal{D}(\varepsilon), \\ \mathcal{T}_3(\varepsilon)(t,s,x_3;\partial_t,\partial_s,\partial_3) & := & \varepsilon^2 T_3(\varepsilon)(\varepsilon\, t,s,x_3;\varepsilon^{-1}\partial_t,\partial_s,\partial_3) \circ \mathcal{D}(\varepsilon). \end{array} \right.$$

Using Taylor expansion in $t = 0$ and x_3, we see that we can associate to this operator a formal series:

$$\boldsymbol{\mathcal{T}}(\varepsilon) = \boldsymbol{\mathcal{T}}^0 + \varepsilon\,\boldsymbol{\mathcal{T}}^1 + \sum_{k\geq 2} \varepsilon^k \boldsymbol{\mathcal{T}}^k,$$

where the operators $\boldsymbol{\mathcal{T}}^k$ are of degree one on $\Sigma^+ \times \partial\omega$, polynomials in x_3 and t and take their values in $(-1,+1) \times \partial\omega$.

Moreover, we find that:

$$\left\{ \begin{array}{lcl} \mathcal{T}_t^0(\boldsymbol{\varphi}) & = & \lambda\partial_3\varphi_3 + (\lambda+2\mu)\partial_t\varphi_t, \\ \mathcal{T}_s^0(\boldsymbol{\varphi}) & = & \mu\partial_t\varphi_s, \\ \mathcal{T}_3^0(\boldsymbol{\varphi}) & = & \mu(\partial_3\varphi_t + \partial_t\varphi_3), \end{array} \right. \tag{5.6}$$

which is the same as for plates, and that:

$$\left\{ \begin{array}{lcl} \mathcal{T}_t^1(\boldsymbol{\varphi}) & = & \lambda(\partial_s\varphi_s - \kappa\varphi_t), \\ \mathcal{T}_s^1(\boldsymbol{\varphi}) & = & \mu(\partial_s\varphi_t + 2\kappa\varphi_s), \\ \mathcal{T}_3^1(\boldsymbol{\varphi}) & = & 2\mu\big((\partial_{rr}\theta)\varphi_t + (\partial_{rs}\theta + \kappa\partial_s\theta)\varphi_s\big) \\ & & +\mu\big(\,(2x_3\partial_{rr}\theta - (\partial_r\theta)^2)\,(\partial_t\varphi_3 + \partial_3\varphi_t) \\ & & +\,(2x_3(\partial_{rs}\theta + \kappa\partial_s\theta) - (\partial_r\theta)\partial_s\theta)\,\partial_3\varphi_s\big). \end{array} \right. \tag{5.7}$$

We remark that $\mathcal{T}_t^1$ and $\mathcal{T}_s^1$ are the same as for plates. Finally, we give explicit formulas for:

$$\left\{ \begin{array}{lcl} \mathcal{T}_t^2(\varphi_3) & = & -\varphi_3\big((\lambda+2\mu)\partial_{rr}\theta + \lambda(\partial_{ss}\theta - \kappa\partial_r\theta)\big) + \lambda\partial_3\varphi_3\big(2x_3\partial_{rr}\theta - (\partial_r\theta)^2\big), \\ \mathcal{T}_s^2(\varphi_3) & = & -2\mu\varphi_3(\partial_{rs}\theta + \kappa\partial_s\theta) + \lambda\partial_3\varphi_3\big(2x_3(\partial_{rs}\theta + \kappa\partial_s\theta) - (\partial_r\theta)\partial_s\theta\big), \\ \mathcal{T}_3^2(\varphi_3) & = & \mu\big(2x_3(\partial_{rs}\theta + \kappa\partial_s\theta) - (\partial_r\theta)\partial_s\theta\big)\partial_s\varphi_3. \end{array} \right.$$

