ANALYSIS OF EXPONENTIAL SPLITTING METHODS FOR INHOMOGENEOUS PARABOLIC EQUATIONS

by

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Abstract. — We analyze the convergence properties of the exponential Lie and Strang splitting applied to inhomogeneous second-order parabolic equations with Dirichlet boundary conditions. A recent result on the smoothing properties of these methods allows us to prove sharp convergence results in L^2 subject to homogeneous Dirichlet boundary conditions. When no source term is present and natural regularity assumptions are imposed on the initial value, we show full-order convergence of both methods. For inhomogeneous equations under natural assumptions on the inhomogeneity, we prove full-order convergence for the exponential Lie splitting, whereas order reduction to 1.25 for the exponential Strang splitting. Furthermore, we give sufficient conditions on the inhomogeneity for full-order convergence of both methods. Our theoretical convergence results also explain the severe order reduction to 0.25 of splitting methods applied to problems involving inhomogeneous Dirichlet boundary conditions. We include numerical experiments to underline the sharpness of our theoretical convergence results.

1. Introduction

The focus of this work lies in a rigorous error analysis of the exponential Lie and Strang splitting applied to second-order inhomogeneous parabolic evolution equations

$$\begin{aligned} &\partial_t w(t,x,y) = \mathcal{L}(\partial_x,\partial_y)w(t,x,y) + \psi(t,x,y), \quad (x,y)\in\Omega=(0,1)^2, \quad t\in(0,T],\\ &w(0,x,y) = w_0(x,y),\\ &w(t,\cdot,\cdot)\big|_{\partial\Omega} = 0 \quad \text{for all } t\in[0,T], \end{aligned} \tag{1}$$

subject to homogeneous Dirichlet boundary conditions. Here, $\mathcal{L}(\partial_x,\partial_y) = \partial_x\big(a(x,y)\partial_x\big) + \partial_y\big(b(x,y)\partial_y\big)$ denotes a strongly elliptic differential operator with sufficiently smooth (positive)

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coefficients a and b. In particular, we elucidate the following phenomena: for general inhomogeneities ψ the exponential Strang splitting naturally induced by the decomposition of $\mathcal{L}$ suffers from an order reduction whereas the exponential Lie splitting is full-order convergent under natural smoothness assumptions on the initial data (see the conditions (13), and Figure 1 (left) in Section 5 for a numerical example). Nevertheless, for a particular choice of ψ, also Strang splitting reaches full second-order convergence, see Figure 2. A recent result on the smoothing properties of exponential splitting methods, see [**14**], allows us to prove this order behavior rigorously. We can show full-order convergence of the exponential Lie splitting method, despite the fact that its local error behaves in $L^2(\Omega)$ like h^{p+1} with $p = 0.25 - \varepsilon$ for arbitrarily small $\varepsilon > 0$. Furthermore we can state a sharp result on the global order reduction to $1.25 - \varepsilon$ for arbitrarily small $\varepsilon > 0$ of the exponential Strang splitting applied to equations with general inhomogeneities ψ.

Operator splitting methods have gained a lot of attention in recent years, as in many situations they allow a keen reduction of the computational cost. For a general introduction to splitting methods, we refer to [**8**], [**12**] and references therein. In particular, exponential splitting methods with relatively bounded operators were studied in [**9**]; the situation of two unbounded operators in the homogeneous parabolic case was studied in [**5**] and [**6**]. However, due to the recent smoothing result, we can improve the error bounds in the homogeneous parabolic setting given in the last two references. Resolvent splitting applied to inhomogeneous equations was discussed in [**13**]. This work differs from [**13**] as we treat exponential splitting methods and are able to state sharp convergence bounds, which allows us to explain the numerically observed order reduction.

Furthermore this work in particular distinguishes from the above as inhomogeneous Dirichlet boundary conditions are included in the analysis. It is well know that inhomogeneous Dirichlet boundary conditions generally lead to a severe order reduction of splitting methods,[1] where for an introductory reading we refer to [**8**, Section IV.2]. Our theoretical analysis allows us a precise explanation of these phenomena, and numerical experiments confirm the sharpness of the derived convergence bounds. Note that for one-step methods a similar order reduction occurs. By certain corrections of the stages, however, this can be avoided, see e.g. [**1**].

