ENERGY CASCADES FOR NLS ON $\mathbb{T}^d$

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ABSTRACT. We consider the nonlinear Schrödinger equation with cubic (focusing or defocusing) nonlinearity on the multidimensional torus. For special small initial data containing only five modes, we exhibit a countable set of time layers in which arbitrarily large modes are created. The proof relies on a reduction to multiphase weakly nonlinear geometric optics, and on the study of a particular two-dimensional discrete dynamical system.

1. INTRODUCTION AND MAIN RESULT

We consider the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = \lambda |u|^2 u, \quad x \in \mathbb{T}^d, \tag{1.1}$$

with $d \geqslant 2$, where the sign of $\lambda \in \{-1, +1\}$ turns out to be irrelevant in the analysis below. In the present analysis, we are interested in the description of some energy exchanges between low and high frequencies for particular solutions of this equation. We will consider solutions with small initial values:

$$u(0, x) = \delta u_0(x), \tag{1.2}$$

where $u_0 \in H^1(\mathbb{T}^d)$ and $0 < \delta \ll 1$. Replacing u with $\delta^{-1}u$, (1.1)–(1.2) is equivalent to

$$i\partial_t u + \Delta u = \lambda\varepsilon |u|^2 u \quad ; \quad u(0, x) = u_0(x), \tag{1.3}$$

where $\varepsilon = \delta^2$. Viewed as an infinite dimensional dynamical system in terms of the Fourier variables of the solution, such an equation is *resonant* in the sense that all the eigenvalues of the Laplace operator are integers only, making possible nontrivial vanishing linear combinations between the frequencies of the linear unperturbed equation ($\varepsilon = 0$). In such a situation, nonlinear perturbation theory cannot be directly applied as in [1, 2, 3, 9, 10, 13, 15]. Let us recall that in all these works, the Laplace operator is slightly perturbed by a typical potential making resonances *generically* disappear. In such situations and when $u_0(x)$ is smooth enough, it is possible to prove the quasi preservation of the *actions* of the solution (the energy $|u_j|^2$ borne by each Fourier coefficient) over very long time: polynomial (of order ε^{-r} for all r) as in [2], exponentially large as in [13], or arbitrary large for a set of specific solutions as in [10].

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In the resonant case considered in this paper, there is *a priori* no reason to observe the long time preservation of the actions of the solution. Despite this fact, Eqn. (1.1) possesses many quasi-periodic solutions (see [4, 20]).

On the other hand, it has been recently shown in [8] that in the defocusing case (Eqn. (1.1) with $\lambda = 1$), solutions exist exhibiting energy transfers between low and high modes which in turn induce a growth in the Sobolev norm H^s with $s > 1$. Strikingly, such phenomenon arises despite the fact that L^2 and H^1 norms of the solution are bounded for all time.

Using a different approach complementary to this last work, the goal of the present analysis is to describe energy exchanges between the actions in the case of a particular explicit initial value $u_0(x)$ made of five low modes. We aim to describe these exchanges both *qualitatively* — description of the dynamics between the actions in a large but finite time interval — and *quantitatively* — determination of the time required to feed some specific modes up to a given threshold.

Since we work on $\mathbb{T}^d$, the solution u takes the form

$$u(t,x) = \sum_{j\in\mathbb{Z}^d} u_j(t)e^{ij\cdot x},$$

where $u_j(t) \in \mathbb{C}$ are the Fourier coefficients of the solution, and as long as t does not exceed the lifespan of u. Here, for $j = (j_1,\dots,j_d) \in \mathbb{Z}^d$ and $x = (x_1,\dots,x_d)$, we have $j\cdot x = j_1x_1+\dots+j_dx_d$. We also set $|j|^2 = j_1^2+\dots+j_d^2 \in \mathbb{N}$. Let us introduce the Wiener algebra W made of functions f on $\mathbb{T}^d$ of the form

$$f(x) = \sum_{j\in\mathbb{Z}^d} b_je^{ij\cdot x}$$

such that $(b_j)_{j\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$. With this space is associated the norm

$$\|f\|_W = \sum_{j\in\mathbb{Z}^d} |b_j|.$$

Our main result is the following:

Theorem 1.1. *Let $d \geqslant 2$, and $u_0 \in C^\infty(\mathbb{T}^d)$ given by*

$$u_0(x) = 1 + 2\cos x_1 + 2\cos x_2.$$

For $\lambda \in \{\pm 1\}$, the following holds. There exist $\varepsilon_0, T, C_0, C > 0$ and a family $(c_j)_{j\in\mathcal{N}_*}$, with $c_j \neq 0$ for all j, such that for $0 < \varepsilon \leqslant \varepsilon_0$, (1.3) *has a unique solution $u \in C([0,T/\varepsilon];W)$, and:*

$$\forall j \in \mathcal{N}_*,\ \forall t \in [0,T/\varepsilon], \quad \left|u_j(t) - c_j(\varepsilon t)^{|j|^2-1}\right| \leqslant (C_0\varepsilon t)^{|j|^2} + C\varepsilon,$$

where the set $\mathcal{N}_*$ is given by

$$\mathcal{N}_* = \{(0,\pm 2^p), (\pm 2^p, 0), (\pm 2^p, \pm 2^p), (\mp 2^p, \pm 2^p),\ p \in \mathbb{N}\} \times \{0_{\mathbb{Z}^{d-2}}\}. \tag{1.4}$$

Arbitrarily high modes appear with equal intensity along a cascade of time layers:

$$\forall\gamma\in]0,1[,\ \forall\theta<\frac{1}{4},\ \forall\alpha>0,\quad \exists\varepsilon_1\in]0,\varepsilon_0],\quad \forall\varepsilon\in]0,\varepsilon_1],$$
$$\forall j\in\mathcal{N}_*,\ |j|<\alpha\left(\log\frac{1}{\varepsilon}\right)^{\theta},\quad \left|u_j\left(\frac{2}{\varepsilon^{1-\gamma/(|j|^2-1)}}\right)\right|\geqslant\frac{\varepsilon^{\gamma}}{4}.$$

This result expresses the possibility of nonlinear exchanges in (1.3): while the high modes in the set $\mathcal{N}_*$ are equal to zero at time $t=0$, they are significantly large in a time that depends on the mode. As this time increases with the size of the mode, this is an *energy cascade* in the sense of [7]. To our knowledge, this result is the first one where such a dynamics is described so precisely as to quantify the time of ignition of different modes.