Taking into account the different scalings that we made, the traction-free condition on the lateral boundary can be written as:

$$\boldsymbol{T}(\varepsilon)\underline{\boldsymbol{u}}(\varepsilon)\big|_{\partial\omega} + \big(\varepsilon^{-1}\mathcal{T}_t(\varepsilon)\boldsymbol{\varphi}(\varepsilon), \varepsilon^{-1}\mathcal{T}_t(\varepsilon)\boldsymbol{\varphi}(\varepsilon), \varepsilon^{-2}\mathcal{T}_t(\varepsilon)\boldsymbol{\varphi}(\varepsilon)\big)\big|_{t=0} = 0. \tag{5.8}$$

In 5.8 we identify the powers of ε and we obtain:

$$\begin{cases} \mathcal{T}_t^0(\boldsymbol{\varphi}^k) + \mathcal{T}_t^1(\boldsymbol{\varphi}^{k-1}) + \mathcal{T}_t^2(\boldsymbol{\varphi}^{k-2}) + \sum_{\ell=3}^k \mathcal{T}_t^\ell(\boldsymbol{\varphi}^{k-\ell}) \\ \qquad\qquad + T_{n;0}(\underline{\boldsymbol{u}}^{k+1}) + T_{n;1}(\underline{\boldsymbol{u}}^k) + \sum_{\ell=2}^{k+1} T_{n;\ell}(\underline{\boldsymbol{u}}^{k+1-\ell}) = 0, \\ \mathcal{T}_s^0(\boldsymbol{\varphi}^k) + \mathcal{T}_s^1(\boldsymbol{\varphi}^{k-1}) + \mathcal{T}_s^2(\boldsymbol{\varphi}^{k-2}) + \sum_{\ell=3}^k \mathcal{T}_s^\ell(\boldsymbol{\varphi}^{k-\ell}) \\ \qquad\qquad + T_{s;0}(\underline{\boldsymbol{u}}^{k+1}) + T_{s;1}(\underline{\boldsymbol{u}}^k) + \sum_{\ell=2}^{k+1} T_{s;\ell}(\underline{\boldsymbol{u}}^{k+1-\ell}) = 0, \\ \mathcal{T}_3^0(\boldsymbol{\varphi}^k) + \mathcal{T}_3^1(\boldsymbol{\varphi}^{k-1}) + \mathcal{T}_3^2(\boldsymbol{\varphi}^{k-2}) + \sum_{\ell=3}^k \mathcal{T}_3^\ell(\boldsymbol{\varphi}^{k-\ell}) \\ \qquad\qquad + T_{3;0}(\underline{\boldsymbol{u}}^k) + T_{3;1}(\underline{\boldsymbol{u}}^{k-1}) + \sum_{\ell=2}^{k} T_{3;\ell}(\underline{\boldsymbol{u}}^{k-\ell}) = 0, \end{cases}$$

where we set $T_{n;\ell} = T_{s;\ell} = T_{3;\ell} = 0$ for ℓ odd. Introducing the expressions of the operators, (see also [2]), we find that:

$$\begin{cases} \begin{array}{lcl} \mathcal{T}_t^0(\boldsymbol{\varphi}^k) & = & -\mathcal{T}_t^1(\boldsymbol{\varphi}^{k-1}) - \mathcal{T}_t^2(\boldsymbol{\varphi}^{k-2}) - \sum_{\ell=3}^k \mathcal{T}_t^\ell(\boldsymbol{\varphi}^{k-\ell}) \\ & & -B_n(\boldsymbol{\zeta}^{k-1}) + 3x_3 M_n(\zeta_3^{k-1}) + E_n(\boldsymbol{v}^{k-1}, \boldsymbol{u}^{k-3} \dots \boldsymbol{u}^0) \\ \mathcal{T}_s^0(\boldsymbol{\varphi}^k) & = & -\mathcal{T}_s^1(\boldsymbol{\varphi}^{k-1}) - \mathcal{T}_s^2(\boldsymbol{\varphi}^{k-2}) - \sum_{\ell=3}^k \mathcal{T}_s^\ell(\boldsymbol{\varphi}^{k-\ell}) \\ & & -B_s(\boldsymbol{\zeta}^{k-1}) + 2\mu x_3(\partial_n + \kappa)\partial_s \zeta_3^{k-1} + E_s(\boldsymbol{v}^{k-1}, \boldsymbol{u}^{k-3} \dots \boldsymbol{u}^0), \\ \mathcal{T}_3^0(\boldsymbol{\varphi}^k) & = & -\mathcal{T}_3^1(\boldsymbol{\varphi}^{k-1}) - \sum_{\ell=2}^k \mathcal{T}_s^\ell(\boldsymbol{\varphi}^{k-\ell}) - (\tilde{\lambda} + 2\mu)\dfrac{x_3^2 - 1}{2}\partial_r(\Delta_* \zeta_3^{k-2}) \\ & & -\dfrac{x_3 + 1}{2}\big(p^\gamma \delta_0^{k-2} n_\gamma + \dfrac{\lambda}{\lambda + 2\mu}\partial_r(p^3 \delta_0^{k-4} - q^3 \delta_0^{k-4})\big) + \displaystyle\int_{-1}^{x_3} n_\gamma f^\gamma \delta_0^{k-2} \\ & & +E_3(\boldsymbol{v}^{k-2}, \boldsymbol{u}^{k-4} \dots \boldsymbol{u}^0), \end{array} \end{cases}$$