For the analysis, the model problem (1) is formulated as an abstract evolution equation in $L^2(\Omega)$, where for convenience we denote by $\|\cdot\|$ the $L^2(\Omega)$ as well as the $L^2(\Omega)$-operator norm. Thus, the model problem reads

$$\begin{aligned} u'(t) &= Lu(t) + g(t) = Au(t) + Bu(t) + g(t), \quad t \in (0,T], \\ u(0) &= u_0 \end{aligned} \tag{2}$$

with $u(t) = w(t,\cdot,\cdot)$, $Au = \partial_x\big(a\partial_x u\big)$, $Bu = \partial_y\big(b\partial_y u\big)$, and $g(t) = \psi(t,\cdot,\cdot)$.

For the numerical time integration, we apply the so called dimension splitting, where the differential operator L is split along its dimensions. Let $h > 0$ denote the step size. At time

[1] *Particular attention will be paid to the order reduction caused by boundary conditions since that is often the main reason for a disappointing convergence behavior with splitting methods*; [**8**, p. 348].

$t_{n+1} = (n+1)h$ the exact solution of (2) is approximated by exponential Lie splitting

$$u_{n+1} = \mathrm{e}^{hA}\mathrm{e}^{hB}\big(u_n + hg(t_n)\big) \tag{3}$$

or exponential Strang splitting

$$u_{n+1} = \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}\Big(\mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A}u_n + hg(t_n + \tfrac{h}{2})\Big), \tag{4}$$

respectively.

In the bounded setting (where L, A, and B are bounded operators) one can easily see via Taylor series expansion that the *classical orders* of the exponential Lie splitting (3) and the exponential Strang splitting (4) are one and two, respectively. Under certain assumptions on the initial value $u_0 = w_0(x,y)$ and the inhomogeneity $g(t) = \psi(t,x,y)$ we will show that full-order convergence of both methods also holds true in our unbounded setting, see Section 3, Theorems 6 and 8, respectively. In particular we can explain the numerically observed full-order convergence of the exponential Strang splitting in [**6**]. In the situation of an arbitrary inhomogeneity g, we will prove full-order convergence for the exponential Lie splitting, whereas order reduction to $1.25-\varepsilon$ for arbitrarily small $\varepsilon > 0$ for the exponential Strang splitting, see Section 3, Theorems 6 and 8, respectively. In Section 4 we explain the disappointing performance of splitting methods applied to equations involving inhomogeneous Dirichlet boundary conditions by applying our newly derived theoretical results, see Corollaries 11 and 13. Finally, in Section 5 we illustrate the sharpness of our theoretical convergence results with numerical experiments.

We commence with a section on the analytic framework.

2. Analytic framework

In order to start with the convergence analysis we need some functional analytic results on the domains of the operators in (2). In the following we set $\Omega = (0,1)^2$ and

$$\begin{aligned}
&\mathcal{D}(L) = H^2(\Omega) \cap H_0^1(\Omega),\\
&\mathcal{D}(A) = \{u \in L^2(\Omega)\,;\ \partial_{xx}u, \partial_x u \in L^2(\Omega),\ u(0,y) = u(1,y) = 0 \ \text{ f.a.e. } y \in (0,1)\},\\
&\mathcal{D}(B) = \{u \in L^2(\Omega)\,;\ \partial_{yy}u, \partial_y u \in L^2(\Omega),\ u(x,0) = u(x,1) = 0 \ \text{ f.a.e. } x \in (0,1)\}.
\end{aligned} \tag{5}$$

We will also need compositions of the operator L and therefore introduce the spaces

$$\mathcal{D}(L^k) = \big\{u \in \mathcal{D}(L^{k-1})\,;\ Lu \in \mathcal{D}(L^{k-1})\big\}, \quad k = 1,2,3,\dots$$

Note that all the appearing operators $(\mathcal{D}(A), A)$, $(\mathcal{D}(B), B)$ and $(\mathcal{D}(L), L)$ generate analytic semigroups of contractions on $L^2(\Omega)$, see [**14**]. For a general introduction to the theory of analytic semigroups it is referred to [**15**]. Furthermore note that $\mathcal{D}(L^\gamma)$ is free from boundary conditions if and only if $\gamma < \frac{1}{4}$, see [**3**].