The proof of this theorem relies on the following ingredients:

- An approximation result showing that the analysis of the dynamics of (1.1) over a time of order $\mathcal{O}(1/\varepsilon)$ can be reduced to the study of an infinite dimensional system for the amplitudes of the Fourier coefficients u_j (in a geometric optics framework). Let us mention that this reduced system exactly corresponds to the *resonant normal form* system obtained after a first order Birkhoff reduction (see [17] for the one dimensional case). We detail this connection between geometric optics and normal forms in the second section.
- A careful study of the dynamics of the reduced system. Here we use the particular structure of the initial value which consists of five modes generating infinitely many new frequencies through the resonances interactions in the reduced system. A Taylor expansion (in the spirit of [5]) then shows how all the frequencies should be *a priori* turned on in finite time. The particular geometry of the energy repartition between the frequencies then makes possible to estimate precisely the evolution of the particular points of the set $\mathcal{N}_*$ in (1.4) and to quantify the energy exchanges between them.

The construction above is very different from the one in [8]. Let us mention that it is also valid only up to a time of order $\mathcal{O}(1/\varepsilon)$ (which explains the absence of difference between the focusing and defocusing cases). After this time, the nature of the dynamics should change completely as all the frequencies of the solution would be significantly present in the system, and the nature of the nonlinearity should become relevant.

Remark 1.2. The proof that we present below remains valid when $\lambda\in\mathbb{C}\setminus\mathbb{R}$, a case where (1.1) has no Hamiltonian structure. In particular, for $\lambda=-i$, (1.1) is dissipative: the L^2-norm of u is a non-increasing function of time. This point of view suggests that our approach does not make it possible to deduce any information concerning the growth of (high order) Sobolev norms of u for large time.

2. An approximation result in geometric optics

For a given element $(\alpha_j)_{j\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$, we define the following infinite dimensional *resonant system*

$$i\dot{a}_j = \lambda \sum_{(k,\ell,m)\in I_j} a_k \overline{a}_\ell a_m \quad ; \quad a_j(0) = \alpha_j. \tag{2.1}$$

where I_j is the set of *resonant* indices (see [17]) associated with j defined by:

$$I_j = \left\{ (k,\ell,m) \in \mathbb{Z}^{3d} \mid j = k-\ell+m, \text{ and } |j|^2 = |k|^2 - |\ell|^2 + |m|^2 \right\}. \tag{2.2}$$

With these notations, we have the following result:

Proposition 2.1. *Let $u_0(x) \in W$ and $(\alpha_j)_{j\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$ its Fourier coefficients. There exists $T > 0$ and a unique analytic solution $(a_j)_{j\in\mathbb{Z}^d} : [0,T] \to \ell^1(\mathbb{Z}^d)$ to the system* (2.1)*. Moreover, there exists $\varepsilon_0(T) > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0(T)$, the exact solution to* (1.3) *satisfies $u \in C([0, \frac{T}{\varepsilon}]; W)$ and there exists C independent of $\varepsilon \in]0, \varepsilon_0(T)]$ such that*

$$\sup_{0\leqslant t\leqslant \frac{T}{\varepsilon}} \|u - v^\varepsilon(t)\|_W \leqslant C\varepsilon.$$

where

$$v^\varepsilon(t,x) = \sum_{j\in\mathbb{Z}^d} a_j(\varepsilon t) e^{ij\cdot x - it|j|^2}. \tag{2.3}$$

Remark 2.2. Even though it is not emphasized in the notation, the function u obviously depends on ε, which is present in (1.3).

We give below a (complete but short) proof of this result using geometric optics. Let us mention however that we can also prove this proposition using a Birkhoff transformation of (1.3) in *resonant normal form* as in [17]. We give some details below. There is also an obvious connection with the *modulated Fourier expansion* framework developed in [14] in the non-resonant case (see also [19]).

2.1. Solution of the resonant system.

The first part of Proposition 2.1 is a consequence of the following result:

Lemma 2.3. *Let $\alpha = (\alpha_j)_{j\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d)$. There exists $T > 0$ and a unique analytic solution $(a_j)_{j\in\mathbb{Z}^d} : [0,T] \to \ell^1(\mathbb{Z}^d)$ to the system* (2.1)*. Moreover, there exists constants M and R such that for all $n \in \mathbb{N}$ and all $s \leqslant T$,*

$$\forall\, j \in \mathbb{Z}^d, \quad \left| \frac{\mathrm{d}^n a_j}{\mathrm{d}t^n}(s) \right| \leqslant M R^n n! \tag{2.4}$$