where E_n , E_s and E_3 are appropriate operators and where p^γ and q^3 are given in (3.16).

5.3. **Recursive equations.** Let $\underline{\boldsymbol{u}}(\varepsilon)$ be a formal series solution of the equations (4.15). This series depends on generators $\boldsymbol{\zeta}^k$. Our aim is to find a formal series $\boldsymbol{\varphi}(\varepsilon)$ solution of the system

$$\begin{cases} \boldsymbol{\mathcal{L}}(\varepsilon)\boldsymbol{\varphi}(\varepsilon) & = & 0 \quad \text{in} \quad \Sigma^+ \times \partial\omega, \\ \boldsymbol{\mathcal{G}}(\varepsilon)\boldsymbol{\varphi}(\varepsilon) & = & 0 \quad \text{on} \quad \gamma_\pm \times \partial\omega, \end{cases} \tag{5.9}$$

with the boundary conditions

$$\underline{\boldsymbol{u}}(\varepsilon)\big|_{\partial\omega} + \big(\varphi_*(\varepsilon), \varepsilon\, \varphi_3(\varepsilon)\big)\big|_{t=0} = 0,$$

in the clamped case, and

$$\boldsymbol{T}(\varepsilon)\underline{\boldsymbol{u}}(\varepsilon)\big|_{\partial\omega} + \big(\varepsilon^{-1}\mathcal{T}_t(\varepsilon)\boldsymbol{\varphi}(\varepsilon), \varepsilon^{-1}\mathcal{T}_t(\varepsilon)\boldsymbol{\varphi}(\varepsilon), \varepsilon^{-2}\mathcal{T}_t(\varepsilon)\boldsymbol{\varphi}(\varepsilon)\big)\big|_{t=0} = 0,$$

in the free case.

The system (5.9) becomes, for $k \geq 0$,

$$\begin{cases} \boldsymbol{\mathcal{L}}^0 \boldsymbol{\varphi}^k & = & -\sum_{\ell=1}^k \boldsymbol{\mathcal{L}}^\ell \boldsymbol{\varphi}^{k-\ell}, \\ \boldsymbol{\mathcal{G}}^0 \boldsymbol{\varphi}^k & = & -\sum_{\ell=1}^k \boldsymbol{\mathcal{G}}^\ell \boldsymbol{\varphi}^{k-\ell}. \end{cases} \tag{5.10}$$

Using the framework of [8], we recall here properties of the operator $(\boldsymbol{\mathcal{L}}^0, \boldsymbol{\mathcal{G}}^0)$ (see also [10, 11]). First of all, we introduce the functional space where the functions $\boldsymbol{\varphi}^k$ will be, (see [8]): Let $\mathfrak{H}(\Sigma^+)$ be the space of $\mathscr{C}^\infty(\Sigma^+)$ functions φ, that are smooth up to any point of the boundary of Σ^+ (except corners) and are exponentially decreasing as $t \to \infty$ in the following sense:

$$\forall i, j, k \in \mathbb{N}, \qquad e^{\delta t}\, t^k\, \partial_t^i \partial_3^j \varphi \in L^2(\Sigma^+)$$