In the following we state some results on the regularity and compatibility between the domains of the full operator L, the split operators A, B, and their compositions, which we will need later on in our convergence analysis. Recall that $\Omega = (0,1)^2$.

Lemma 1. — *The following regularity results hold*

$$\mathcal{D}(L^2) \subset H^4(\Omega) \quad \textit{and} \quad \mathcal{D}(L^3) \subset H^{5-\varepsilon}(\Omega) \quad \textit{for all } \varepsilon > 0.$$

Proof. — This result can be derived from an analysis of the classical compatibility conditions for a strongly elliptic operator with Dirichlet boundary conditions on a square domain, see for instance [**4**]. A complete proof can be found in [**7**]. □

As the operator $(\mathcal{D}(L), L)$ is considered on the unit square Ω, i.e., a corner domain, the standard regularity results do not hold true. This is due to the so called corner singularities that arise when solving partial differential equations on non-smooth domains.

Lemma 2. — *The domains of the operators satisfy the following compatibility condition*

$$\mathcal{D}(L^2) \subseteq \mathcal{D}(A^2) \cap \mathcal{D}(AB) \cap \mathcal{D}(BA) \cap \mathcal{D}(B^2).$$

Proof. — From Lemma 1 we have

$$\mathcal{D}(L^2) \subseteq \left\{ u \in H^4(\Omega) \cap H_0^1(\Omega)\,;\, Lu|_{\partial\Omega} = 0 \right\}.$$

Let $u \in \mathcal{D}(L^2)$. Then $Au, Bu \in H^2(\Omega)$. As $\Omega \subset \mathbb{R}^2$, we have $H^4(\Omega) \cap H_0^1(\Omega) \subset \mathcal{C}^2(\overline{\Omega}) \cap H_0^1(\Omega)$ using standard Sobolev embeddings results. Hence as $u \in H^4(\Omega) \cap H_0^1(\Omega)$ we have for $x_0, y_0 \in \{0, 1\}$

$$\begin{aligned} Au(x,y)|_{y=y_0} &= \partial_x\big(a(x,y_0)\partial_x\big)u(x,y_0) = 0, \\ Bu(x,y)|_{x=x_0} &= \partial_y\big(b(x_0,y)\partial_y\big)u(x_0,y) = 0. \end{aligned} \tag{6}$$

Thus, as $0 = Lu|_{\partial\Omega} = Au|_{\partial\Omega} + Bu|_{\partial\Omega}$ we have, by using (6), $Au|_{\partial\Omega} = Bu|_{\partial\Omega} = 0$, i.e., $Au, Bu \in \mathcal{D}(A) \cap \mathcal{D}(B)$. □

Lemma 3. — *Let $f \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$. Then, for every $\lambda > 0$, the solution u of the elliptic problem*

$$\begin{aligned} &(\lambda I - A)u = f \quad \textit{in } \Omega, \\ &u(0,y) = u(1,y) = 0 \quad \textit{for almost every } y \in (0,1) \end{aligned} \tag{7}$$

lies in $\mathcal{D}(AB) \cap \mathcal{D}(BA)$.