Proof. In [6], the existence of a time T_1 and continuity in time of the solution $a(t) = (a_j(t))_{j\in\mathbb{Z}^d}$ in ℓ^1 is proved. As ℓ^1 is an algebra, a bootstrap argument shows that $a(t) \in C^\infty\left([0,T_1]; \ell^1(\mathbb{Z}^d)\right)$. From (2.1) we immediately obtain for $s \in [0, T_1]$,

$$\|\dot{a}(s)\|_{\ell^1} \leqslant 3\|a(s)\|^3_{\ell^1},$$

and by induction

$$\left\|a^{(n)}(s)\right\|_{\ell^1} \leqslant 3 \cdot 5 \cdot \dots \cdot (2n+1)\|a(s)\|_{\ell^1}^{2n+1},$$

where $a^{(n)}(t)$ denote the n-th derivative of $a(t)$ with respect to time. This implies

$$\left\|a^{(n)}(0)\right\|_{\ell^1} \leqslant 3 \cdot 5 \cdot \dots \cdot (2n+1)\|\alpha\|_{\ell^1}^{2n+1} \leqslant \|\alpha\|_{\ell^1} n! \left(3\|\alpha\|_{\ell^1}^2\right)^n,$$

which shows the analyticity of a for $t \leqslant T_2 = \frac{1}{6}\|\alpha\|_{\ell^1}^{-2}$. The estimate (2.4) is then a standard consequence of Cauchy estimates applied to the complex power series $\sum_{n\in\mathbb{N}} \frac{1}{n!} a^{(n)}(0) z^n$ defined in the ball $B(0, 2T)$ where $2T = \min(T_1, T_2)$. □

2.2. **Geometric optics.**

Let us introduce the scaling

$$\mathtt{t} = \varepsilon t, \quad \mathtt{x} = \varepsilon x, \quad u(t,x) = \mathtt{u}^\varepsilon(\varepsilon t, \varepsilon x). \tag{2.5}$$

Then (1.3) is *equivalent* to:

$$i\varepsilon\partial_{\mathtt{t}}\mathtt{u}^\varepsilon + \varepsilon^2\Delta\mathtt{u}^\varepsilon = \lambda\varepsilon|\mathtt{u}^\varepsilon|^2\mathtt{u}^\varepsilon \quad ; \quad \mathtt{u}^\varepsilon(0,\mathtt{x}) = u_0\left(\frac{\mathtt{x}}{\varepsilon}\right) = \sum_{j\in\mathbb{Z}^d} \alpha_j e^{ij\cdot\mathtt{x}/\varepsilon}. \tag{2.6}$$

In the limit $\varepsilon \to 0$, multiphase geometric optics provides an approximate solution for (2.6). The presence of the factor ε in front of the nonlinearity has two consequences: in the asymptotic regime $\varepsilon \to 0$, the eikonal equation is the same as in the linear case $\lambda = 0$, but the transport equation describing the evolution of the amplitude is nonlinear. This explains why this framework is referred to as *weakly nonlinear geometric optics*. Note that simplifying by ε in the Schrödinger equation (2.6), we can view the limit $\varepsilon \to 0$ as a small dispersion limit, as in e.g. [16].

We sketch the approach described more precisely in [6]. The approximate solution provided by geometric optics has the form

$$\mathtt{v}^\varepsilon(\mathtt{t},\mathtt{x}) = \sum_{j\in\mathbb{Z}^d} a_j(\mathtt{t}) e^{\phi_j(\mathtt{t},\mathtt{x})/\varepsilon}, \tag{2.7}$$

where we demand $\mathtt{u}^\varepsilon = \mathtt{v}^\varepsilon$ at time $\mathtt{t} = 0$, that is

$$a_j(0) = \alpha_j \quad ; \quad \phi_j(0,\mathtt{x}) = j \cdot \mathtt{x}.$$

Plugging this ansatz into (2.6) and ordering the powers of ε, we find, for the $\mathcal{O}(\varepsilon^0)$ term:

$$\partial_{\mathtt{t}}\phi_j + |\nabla\phi_j|^2 = 0 \quad ; \quad \phi_j(0,\mathtt{x}) = j \cdot \mathtt{x}.$$

The solution is given explicitly by

$$\phi_j(\mathtt{t},\mathtt{x}) = j \cdot \mathtt{x} - \mathtt{t}|j|^2. \tag{2.8}$$

The amplitude a_j is given by the $\mathcal{O}(\varepsilon^1)$ and is given by equation (2.1) after projecting the wave along the oscillation $e^{i\phi_j/\varepsilon}$ according to the the set of *resonant* phases given by I_j (Eqn. (2.2)). By doing so, we have dropped the oscillations of the form $e^{i(k\cdot\mathtt{x}-\omega\mathtt{t})/\varepsilon}$ with $\omega \neq |k|^2$, generated by nonlinear interaction: the phase $k\cdot\mathtt{x} - \omega\mathtt{t}$ does not solve the eikonal equation, and the corresponding term is negligible in the limit $\varepsilon \to 0$ thanks to a non-stationary phase argument.

Proposition 2.1 is a simple corollary of the following result that is established in [6]. We sketch the proof in Appendix A.

Proposition 2.4. *Let* $(\alpha_j) \in \ell^1(\mathbb{Z}^d)$, *and* $\mathtt{v}^\varepsilon$ *be defined by* (2.7) *and* (2.8). *Then there exists* $\varepsilon_0(T) > 0$ *such that for* $0 < \varepsilon \leqslant \varepsilon_0(T)$, *the exact solution to* (2.6) *satisfies* $\mathtt{u}^\varepsilon \in C([0,T];W)$, *where* T *is given by Lemma 2.3. In addition,* $\mathtt{v}^\varepsilon$ *approximates* $\mathtt{u}^\varepsilon$ *up to* $\mathcal{O}(\varepsilon)$: *there exists* C *independent of* $\varepsilon \in]0, \varepsilon_0(T)]$ *such that*

$$\sup_{0\leqslant t\leqslant T} \|\mathtt{u}^\varepsilon(t) - \mathtt{v}^\varepsilon(t)\|_W \leqslant C\varepsilon.$$

2.3. **Link with normal forms.** Viewed as an infinite dimensional Hamiltonian system, (1.3) can also be interpreted as the equation associated with the Hamiltonian

$$H^\varepsilon(u,\bar u) = H_0 + \varepsilon P := \sum_{j\in\mathbb{Z}^d} |j|^2|u_j|^2 + \varepsilon\frac{\lambda}{2}\sum_{k+m=j+\ell} u_k u_m \bar u_\ell \bar u_j.$$

that is $iu_j = \partial_{\bar u_j} H(u,\bar u)$, see for instance the presentations in [1, 15] and [13].