where $\delta > 0$ is a fixed number smaller than the smallest exponent arising from the Papkovich-Fadle eigen- functions, see [15]. Denoting by ρ the distance between the two corners of Σ^+, we prescribe the following behaviour at the corners for the elements of $\mathfrak{H}(\Sigma^+)$:

$$\varphi \in L^2(\Sigma^+) \quad \text{and} \quad \forall i, j \in \mathbb{N},\ i + j \neq 0, \qquad \rho^{i+j-1}\, \partial_t^i \partial_3^j \varphi \in L^2(\Sigma^+).$$

Then we define the corresponding displacement space $\boldsymbol{\mathfrak{H}}(\Sigma^+) := \mathfrak{H}(\Sigma^+)^3$. Our formal series $\boldsymbol{\varphi}(\varepsilon)$ will have its coefficients in $\mathscr{C}^\infty\big(\partial\omega, \boldsymbol{\mathfrak{H}}(\Sigma^+)\big)$.

Let $\mathfrak{K}(\Sigma^+)$ be the space of triples $(\psi, \psi^\pm) \in \mathscr{C}^\infty(\Sigma^+) \times \mathscr{C}^\infty(\gamma_\pm)$ that satisfy

$$\forall i, j, k \in \mathbb{N}, \qquad e^{\delta t}\, t^k\, \partial_t^i \partial_3^j \psi \in L^2(\Sigma^+) \quad \text{and} \quad e^{\delta t}\, t^k\, \partial_t^i \psi^\pm \in L^2(\gamma_\pm)$$

and

$$\forall i, j \in \mathbb{N}, \qquad \rho^{i+j+1}\, \partial_t^i \partial_3^j \psi \in L^2(\Sigma^+) \quad \text{and} \quad \rho^{i+j+1/2}\, \partial_t^i \psi^\pm \in L^2(\gamma_\pm).$$

Then we define the corresponding displacement space:

$$\boldsymbol{\mathfrak{K}}(\Sigma^+) := \big\{ \boldsymbol{\Psi} = (\boldsymbol{\psi}, \boldsymbol{\psi}^\pm) \in \mathfrak{K}(\Sigma^+)^3 \big\}.$$

According to [11] the operator $(\boldsymbol{\mathcal{L}}^0, \boldsymbol{\mathcal{G}}^0)$ has similar properties in both the clamped and free cases. We recall here those that we need; compare [11, section 5].

Proposition 5.1. *There exists a four-dimensional space $\mathscr{Z}$ of polynomials, such that if $\boldsymbol{\Psi}$ belongs to $\mathscr{C}^\infty\big(\partial\omega, \boldsymbol{\mathfrak{K}}(\Sigma^+)\big)$ and $\boldsymbol{v}$ belongs to $\mathscr{C}^\infty(\overline{\Gamma}_0)^3$, then there exist an unique $\boldsymbol{\varphi} \in \mathscr{C}^\infty\big(\partial\omega, \boldsymbol{\mathfrak{H}}(\Sigma^+)\big)$ and an unique $\boldsymbol{Z} \in \mathscr{C}^\infty(\partial\omega, \mathscr{Z})$ such that*

$$\left\{ \begin{array}{rcll} \boldsymbol{\mathcal{L}}^0(\boldsymbol{\varphi}) & = & \boldsymbol{\Psi} & \text{in} \quad \Sigma^+ \times \partial\omega, \\ \boldsymbol{\mathcal{G}}^0(\boldsymbol{\varphi}) & = & 0 & \text{on} \quad \gamma_\pm \times \partial\omega, \\ \boldsymbol{\mathcal{H}}_0(\boldsymbol{\varphi} - \boldsymbol{Z})\big|_{t=0} + \boldsymbol{v}\big|_{\Gamma_0} & = & 0, & \end{array} \right.$$

where $\boldsymbol{\mathcal{H}}_0 = \mathrm{Id}$ in the clamped case, and $\boldsymbol{\mathcal{H}}_0 = \boldsymbol{\mathcal{T}}^0$ in the free case.