Proof. — For u solving (7) and $f \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ we have $u \in \mathcal{D}(B)$, see [**14**]. Thus, $Au = \lambda u - f \in \mathcal{D}(B)$, i.e., $u \in \mathcal{D}(BA)$. Next we show that $u \in \mathcal{D}(AB)$. For this, we compute

$$Bu = \partial_y(b\partial_y u) = (\partial_y b)\partial_y\big((\lambda I - A)^{-1} f\big) + b\partial_{yy}\big((\lambda I - A)^{-1} f\big). \tag{8}$$

The relation $\partial_y(\lambda I - A)^{-1} = (\lambda I - A)^{-1}\partial_x\big((\partial_y a)\partial_x\big)(\lambda I - A)^{-1}$ (see [**14**, Section 7]) shows that

$$\partial_y(\lambda I - A)^{-1},\ \partial_{yy}(\lambda I - A)^{-1} \,:\, L^2(\Omega) \to \mathcal{D}(A) \subset L^2(\Omega)$$

for smooth coefficients a. Consequently $Bu \in \mathcal{D}(A)$, which proves the lemma. □

As $\mathcal{D}(L^2) \subseteq \mathcal{D}(AB) \cap \mathcal{D}(BA)$ the above lemma in particular implies that for all $\lambda > 0$

$$(\lambda I - A)^{-1}\big|_{\mathcal{D}(L^2)},\ (\lambda I - B)^{-1}\big|_{\mathcal{D}(L^2)} : \mathcal{D}(L^2) \subset L^2(\Omega) \to \mathcal{D}(AB) \cap \mathcal{D}(BA) \subset L^2(\Omega).$$

Thus, by using Cauchy's integral formula we can conclude that

$$\mathrm{e}^{tA}\big|_{\mathcal{D}(L^2)},\ \mathrm{e}^{tB}\big|_{\mathcal{D}(L^2)} : \mathcal{D}(L^2) \subset L^2(\Omega) \to \mathcal{D}(AB) \cap \mathcal{D}(BA) \subset L^2(\Omega)$$

uniformly in $t > 0$. Hence, by the closed graph theorem there exists a constant C such that for any $h > 0$, $\sup_{s,\tau\in[0,h]} \|[A,B]\mathrm{e}^{sA}\mathrm{e}^{\tau B}L^{-2}\| \leq C$. Furthermore, as for $0 \leq \gamma < \frac{5}{4}$, $\mathcal{D}(L^\gamma)$ is an invariant subspace under the semigroups generated by the split operators, we have for $0 \leq \varepsilon < \frac{1}{4}$ that $\sup_{s,\tau\in[0,h]} \|L^{-1+\varepsilon}[A,B]\mathrm{e}^{sA}\mathrm{e}^{\tau B}L^{-1-\varepsilon}\| \leq C$ uniformly in $h > 0$ (for some constant depending on ε).

Next we study the regularity of the solution of our model problem (2). Its exact solution is given by the variation-of-constants formula

$$u(t) = \mathrm{e}^{tL}u_0 + \int_0^t \mathrm{e}^{(t-\tau)L}g(\tau)\,\mathrm{d}\tau, \qquad 0 \leq t \leq T.$$

At time $t_{n+1} = t_n + h$, with a step size $h > 0$, the solution can be rewritten as

$$u(t_{n+1}) = \mathrm{e}^{hL}u(t_n) + \int_0^h \mathrm{e}^{(h-s)L}g(t_n + s)\,\mathrm{d}s.$$

Taylor expansion of $g(t_n + s)$ at t_n yields

$$u(t_{n+1}) = \mathrm{e}^{hL}u(t_n) + \int_0^h \mathrm{e}^{(h-s)L}\Big(g(t_n) + sg'(t_n) + \int_{t_n}^{t_n+s}(t_n + s - \tau)g''(\tau)\,\mathrm{d}\tau\Big)\mathrm{d}s.$$

We introduce the following notation (see [**5**]):

Definition 4. — For $h > 0$ we define the bounded operators λ_j by setting

$$\lambda_j = \frac{1}{h^j}\int_0^h \mathrm{e}^{(h-s)L}\frac{s^{j-1}}{(j-1)!}\,\mathrm{d}s, \qquad j \geq 1$$

and $\lambda_0 = \mathrm{e}^{hL}$. We likewise define $\alpha_0 = \mathrm{e}^{hA}$, $\beta_0 = \mathrm{e}^{hB}$ and for $j \geq 1$

$$\alpha_j = \frac{1}{h^j}\int_0^h \mathrm{e}^{(h-s)A}\frac{s^{j-1}}{(j-1)!}\,\mathrm{d}s, \qquad \beta_j = \frac{1}{h^j}\int_0^h \mathrm{e}^{(h-s)B}\frac{s^{j-1}}{(j-1)!}\,\mathrm{d}s.$$