In this setting, the Birkhoff normal form approach consists in searching a transformation $\tau(u) = u + \mathcal{O}(\varepsilon u^3)$ close to the identity over bounded set in the Wiener algebra W, and such that in the new variable $v = \tau(u)$, the Hamiltonian $K^\varepsilon(v,\bar v) = H^\varepsilon(u,\bar u)$ takes the form $K^\varepsilon = H_0 + \varepsilon Z + \varepsilon^2 R$ where Z is expected to be as simple as possible and $R = \mathcal{O}(u^6)$. Searching τ as the time $t = \varepsilon$ flow of an unknown Hamiltonian χ, we are led to solving the *homological* equation

$$\{H_0, \chi\} + Z = P,$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the Poisson bracket of the underlying (complex) Hamiltonian structure. Now with unknown Hamiltonians $\chi(u,\bar u) = \sum_{k,m,j,\ell} \chi_{km\ell j} u_k u_m \bar u_\ell \bar u_j$ and $Z(u,\bar u) = \sum_{k,m,j,\ell} Z_{km\ell j} u_k u_m \bar u_\ell \bar u_j$, the previous relation can be written

$$(|k|^2 + |m|^2 - |j|^2 - |\ell|^2)\chi_{km\ell j} + Z_{km\ell j} = P_{km\ell j},$$

where

$$P_{km\ell j} = \begin{cases} 1 & \text{if} \quad k+m-j-\ell = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The solvability of this equation relies precisely on the resonant relation $|k|^2 + |m|^2 = |j|^2 + |\ell|^2$: For non resonant indices, we can solve for $\chi_{km\ell j}$ and set $Z_{km\ell j} = 0$, while for resonant indices, we must take $Z_{km\ell j} = P_{km\ell j}$. Note that here there is no small divisors issues, as the denominator is always an integer (0 or greater than 1).

Hence we see that up to $\mathcal{O}(\varepsilon^2)$ terms as long as the solution remains bounded in W, the dynamics in the new variables will be close to the dynamics associated with the Hamiltonian

$$K_1^\varepsilon(u,\bar u) = H_0 + \varepsilon Z := \sum_{j\in\mathbb{Z}^d} |j|^2|u_j|^2 + \varepsilon\frac{\lambda}{2}\sum_{\substack{k+m=j+\ell\\ |k|^2+|m|^2=|j|^2+|\ell|^2}} u_k u_m \bar u_\ell \bar u_j.$$

At this point, let us observe that H_0 and Z commute: $\{H_0, Z\} = 0$, and hence the dynamics of K_1^ε is the simple superposition of the dynamics of H_0 (the phase oscillation (2.8)) to the dynamics of εZ (the resonant system (2.1)). Hence we easily calculate that $v^\varepsilon(t,x)$ defined in (2.3) is the exact solution of the Hamiltonian K_1^ε. In other words, it is the solution of the *first resonant normal form* of the system (1.3).

The approximation result can then easily be proved using estimates on the remainder terms (that can be controlled in the Wiener algebra, see [13]), in combination with Lemma 2.3 which ensures the stability of the solution of K_1^ε and a uniform bound in the Wiener algebra over a time of order $1/\varepsilon$.

3. An iterative approach

We now turn to the analysis of the resonant system (2.1). The main remark for the forthcoming analysis is that new modes can be generated by nonlinear interaction: we may have $a_j \neq 0$ even though $\alpha_j = 0$. We shall view this phenomenon from a dynamical point of view. As a first step, we recall the description of the sets of resonant phases, established in [8] in the case $d = 2$ (the argument remains the same for $d \geqslant 2$, see [6]):

Lemma 3.1. *Let $j \in \mathbb{Z}^d$. Then, $(k, \ell, m) \in I_j$ precisely when the endpoints of the vectors k, ℓ, m, j form four corners of a non-degenerate rectangle with ℓ and j opposing each other, or when this quadruplet corresponds to one of the two following degenerate cases: $(k = j, m = \ell)$, or $(k = \ell, m = j)$.*

As a matter of fact, (the second part of) this lemma remains true in the one-dimensional case $d = 1$. A specifity of that case, though, is that the associated transport equations show that no mode can actually be created [6]. The reason is that Lemma 3.1 implies that when $d = 1$, (2.1) takes the form $i\dot{a}_j = M_j a_j$ for some (smooth and real-valued) function M_j whose exact value is unimportant: if $a_j(0) = 0$, then $a_j(t) = 0$ for all t. In the present paper, on the contrary, we examine precisely the appearance of new modes.