In the next section, we will show how the condition in the space $\mathscr{Z}$ will give boundary conditions on the generators $\boldsymbol{\zeta}^k$ in order to obtain the existence of the terms $\boldsymbol{\varphi}^k$.

6. Clamped shallow shells

In this case, the space $\mathscr{Z}$ of Proposition 5.1 is spanned by the four elements

$$\boldsymbol{Z}_{\mathrm{D}}^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{Z}_{\mathrm{D}}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \boldsymbol{Z}_{\mathrm{D}}^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \boldsymbol{Z}_{\mathrm{D}}^2 = \begin{pmatrix} -x_3 \\ 0 \\ t \end{pmatrix}. \tag{6.1}$$

In the case of Dirichlet boundary conditions on Γ_0, we must have for $k \geq 1$,

$$\begin{array}{rcl} \varphi_t^k & = & -(\zeta_n^k - x_3\partial_n\zeta_3^k + v_n^k), \\ \varphi_s^k & = & -(\zeta_s^k - x_3\partial_s\zeta_3^k + v_s^k), \\ \varphi_3^k & = & -(\zeta_3^{k+1} + v_3^{k+1}). \end{array} \tag{6.2}$$

We then have the following proposition:

Proposition 6.1. *Let $\boldsymbol{\zeta}^k = (\zeta_*, \zeta_3)$ be a family of generators satisfying the relations $\boldsymbol{P}(\boldsymbol{\zeta}^k) = \boldsymbol{r}^k$ of Theorem* 4.5*, and let $\underline{\boldsymbol{u}}^k$ be the displacement constructed in this theorem. Then there exist $\boldsymbol{h}^k = (h_r^k, h_s^k, h_3^k, h_n^k) \in \mathscr{C}^\infty(\partial\omega)^4$ depending only on $\boldsymbol{f}$ and on $\boldsymbol{\zeta}^\ell$, $0 \leq \ell \leq k-1$, such that if $\boldsymbol{\zeta}^k$ satisfy the conditions*

$$(\zeta_r^k, \zeta_s^k, \zeta_3^k, \zeta_3^k)\big|_{\partial\omega} = \boldsymbol{h}^k,$$

then there exist boundary layer profiles $\boldsymbol{\varphi}^k$ satisfying equations (5.4)*.*

Proof. For all $k \geq 0$, we search an element $\boldsymbol{\varphi}^k \in \mathscr{C}^\infty\big(\partial\omega, \boldsymbol{\mathfrak{K}}(\Sigma^+)\big)$ such that

$$\left\{\begin{array}{rcl} \boldsymbol{\mathcal{L}}^0(\boldsymbol{\varphi}^k) & = & -\sum_{\ell=1}^k \boldsymbol{\mathcal{L}}^\ell(\boldsymbol{\varphi}^{k-\ell}) \quad \text{in} \quad \Sigma^+ \times \partial\omega, \\ \boldsymbol{\mathcal{G}}^0(\boldsymbol{\varphi}^k) & = & -\sum_{\ell=1}^k \boldsymbol{\mathcal{G}}^\ell(\boldsymbol{\varphi}^{k-\ell}) \quad \text{on} \quad \gamma_\pm \times \partial\omega, \\ \varphi_t^k\big|_{t=0} + (\zeta_n^k - x_3\partial_n\zeta_3^k + v_n^k)\big|_{\partial\omega} & = & 0, \\ \varphi_s^k\big|_{t=0} + (\zeta_s^k - x_3\partial_s\zeta_3^k + v_s^k)\big|_{\partial\omega} & = & 0, \\ \varphi_3^k\big|_{t=0} + (\zeta_3^{k+1} + v_3^{k+1})\big|_{\partial\omega} & = & 0. \end{array}\right.$$

Hence, using Proposition 5.1 and the expression of the basis (6.1), we see that we have the existence of $\boldsymbol{\varphi}^k$ if and only if there exist functions h_r^k, h_s^k, h_3^k and h_n^{k+1} on $\partial\omega$ such that

$$\zeta_r^k\big|_{\partial\omega} = h_r^k, \quad \zeta_s^k\big|_{\partial\omega} = h_s^k, \quad \zeta_3^k\big|_{\partial\omega} = h_3^k, \quad \text{and} \quad \zeta_n^{k+1}\big|_{\partial\omega} = h_n^{k+1}.$$