With this notation the exact solution of (2) can be written as

$$u(t_{n+1}) = \lambda_0 u(t_n) + h\lambda_1 g(t_n) + h^2\lambda_2 g'(t_n) + R_{3,n}(h), \tag{9}$$

with the remainder

$$R_{3,n}(h) = \int_0^h \mathrm{e}^{(h-s)L}\Big(\int_{t_n}^{t_n+s}(t_n + s - \tau)g''(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s. \tag{10}$$

We note for later use that the following relations hold on $L^2(\Omega)$

$$\lambda_j = \frac{1}{j!} I + hL\lambda_{j+1}, \qquad j \geq 0, \tag{11}$$

which can easily be seen by integration by parts.

3. Convergence results

In order to state our sharp convergence results we need the following lemma about the smoothing property of splitting methods recently derived in [**14**].

Lemma 5. — *For $0 \leq \alpha < 1$, the following smoothing properties hold*

$$\|(-L)^\alpha \big(\mathrm{e}^{hA}\mathrm{e}^{hB}\big)^n\| \leq C t_n^{-\alpha}, \qquad 0 < t_n = nh \leq T,$$
$$\|(-L)^\alpha \big(\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{hB}\mathrm{e}^{\frac{h}{2}A}\big)^n\| \leq C t_n^{-\alpha}, \qquad 0 < t_n = nh \leq T,$$

where the constant C can be chosen uniformly on $[0,T]$ and, in particular, independently of $n \geq 1$ and $h > 0$.

Now we are able to state the main convergence results for the exponential Lie and Strang splitting, see Sections 3.1 and 3.2, respectively.

3.1. The exponential Lie splitting. — As a numerical scheme for solving (2), (5), we now consider the exponential Lie splitting method for inhomogeneous equations, i.e., the numerical approximation to the exact solution at time $t_{n+1} = t_n + h$ is given by

$$u_{n+1} = \mathrm{e}^{hA}\mathrm{e}^{hB}\big(u_n + hg(t_n)\big) \tag{12}$$

with initial condition $u_0 = u(0)$. Under natural regularity assumptions on the inhomogeneity g and initial data u_0, i.e.,

$$g \in \mathcal{C}([0,T];\mathcal{D}(L^\gamma)) \cap \mathcal{C}^1([0,T];L^2(\Omega)), \quad u_0 \in \mathcal{D}(L^{1+\gamma}) \tag{13}$$

we obtain first-order convergence. Note that (13) in particular implies $u \in \mathcal{C}^1([0,T];\mathcal{D}(L^\gamma))$, see [**11**].

Theorem 6. — *Let $u(t)$ denote the exact solution of* (2), (5), *and let the data satisfy* (13) *for some $\gamma > 0$. Then the global error of the exponential Lie splitting method* (12) *satisfies the following bound: for all fixed T there exists a constant C such that*

$$\|u(t_n) - u_n\| \leq Ch \sup_{t\in[0,T]} \Big(\|L^\gamma u'(t)\| + \|L^\gamma g(t)\| + \|g'(t)\|\Big) \tag{14}$$

holds for all $h > 0$ and all $n \in \mathbb{N}$ with $t_n = nh \leq T$.

We recall that the domain $\mathcal{D}(L^\gamma)$ is free of boundary conditions for $\gamma < \frac{1}{4}$. Thus full-order convergence of the exponential Lie splitting only requires some additional smoothness in space of the inhomogeneity g.