Introduce the set of initial modes:

$$J_0 = \{j \in \mathbb{Z}^d \mid \ \alpha_j \neq 0\}.$$

In view of (2.1), modes which appear after one iteration of Lemma 3.1 are given by:

$$J_1 = \{j \in \mathbb{Z}^d \setminus J_0 \mid \ \dot{a}_j(0) \neq 0\}.$$

One may also think of J_1 in terms of Picard iteration. Plugging the initial modes (from J_0) into the nonlinear Duhamel's term and passing to the limit $\varepsilon \to 0$, J_1 corresponds to the new modes resulting from this manipulation. More generally, modes appearing after k iterations exactly are characterized by:

$$J_k = \left\{ j \in \mathbb{Z}^d \setminus \bigcup_{\ell=0}^{j-1} J_\ell \mid \ \frac{\mathrm{d}^k}{\mathrm{d}t^k} a_j(0) \neq 0 \right\}.$$

4. A PARTICULAR DYNAMICAL SYSTEM

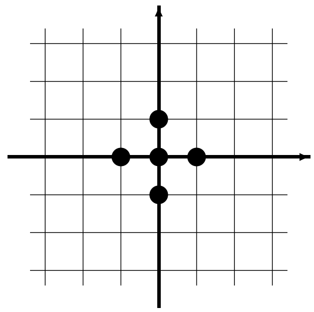
We consider the initial datum

$$u_0(x) = 1 + 2\cos x_1 + 2\cos x_2 = 1 + e^{ix_1} + e^{-ix_1} + e^{ix_2} + e^{-ix_2}. \tag{4.1}$$

The corresponding set of initial modes is given by

$$J_0 = \{(0,0),(1,0),(-1,0),(0,1),(0,-1)\} \times \{0_{\mathbb{Z}^{d-2}}\}.$$

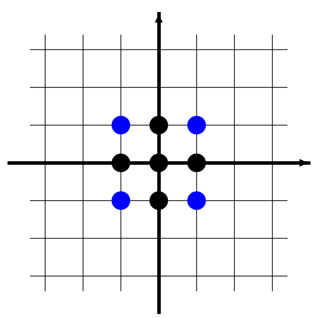
It is represented on the following figure:



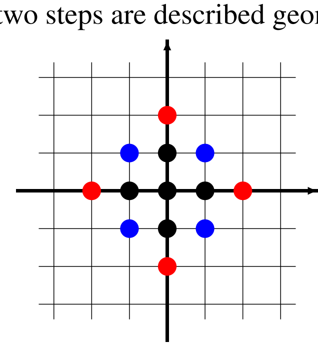
In view of Lemma 3.1, the generation of modes affects only the first two coordinates: the dynamical system that we study is two-dimensional, and we choose to drop out the last $d-2$ coordinates in the sequel, implicitly equal to $0_{\mathbb{Z}^{d-2}}$. After one iteration of Lemma 3.1, four points appear:

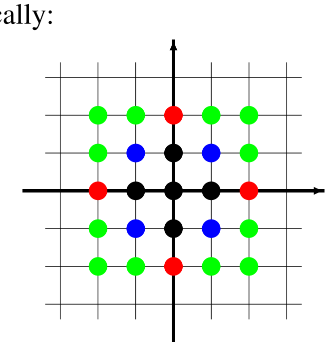
$$J_1 = \{(1,1),(1,-1),(-1,-1),(-1,1)\},$$

as plotted below.



The next two steps are described geometrically:





As suggested by these illustrations, we can prove by induction:

Lemma 4.1. *Let* $p \in \mathbb{N}$.

- *The set of relevant modes after* $2p$ *iterations is the square of length* 2^p *whose diagonals are parallel to the axes:*
$$\mathcal{N}^{(2p)} := \bigcup_{\ell=0}^{2p} J_\ell = \{(j_1, j_2) \mid |j_1| + |j_2| \leqslant 2^p\}.$$
- *The set of relevant modes after* $2p+1$ *iterations is the square of length* 2^{p+1} *whose sides are parallel to the axes:*
$$\mathcal{N}^{(2p+1)} := \bigcup_{\ell=0}^{2p+1} J_\ell = \{(j_1, j_2) \mid \max(|j_1|, |j_2|) \leqslant 2^p\}.$$

After an infinite number of iterations, the whole lattice $\mathbb{Z}^2$ is generated:
$$\bigcup_{k \geqslant 0} \mathcal{N}^{(k)} = \mathbb{Z}^2 \times \{0_{\mathbb{Z}^{d-2}}\}.$$
Among these sets, our interest will focus on *extremal modes*: for $p \in \mathbb{N}$,
$$\mathcal{N}_*^{(2p)} := \{(j_1, j_2) \in \{(0, \pm 2^p), (\pm 2^p, 0)\}\},$$
$$\mathcal{N}_*^{(2p+1)} := \{(j_1, j_2) \in \{(\pm 2^p, \pm 2^p), (\mp 2^p, \pm 2^p)\}\}.$$
These sets correspond to the edges of the squares obtained successively by iteration of Lemma 3.1 on J_0. The set $\mathcal{N}_*$ defined in Theorem 1.1 corresponds to
$$\mathcal{N}_* = \bigcup_{k \geqslant 0} \mathcal{N}_*^{(k)}.$$
The important property associated to these extremal points is that they are generated in a unique fashion:

Lemma 4.2. *Let* $n \geqslant 1$, *and* $j \in \mathcal{N}_*^{(n)}$. *There exists a unique pair* $(k, m) \in \mathcal{N}^{(n-1)} \times \mathcal{N}^{(n-1)}$ *such that* j *is generated by the interaction of the modes* 0, k *and* m, *up to the permutation of* k *and* m. *More precisely,* k *and* m *are extremal points generated at the previous step:* $k, m \in \mathcal{N}_*^{(n-1)}$.

Note however that points in $\mathcal{N}_*^{(n)}$ are generated in a non-unique fashion by the interaction of modes in $\mathbb{Z}^d$. For instance, $(1,1) \in J_1$ is generated after one step only by the interaction of $(0,0)$, $(1,0)$ and $(0,1)$. On the other hand, we see that after two iterations, $(1,1)$ is fed also by the interaction of the other three points in $\mathcal{N}_*^{(1)}$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$. After three iterations, there are even more three waves interactions affecting $(1,1)$.