The fact that the field $\boldsymbol{v}^\ell$ depends only on the $\boldsymbol{\zeta}^i$ for $i < \ell - 1$ ends the proof. ☐

For the first terms of the asymptotic, we show, as in [11], that

$$\zeta_n^0 = \zeta_s^0 = \zeta_3^0 = \partial_n\zeta_3^0 = 0 \quad \text{on} \quad \partial\omega. \tag{6.3}$$

We study now the fields $\boldsymbol{\zeta}^1$ and $\boldsymbol{\varphi}^1$. We easily get that

$$\zeta_3^1 = 0 \quad \text{on} \quad \partial\omega. \tag{6.4}$$

Note that, since the operator $(\boldsymbol{\mathcal{L}}^0, \boldsymbol{\mathcal{G}}^0)$ is the same for all functions θ, it is the same as for plates, and we have a splitting between the operator $(\mathcal{L}_t^0, \mathcal{L}_3^0; \mathcal{G}_t^0, \mathcal{G}_3^0)$ acting on (φ_t, φ_3) and the operator $(\mathcal{L}_s^0; \mathcal{G}_s^0)$ acting on φ_s. Hence, as for plates (see Proposition 4.4 in [11]) we can show that the boundary condition imposed to ζ_s^1 and the term φ_s^1 are:

$$\zeta_s^1\big|_{\partial\omega} = 0 \quad \text{and} \quad \varphi_s^1 = 0. \tag{6.5}$$

We recall the notations used in [11], viz.,

$$\overline{p}_1(x_3) = -\frac{\lambda}{\lambda + 2\mu}x_3, \quad \overline{p}_2(x_3) = \frac{\lambda}{2(\lambda + 2\mu)}\left(x_3^2 - \frac{1}{3}\right).$$

Then relation (4.28) can be written as $v_3^2 = \overline{p}_2\Delta\zeta_3^0 + \overline{p}_1(\operatorname{div}_*\zeta_*^0 - \Delta\theta\zeta_3^0)$. Following exactly the same computations as in the case of plates, we show that we have

$$\left\{\begin{array}{rcl} \zeta_n^1\big|_{\partial\omega} &=& c_1^1\operatorname{div}_*\zeta_*^0\big|_{\partial\omega} , \\ \partial_n\zeta_3^1\big|_{\partial\omega} &=& c_4^1\Delta_*\zeta_3^0\big|_{\partial\omega} , \\ \zeta_3^2\big|_{\partial\omega} &=& c_3^1\Delta_*\zeta_3^0\big|_{\partial\omega} , \end{array}\right. \tag{6.6}$$

where c_1^1 , c_4^1 , c_3^1 are the same constants as in [11]. We can also give an expression of the boundary layer terms φ_t^1 and φ_3^1 , but these are exactly the same as in equations (6.5) and (6.7) in [11].

Finally, using Theorem 4.5 and Proposition 6.1, we prove Theorem 3.1 by using classical energy estimates (see [9, 8]).

7. FREE SHALLOW SHELLS

In this case, we can find a basis $(\boldsymbol{Z}_{\mathrm{N}}^1, \boldsymbol{Z}_{\mathrm{N}}^2, \boldsymbol{Z}_{\mathrm{N}}^3, \boldsymbol{Z}_{\mathrm{N}}^4)$ of the space $\mathscr{Z}$ in Proposition 5.1, and moreover, give directly the expression of $\boldsymbol{Z}$ in this basis with respect to the right-hand sides $\boldsymbol{\Psi}$ and $\boldsymbol{v}$ (see [11, 8]). Hence, by doing the same computations as in the citated papers, we establish the following proposition:

Proposition 7.1. *Let* $\boldsymbol{\zeta}^k = (\zeta_*, \zeta_3)$ *be a family of generators satisfying the relations* $\boldsymbol{P}(\boldsymbol{\zeta}^k) = \boldsymbol{r}^k$ *of Theorem* 4.5*, and let* $\underline{\boldsymbol{u}}^k$ *be the displacement constructed in this Theorem. Then there exist* $\boldsymbol{g}^k = (g_n^k, g_s^k, g_3^k, g_m^k) \in (\mathscr{C}^\infty(\gamma))^4$ *depending only on* $\boldsymbol{f}$ *and on* $\boldsymbol{\zeta}^\ell$ *,* $0 \le \ell \le k-1$ *, such that, if* $\boldsymbol{\zeta}^k$ *satisfy conditions*

$$\big(B_n(\boldsymbol{\zeta}^k), B_s(\boldsymbol{\zeta}^k), N_n(\zeta_3^k), M_n(\zeta_3^k)\big)\big|_{\partial\omega} = \boldsymbol{g}^k \quad \textit{on} \quad \partial\omega,$$

then there exist boundary layer profiles $\boldsymbol{\varphi}^k$ *satisfying equations* (5.8)*. Moreover, conditions* (3.14) *are satisfied.*

Proof. The proof is the same as for plates (see [11, 8]). In order to prove that the condition (3.14) is satisfied at k-th order, we construct three-dimensional displacement satisfying the boundary conditions and the outer and inner equations up to the order k , and we use the compatibility conditions (2.11), see [8].

For $k = 0$, we find:

$$B_n(\boldsymbol{\zeta}^0) = B_s(\boldsymbol{\zeta}^0) = M_n(\zeta_3^0) = 0 \quad \text{and} \quad N_n(\zeta_3^0) = -\tfrac{1}{2}n_\gamma q^\gamma \quad \text{on} \quad \partial\omega. \tag{7.1}$$

Therefore, $\boldsymbol{\zeta}^0$ has to solve a two-dimensional problem of the type (3.12) for $\boldsymbol{r}^0 = \frac{1}{2}(p^\alpha, \partial_\alpha q^\alpha + p^3)$ and $\boldsymbol{g}^0 = (0, 0, -\frac{1}{2}n_\beta q^\beta, 0)$. Then the compatibility condition (3.14) has to be satisfied

$$\forall \boldsymbol{\eta} \in \mathcal{K}(\omega), \qquad \int_\omega p^\alpha\eta_\alpha + \int_\omega (\partial_\alpha q^\alpha + p^3)\eta_3 - \int_\gamma n_\beta q^\beta \eta_3 = 0. \tag{7.2}$$

Using Green's formula and the definitions of p^i and q^α , this compatibility condition becomes:

$$\forall \boldsymbol{\eta} \in \mathcal{K}(\omega), \qquad \int_\Omega f^i\eta_i - x_3 f^\alpha \partial_\alpha \eta_3 = 0. \tag{7.3}$$

It is enough to check that it holds for a basis in $\mathcal{K}(\omega)$, for example for the vectors (3.10). We have the following result:

Lemma 7.2. *Let* $\boldsymbol{v}_i^R(\varepsilon)$ *,* $i \in \{1, 2, \dots, 6\}$ *be a basis of* $\mathcal{R}(\varepsilon, \Omega)$ *. The following expansion holds*

$$\boldsymbol{v}_i^R(\varepsilon) \sim \sum_{k \geq 0} \boldsymbol{v}_i^{R;2k} \varepsilon^{2k}, \tag{7.4}$$

with

$$\boldsymbol{v}_1^{R;0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_2^{R;0} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_3^{R;0} = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_4^{R;0} = \begin{pmatrix} \partial_1\theta \\ \partial_2\theta \\ 1 \end{pmatrix},$$

$$\boldsymbol{v}_5^{R;0} = \begin{pmatrix} x_2\partial_1\theta \\ x_2\partial_2\theta - \theta - x_3 \\ x_2 \end{pmatrix}, \quad \boldsymbol{v}_6^{R;0} = \begin{pmatrix} x_1\partial_1\theta - \theta - x_3 \\ x_1\partial_2\theta \\ x_1 \end{pmatrix}.$$

Proof. The proof uses the rigid displacements lemma in curvilinear coordinates, see [5] and Proposition4.2. □

Therefore, the compatibility condition (7.3) for the two-dimensional basis (3.10) follows if we identify the coefficient of ε^0 in the three-dimensional compatibility condition (2.11) for the basis given by Lemma 7.2. □

Using energy estimate (exactly like for plates), we then obtain the result of Theorem 3.1.