Proof. — In the following we will denote by S the exponential Lie splitting, i.e., $S = \mathrm{e}^{hA}\mathrm{e}^{hB}$. The exact solution of (2) is given by

$$u(t_{n+1}) = \lambda_0 u(t_n) + h\lambda_1 g(t_n) + R_{2,n}(h), \tag{15}$$

where we have set

$$R_{2,n}(h) = \int_0^h \mathrm{e}^{(h-s)L}\Big(\int_{t_n}^{t_n+s} g'(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s. \tag{16}$$

In order to prove convergence of the method, we first take a look at the error $e_{n+1} = u_{n+1} - u(t_{n+1})$. Subtracting (15) from (12) leads to

$$e_{n+1} = Se_n + (S-\lambda_0)u(t_n) + h(S-\lambda_1)g(t_n) - R_{2,n}(h). \tag{17}$$

Using the recurrence relation (11), namely $\lambda_1 = \frac{1}{h}(\lambda_0 - I)$, as well as the equation in the form $u(t_n) + L^{-1}g(t_n) = L^{-1}u'(t_n)$, we obtain

$$e_{n+1} = Se_n + S\left(u(t_n) + hg(t_n)\right) - \lambda_0 L^{-1}g(t_n) + L^{-1}g(t_n) - R_{2,n}(h). \tag{18}$$

The convergence analysis of the exponential Lie splitting for the homogeneous problem stated in [**5**, proof of Thm. 3.4] allows us to express the identity operator I restricted to the domain $\mathcal{D}(L)$ as

$$I\big|_{\mathcal{D}(L)} = S(I - hL) + h^2\mathcal{K}_2, \tag{19}$$

where the operator $\mathcal{K}_2$ is such that $h\|\mathcal{K}_2 L^{-1}\| \le C$ for some constant independent of h. Inserting the representation (19) of the identity operator in front of the terms with L^{-1} in (18) yields

$$\begin{aligned} e_{n+1} &= Se_n + S\left(u(t_n) + hg(t_n)\right) - I\lambda_0 L^{-1}g(t_n) + IL^{-1}g(t_n) - R_{2,n}(h) \\ &= Se_n + S(I - \lambda_0 + hL\lambda_0)L^{-1}u'(t_n) - h^2\mathcal{K}_2\lambda_0 L^{-1}u'(t_n) + h^2\mathcal{K}_2 L^{-1}g(t_n) - R_{2,n}(h). \end{aligned}$$

Using once more the recurrence relation (11) we obtain

$$e_{n+1} = Se_n + h^2 S(\lambda_1 - \lambda_2)Lu'(t_n) - h^2\mathcal{K}_2\lambda_0 L^{-1}u'(t_n) + h^2\mathcal{K}_2 L^{-1}g(t_n) - R_{2,n}(h). \tag{20}$$

Solving the above error recursion we obtain with the aid of Lemma 5 for $0 < \gamma < 1$

$$\begin{aligned} \|e_{n+1}\| &\le h^2\sum_{k=0}^{n-1}\|S^{n-k}L^{1-\gamma}\|\|L^{-1+\gamma}J_k\| + h^2\|J_n\| + \sum_{k=0}^{n-k}\|S^{n-k}\|\|R_{2,k}(h)\| \\ &\le Ch\sum_{k=0}^{n-1}\frac{h}{((n-k)h)^{1-\gamma}}\|L^{-1+\gamma}J_k\| + h^2\|J_n\| + \sum_{k=0}^{n-k}\|R_{2,k}(h)\|, \end{aligned} \tag{21}$$

where we have set

$$J_k = S(\lambda_1 - \lambda_2)Lu'(t_k) - \mathcal{K}_2\lambda_0 L^{-1}u'(t_k) + \mathcal{K}_2 L^{-1}g(t_k).$$

Using the domain invariance of $\mathcal{D}(L^{1-\gamma})$ for $0 < \gamma < 1$ under the semigroups, we have

$$\|L^{-1+\gamma}J_k\| \le C\Big(\|L^{\gamma}u'(t_k)\| + \|L^{\gamma}g(t_k)\|\Big).$$

Note that $h\|\mathcal{K}_2 L^{-1}\|$ is bounded and thus $h^2\|J_n\| \leq Ch$. Furthermore $\|R_{2,k}(h)\| \leq Ch^2 \sup_{t\in[0,T]}\|g'(t)\|$, see (16). Thus, by interpreting the sum in (21) as a lower Riemann sum, we obtain the desired convergence result (14). □