Remark 4.3. According to the numerical experiment performed in the last section, it seems that *all* modes — and not only the extremal ones — receive some energy in the time interval $[0, T/\varepsilon]$. However the dynamics for the other modes is much more complicated to understand, as non extremal points of $\mathcal{N}^{(n+1)}$ are in general generated by several triplets of points in $\mathcal{N}^{(n)}$.

5. Proof of Theorem 1.1

Since the first part of Theorem 1.1 has been established at the end of §2, we now focus our attention on the estimates announced in Theorem 1.1.

In view of the geometric analysis of the previous section, we will show that can compute the first non-zero term in the Taylor expansion of solution $a_m(t)$ of (2.1) at $t=0$, for $m \in \mathcal{N}_*$. Let $n \geqslant 1$ and $j \in \mathcal{N}_*^{(n)}$. Note that since we have considered initial coefficients which are all equal to one — see (4.1) — and because of the symmetry in (2.1), the coefficients $a_j(t)$ do not depend on $j \in \mathcal{N}_*^{(n)}$ but only on n.

Hence we have

$$a_j(t) = \frac{t^{\alpha(n)}}{\alpha(n)!}\frac{\mathrm{d}^{\alpha(n)}a_j}{\mathrm{d}t^{\alpha(n)}}(0) + \frac{t^{\alpha(n)+1}}{\alpha(n)!}\int_0^1 (1-\theta)^{\alpha(n)}\frac{\mathrm{d}^{\alpha(n)+1}a_j}{\mathrm{d}t^{\alpha(n)+1}}(\theta t)d\theta,$$

for some $\alpha(n) \in \mathbb{N}$ still to be determined.

First, Eqn. (2.4) in Lemma 2.3 ensures that there exists $C_0 > 0$ *independent of* j *and* n such that

$$r_j(t) = \frac{t^{\alpha(n)+1}}{\alpha(n)!}\int_0^1 (1-\theta)^{\alpha(n)}\frac{d^{\alpha(n)+1}a_j}{dt^{\alpha(n)+1}}(\theta t)d\theta$$

satisfies

$$|r_j(t)| \leqslant (C_0 t)^{\alpha(n)+1}. \tag{5.1}$$

Next, we write

$$a_j(t) = c(n)t^{\alpha(n)} + r_j(t), \tag{5.2}$$

and we determine $c(n)$ and $\alpha(n)$ thanks to the iterative approach analyzed in the previous paragraph. In view of Lemma 4.2, we have

$$i\dot{a}_j = 2\lambda c(n-1)^2 t^{2\alpha(n-1)} + \mathcal{O}\left(t^{2\alpha(n-1)+1}\right),$$

where the factor 2 accounts for the fact that the vectors k and m can be exchanged in Lemma 4.2. We infer the relations:

$$\alpha(n) = 2\alpha(n-1)+1 \quad ; \quad \alpha(0)=0.$$

$$c(n) = -2i\lambda\frac{c(n-1)^2}{2\alpha(n-1)+1} \quad ; \quad c(0)=1.$$

We first derive

$$\alpha(n) = 2^n - 1.$$

We can then compute, $c(1) = -2i\lambda$, and for $n \geqslant 1$:

$$c(n+1) = i\frac{(2\lambda)^{\sum_{k=0}^n 2^k}}{\prod_{k=1}^{n+1}(2^k-1)^{2^{n+1-k}}} = i\frac{(2\lambda)^{2^{n+1}-1}}{\prod_{k=1}^{n+1}(2^k-1)^{2^{n+1-k}}}.$$

We can then infer the first estimate of Theorem 1.1: by Proposition 2.4, there exists C independent of j and ε such that for $0 < \varepsilon \leqslant \varepsilon_0(T)$,

$$|u_j(t) - a_j(\varepsilon t)| \leqslant C\varepsilon, \quad 0 \leqslant t \leqslant \frac{T}{\varepsilon}.$$

We notice that since for $j \in \mathcal{N}_*^{(n)}$, $|j| = 2^{n/2}$, regardless of the parity of n, we have $\alpha(n) = |j|^2 - 1$. For $j \in \mathcal{N}_*$, we then use (5.2) and (5.1), and the estimate follows, with $c_j = c(n)$.

To prove the last estimate of Theorem 1.1, we must examine more closely the behavior of $c(n)$. Since $2^k - 1 \leqslant 2^k$ for $k \geqslant 1$, we have the estimate

$$|c(n+1)| \geqslant \frac{2^{2^{n+1}-1}}{2^{\sum_{k=1}^{n+1} k 2^{n+1-k}}}.$$

Introducing the function

$$f_{n+1}(x) = \sum_{k=1}^{n+1} x^k = \frac{1-x^{n+1}}{1-x}x, \quad x \neq 1,$$

we have

$$\sum_{k=1}^{n+1} k 2^{n+1-k} = 2^n f_{n+1}'\left(\frac{1}{2}\right) = 2^{n+2} - n - 3,$$

and the (rough) bound

$$|c(n+1)| \geqslant 2^{2^{n+1}-1-2^{n+2}+n+3} = 2^{-2^{n+1}+n+2} \geqslant 2^{-2^{n+1}}.$$

We can now gather all the estimates together:

$$\begin{aligned} |u_j(t)| &\geqslant \left| c(n) \, (\varepsilon t)^{\alpha(n)} \right| - (C_0 \varepsilon t)^{\alpha(n)+1} - C\varepsilon \\ &\geqslant \frac{1}{2}\left(\frac{\varepsilon t}{2}\right)^{2^n-1} - (C_0\varepsilon t)^{2^n} - C\varepsilon \\ &\geqslant \frac{1}{2}\left(\frac{\varepsilon t}{2}\right)^{2^n-1} \left(1 - (2C_0)^{2^n} \varepsilon t\right) - C\varepsilon. \end{aligned} \tag{5.3}$$