REFERENCES

[1] S. Agmon, A. Douglis, L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. *Comm. Pure Appl. Math.* **17** (1964) 35–92.

[2] G. Andreoiu. Analyse des coques faiblement courbées. Thèse, Université Pierre et Marie Curie, Paris 1999.

[3] G. Andreoiu. Comparaison entre modèles bidimensionnels de coques faiblement courbées. *C. R. Acad. Sci. Paris, Sér. I* **329** (1999) 339–342.

[4] S. Busse, P. G. Ciarlet, B. Miara. Justification d'un modele lineaire bi-dimensionnel de coques "faiblement courbees" en coordonnees curvilignes. *RAIRO Modél. Math. Anal. Numer.* **31 (3)** (1997) 409–434.

[5] P. G. Ciarlet. *Mathematical Elasticity. Vol. III, Theory of Shells.* North-Holland, Amsterdam 2000.

[6] P. G. Ciarlet, B. Miara. Justification of the two-dimensional equations of a linearly elastic shallow shell. *Comm. Pure Appl. Math.* **45 (3)** (1992) 327–360.

[7] P. G. Ciarlet, J. C. Paumier. A justification of the marguerre-von- kármán equations. *Computational Mechanics* **1** (1986) 177–202.

[8] M. Dauge, , I. Djurdjevic, E. Faou, A. Rössle. Eigenmodes asymptotic in thin elastic plates. *J. Maths. Pures Appl.* **78** (1999) 925–964.

[9] M. Dauge, I. Gruais. Asymptotics of arbitrary order for a thin elastic clamped plate. I: Optimal error estimates. *Asymptotic Analysis* **13** (1996) 167–197.

[10] M. Dauge, I. Gruais. Asymptotics of arbitrary order for a thin elastic clamped plate. II: Analysis of the boundary layer terms. *Asymptotic Analysis* **16** (1998) 99–124.

[11] M. Dauge, I. Gruais, A. Rössle. The influence of lateral boundary conditions on the asymptotics in thin elastic plates. To appear in *SIAM Jour. of Math. Anal.* (1999).

[12] M. P. do Carmo. *Differential geometry of curves and surfaces.* Prentice Hall 1976.

[13] G. Duvaut, J.-L. Lions. *Les Inéquations en Mécanique et en Physique*. Dunod, Paris 1972.

[14] E. Faou. Elasticité linéarisée tridimensionnelle pour une coque mince : Résolution en série formelle en puissances de l'épaisseur. To appear in *C. R. Acad. Sc. Paris, Sér. I* (2000).

[15] R. D. Gregory, F. Y. Wan. Decaying states of plane strain in a semi-infinite strip and boundary conditions for plate theory. *J. Elasticity* **14** (1984) 27–64.

[16] W. T. Koiter. On the foundations of the linear theory of thin elastic shells: I. *Proc. Kon. Ned. Akad. Wetensch., Ser.B* **73** (1970) 169–182.

[17] V. G. Maz'ya, S. A. Nazarov, B. A. Plamenevskii. *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten II.* Mathematische Monographien, Band 83. Akademie Verlag, Berlin 1991.

[18] S. A. Nazarov, I. S. Zorin. Edge effect in the bending of a thin three-dimensional plate. *Prikl. Matem. Mekhan.* **53 (4)** (1989) 642–650. English translation *J. Appl. Maths. Mechs.* (1989) 500–507.

[19] J. Pitkäranta, A.-M. Matache, C. Schwab. Fourier mode analysis of layers in shallow shell deformations. *Preprint ETH Zürich.* (1999).

[20] J. Pitkäranta, E. Sanchez-Palencia. On the asymptotic behaviour of sensitive shells with small thickness. *C. R. Acad. Sci. Paris, Sér. II* **325** (1997) 127–134.

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