Remark 7. — Note that for $g \in \mathcal{D}(L^\gamma)$ with $\gamma < \frac{1}{4}$, the *local error* of the Lie splitting method is only of order $h^{1+\gamma}$. This can be seen from the error recurrence (20) as the term $h^2\mathcal{K}_2 L^{-1} g(t_n)$ can only be bounded by $h^{1+\gamma}\|L^\gamma g(t_n)\|$. The full convergence of the Lie splitting is due to the smoothing property (Lemma 5).

3.2. The exponential Strang splitting. — In this section we are interested in applying a second-order exponential splitting method to the inhomogeneous problem (2), (5). As numerical approximation u_{n+1} to the exact solution $u(t_{n+1})$ we choose the following scheme

$$u_{n+1} = \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}\left(\mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A}u_n + hg\big(t_n + \frac{h}{2}\big)\right), \tag{22}$$

which is an exponential midpoint rule.

In contrast to the Lie splitting method, we have to be more careful here with the regularity assumptions on the data in order to avoid unnatural conditions. We will show below that if the data satisfies for some $\gamma > 0$ and some $0 \leq \nu \leq 1$ the assumptions

$$g \in \mathcal{C}([0,T];\mathcal{D}(L^{\nu+\gamma})) \cap \mathcal{C}^1([0,T];\mathcal{D}(L^\gamma)) \cap \mathcal{C}^2([0,T];L^2(\Omega)), \quad u_0 \in \mathcal{D}(L^{1+\nu+\gamma}) \tag{23}$$

we obtain convergence order of $1+\nu$ for the exponential Strang splitting. Note that (23) implies $u \in \mathcal{C}^2([0,T];\mathcal{D}(L^{\gamma+\nu-1}))$, see [**11**].

Theorem 8. — *Let $u(t)$ denote the exact solution of* (2), (5), *and let the data satisfy* (23) *for some $\gamma > 0$ and some $0 \leq \nu \leq 1$. Then the global error of the exponential Strang splitting method* (22) *satisfies the following bound: for all fixed T there exists a constant C such that*

$$\|u(t_n) - u_n\| \leq Ch^{1+\nu} \sup_{t\in[0,T]} \Big(\|L^{\gamma+\nu-1}u''(t)\| + \|L^{\gamma+\nu}g(t)\| + \|L^\gamma g'(t)\| + \|g''(t)\|\Big) \tag{24}$$

holds for all $h > 0$ and all $n \in \mathbb{N}$ with $t_n = nh \leq T$.

Remark 9. — Note that full-order convergence of the exponential Strang splitting (22) requires the choice $\nu = 1$ in Theorem 8. In particular the inhomogeneity $\psi(t,\cdot,\cdot) = g(t)$ needs to vanish at the boundary, i.e., $\psi(t,\cdot,\cdot)|_{\partial\Omega} = 0$.

Remark 10. — Full-order convergence of the exponential Strang splitting (22) without unnatural assumptions on the inhomogeneity $g(t)$ can be obtained when considering the evolution equation (2) in the extrapolation space $(\mathcal{D}(L)^*, \|L^{-1}\cdot\|)$. We refer to [**2**] for an introduction to scales of Banach spaces. Precisely, we can adapt the previous proof and show that the global error measured in the $\|L^{-1}\cdot\|$ norm satisfies the following bound for any $\gamma > 0$:

$$\|L^{-1}(u(t_n) - u_n)\| \leq Ch^2 \sup_{t\in[0,T]} \Big(\|L^{\gamma-1}u''(t)\| + \|L^\gamma g(t)\| + \|L^{\gamma-1}g'(t)\| + \|L^{-1}g''(t)\|\Big)$$

with a constant C that can be chosen uniformly with respect to h on $[0,T]$

Proof of Theorem 8. — Let $S = \mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{hB}\mathrm{e}^{\frac{h}{2}A}$. We start off with the analysis of the local error $e_{n+1} = u_{n+1} - u(t_{n+1})$. Subtracting the exact solution (9) from the numerical approximation (22) we obtain