To conclude, we simply consider t such that

$$\left(\frac{\varepsilon t}{2}\right)^{2^n-1} = \varepsilon^\gamma, \text{ that is } t = \frac{2}{\varepsilon^{1-\gamma/\alpha(n)}}. \tag{5.4}$$

Hence for the time t given in (5.4), since $\alpha(n) = |j|^2 - 1$, we have

$$\begin{aligned} (2C_0)^{2^n} \varepsilon t &= (2C_0)^{|j|^2} \varepsilon^{\gamma/(|j|^2-1)} \\ &= \exp\left(|j|^2 \log(2C_0) - \frac{\gamma}{|j|^2-1} \log\left(\frac{1}{\varepsilon}\right)\right). \end{aligned}$$

Assuming the spectral localization

$$|j| \leqslant \alpha \left(\log \frac{1}{\varepsilon}\right)^\theta,$$

we get for ε small enough

$$(2C_0)^{2^n} \varepsilon t \leqslant \exp\left(\alpha^2 \left(\log\frac{1}{\varepsilon}\right)^{2\theta} \log(2C_0) - \frac{\gamma}{\alpha^2}\left(\log \frac{1}{\varepsilon}\right)^{1-2\theta}\right).$$

The argument of the exponential goes to $-\infty$ as $\varepsilon \to 0$ provided that

$$\gamma > 0 \quad \text{and} \quad \theta < \frac{1}{4},$$

in which case we have $1 - (2C_0)^{2^n}\varepsilon t > 3/4$ for ε sufficiently small. Inequality (5.3) then yields the result, owing to the fact that $C\varepsilon$ is negligible compared to ε^γ when $0 \leqslant \gamma < 1$.

Finally, we note that the choice (5.4) is consistent with $\varepsilon t \in [0, T]$ for $\varepsilon \leqslant \varepsilon_0$, for some $\varepsilon_0 > 0$ uniform in j satisfying the above spectral localization, since

$$\varepsilon^{\gamma/\alpha(n)} = e^{-\frac{\gamma}{\alpha(n)}\log\frac{1}{\varepsilon}} \leqslant \exp\left(-\frac{\gamma}{\alpha^2}\left(\log\frac{1}{\varepsilon}\right)^{1-2\theta}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

6. Numerical illustration

We consider the equation (1.1) in the defocusing case ($\lambda = 1$) on the two-dimensional torus, $d = 2$. We take $u(0,x) = \delta(1 + 2\cos(x_1) + 2\cos(x_2))$ with $\delta = 0.0158$. With the previous notations, this corresponds to $1/\varepsilon = \delta^{-2} \simeq 4.10^3$. In Figure 1, we plot the evolution of the logarithms of the Fourier modes $\log|u_j(t)|$ for $j = (0, n)$, with $n = 0, \dots, 15$. We observe the energy exchanges between the modes. Note that all the modes (and not only the extremal modes in the set $\mathcal{N}_*$) gain some energy, but that after some time there is a stabilization effect (all the modes are turned on) and the energy exchanges are less significant.

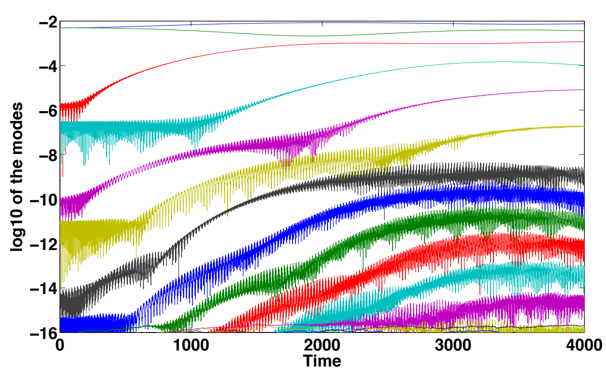


FIGURE 1. Evolution of the Fourier modes of the resonant solution in logarithmic scale: $u_0(x) = 1 + 2\cos(x_1) + 2\cos(x_2)$

In contrast, we plot in Figure 2 the solution corresponding to the initial value $u(0,x) = \delta(2\cos(x_1) + 2\cos(x_2))$ with the same δ. In this situation, no energy exchanges are observed after a relatively long time. Note that in this case, the initial data is made of the 4 modes $\{j \in \mathbb{Z}^2 \,||j| = 1\}$ forming a square in $\mathbb{Z}^2$. This set is closed for the resonance relation, so no energy exchange is expected in the time scale $\mathcal{O}(1/\varepsilon)$. We notice that the solution of (2.1) is given explicitly by $a_j(t) = 0$ for $|j| \neq 1$, and $a_j(t) = \exp(9it)$ for $|j| = 1$.

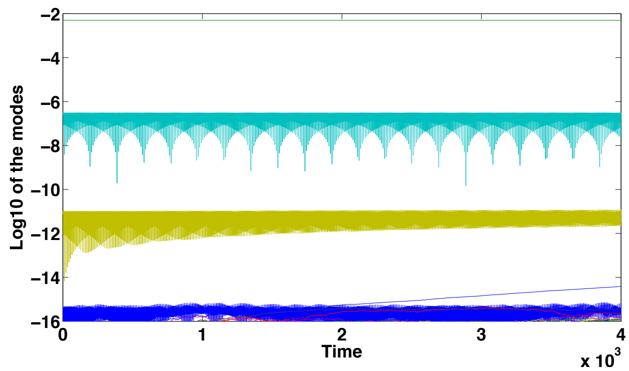


FIGURE 2. Evolution of the Fourier modes of the nonresonant solution in logarithmic scale: $u_0(x) = 2\cos(x_1) + 2\cos(x_2)$

The numerical scheme is a splitting time integrator based on the decomposition between the Laplace operator and the nonlinearity in combination with a Fourier pseudo-spectral collocation method (see for instance [18] and [11, Chap IV] for convergence results in the case of (1.1)). While the Laplace operator part $i\partial_t u = -\Delta u$ can be integrated exactly in Fourier, the solution of the nonlinear part $i\partial_t u = |u|^2 u$ starting in $v(x)$ is given explicitly by the formula $u(t,x) = \exp(-it|v(x)|^2)v(x)$. The fast Fourier transform algorithm allows an easy implementation of the algorithm. The stepsize used is $\tau = 0.001$ and a 128×128 grid is used.

Note that using the framework of [12, 11], we can prove that the numerical solution can be interpreted as the exact solution of a modified Hamiltonian of the form

$$\sum_{j \in B_K} |j|^2 |u_j|^2 + \frac{\lambda}{2} \sum_{\substack{(k,m,\ell,j) \in B_K \\ k+m-\ell-j \in K\mathbb{Z}^2}} \frac{i\tau\omega_{km\ell j}}{\exp(i\tau\omega_{km\ell j}) - 1} u_k u_m \bar{u}_\ell \bar{u}_j + \mathcal{O}(\tau),$$

where $\omega_{km\ell j} = |k|^2 + |m|^2 - |\ell|^2 - |j|^2$ and B_K the grid of frequencies $j = (j_1, j_2)$ such that j_1 and j_2 are less than $K/2 = 64$. Note that this energy is well defined as $\tau\omega_{km\ell j}$ is never a multiple of 2π, and that the frequencies of the linear operator of this modified energy carry on the same resonance relations (at least for relatively low modes). This partly explains why the cascade effect due to the resonant frequencies should be correctly reproduced by the numerical simulations.

APPENDIX A. SKETCH OF THE PROOF OF PROPOSITION 2.4

By construction, the approximate solution v^ε solves

$$i\varepsilon\partial_t \mathrm{v}^\varepsilon + \varepsilon^2 \Delta \mathrm{v}^\varepsilon = \lambda\varepsilon|\mathrm{v}^\varepsilon|^2 \mathrm{v}^\varepsilon + \lambda\varepsilon r^\varepsilon,$$

where the source term r^ε correspond to non-resonant interaction terms which have been discarded:

$$r^\varepsilon(t,x) = \sum_{j\in\mathbb{Z}^d}\sum_{(k,\ell,m)\notin I_j} a_k(t)\overline{a}_\ell(t)a_m(t)e^{i(\phi_k(t,x)-\phi_\ell(t,x)+\phi_m(t,x))/\varepsilon}.$$

We write

$$\phi_k(t,x)-\phi_\ell(t,x)+\phi_m(t,x) = \kappa_{k,\ell,m}\cdot x - \omega_{k,\ell,m}t,$$

with $\kappa_{k,\ell,m}\in\mathbb{Z}^d$, $\omega_{k,\ell,m}\in\mathbb{Z}$ and $|\kappa_{k,\ell,m}|^2\neq\omega_{k,\ell,m}$, hence

$$\big||\kappa_{k,\ell,m}|^2-\omega_{k,\ell,m}\big| \geqslant 1. \tag{A.1}$$

The error term $\mathtt{w}^\varepsilon = \mathtt{u}^\varepsilon - \mathtt{v}^\varepsilon$ solves

$$i\varepsilon\partial_t \mathtt{w}^\varepsilon + \varepsilon^2\Delta \mathtt{w}^\varepsilon = \lambda\varepsilon\left(|\mathtt{w}^\varepsilon+\mathtt{v}^\varepsilon|^2\left(\mathtt{w}^\varepsilon+\mathtt{v}^\varepsilon\right) - |\mathtt{v}^\varepsilon|^2\mathtt{v}^\varepsilon\right) - \lambda\varepsilon r^\varepsilon \quad ; \quad \mathtt{w}^\varepsilon_{|t=0}=0.$$

By Duhamel's principle, this can be recasted as

$$\begin{aligned}\mathtt{w}^\varepsilon(t) = &-i\lambda\int_0^t e^{i\varepsilon(t-s)\Delta}\left(|\mathtt{w}^\varepsilon+\mathtt{v}^\varepsilon|^2\left(\mathtt{w}^\varepsilon+\mathtt{v}^\varepsilon\right)-|\mathtt{v}^\varepsilon|^2\mathtt{v}^\varepsilon\right)(s)ds\\ &+i\lambda\int_0^t e^{i\varepsilon(t-s)\Delta}r^\varepsilon(s)ds.\end{aligned}$$

Denote

$$R^\varepsilon(t) = \int_0^t e^{i\varepsilon(t-s)\Delta}r^\varepsilon(s)ds.$$

Since W is an algebra, and the norm in W controls the L^∞-norm, it suffices to prove

$$\|R^\varepsilon\|_{L^\infty([0,T];W)} = \mathcal{O}(\varepsilon).$$

We compute

$$R^\varepsilon(t,x) = \sum_{j\in\mathbb{Z}^d}\sum_{(k,\ell,m)\notin I_j} b_{k,\ell,m}(t,x),$$

where

$$b_{k,\ell,m}(t,x) = \int_0^t a_k(s)\overline{a}_\ell(s)a_m(s)\exp\left(i\frac{\kappa_{k,\ell,m}\cdot x + |\kappa_{k,\ell,m}|^2 s - \omega_{k,\ell,m}s}{\varepsilon}\right)ds.$$

Proposition 2.4 then follows from one integration by parts (integrate the exponential), along with (A.1) and Lemma 2.3.

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