$$\begin{aligned} e_{n+1} = Se_n + \big(S - \lambda_0\big)u(t_n) + h\big(\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B} - \lambda_1\big)g(t_n) \\ + h^2\big(\tfrac{1}{2}\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B} - \lambda_2\big)g'(t_n) - R_{3,n}(h) \end{aligned} \tag{25}$$

with $R_{3,n}$ given in (10). In the following we set

$$E_n(h) = \big(S - \lambda_0\big)u(t_n) + h\big(\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B} - \lambda_1\big)g(t_n) + h^2\big(\tfrac{1}{2}\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B} - \lambda_2\big)g'(t_n). \tag{26}$$

With the identities $\lambda_1 = \frac{1}{h}(\lambda_0 - I)L^{-1}$ and $\lambda_2 = \frac{1}{h^2}(\lambda_0 - I - hL)L^{-2}$ we obtain

$$\begin{aligned} E_n(h) = Su(t_n) - \lambda_0 L^{-2}u''(t_n) + h\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}g(t_n) + L^{-1}g(t_n) \\ + \frac{h^2}{2}\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}g'(t_n) + L^{-2}g'(t_n) + hL^{-1}g'(t_n). \end{aligned} \tag{27}$$

Note that the following representations of the identity operator are valid on $\mathcal{D}(L)$ and $\mathcal{D}(L^2)$, respectively, see [**5**],

$$\begin{aligned} I|_{\mathcal{D}(L)} &= S(I - hL) + \frac{h^2}{2}\mathcal{K}_2, \\ I|_{\mathcal{D}(L^2)} &= S(I - hL + \frac{h^2}{2}L^2) + h^3\mathcal{K}_3, \end{aligned} \tag{28}$$

where the operators $\mathcal{K}_2$ and $\mathcal{K}_3$ satisfy the following bounds: for all ν with $0 \le \nu \le 1$

$$h^2\|\mathcal{K}_2 L^{-1}f\| \le Ch^{1+\nu}\|L^\nu f\|, \tag{29}$$

and for all $\gamma > 0$ and ν with $0 \le \nu \le 1$

$$h^3\|L^{-1+\gamma}\mathcal{K}_3 L^{-1}f\| \le Ch^{2+\nu}\|L^{\gamma+\nu}f\| \tag{30}$$

for some constants independent of h and for all functions f. Using the first representation of the identity operator given in (28) in front of the terms with L^{-1} and L^{-2} except for the term $\lambda_0 L^{-2}u''(t_n)$ in (27) we obtain

$$\begin{aligned} E_n(h) =& (S - \lambda_0)L^{-2}u''(t_n) + h\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}(I - \mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A})g(t_n) \\ &+ \tfrac{h^2}{2}\mathcal{K}_2 L^{-1}\big(g(t_n) + L^{-1}g'(t_n) + hg'(t_n)\big) + \tfrac{h^2}{2}\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}g'(t_n) - h^2 Sg'(t_n). \end{aligned} \tag{31}$$

From the convergence analysis of the exponential Lie splitting with h replaced by $\frac{h}{2}$ we know that

$$I - \mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A} = -\frac{h}{2}\mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A}L + \frac{h^2}{4}\mathcal{K}_2, \tag{32}$$

see (19). Furthermore, by [**5**] we have for all functions f and all ν with $0 \le \nu \le 1$

$$\|(S - \lambda_0)L^{-2}f\| \le Ch^{1+\nu}\|L^{\nu-1}f\| \tag{33}$$

for some constant C independent of h and f. Thus, using (32) as well as the bounds (29) and (33) in (31), we obtain that for $0 \le \nu \le 1$

$$\|E_n(h)\| \le Ch^{1+\nu}\Big(\|L^{\nu-1}u''(t_n)\| + \|L^{\nu}g(t_n)\| + \|g'(t_n)\|\Big). \tag{34}$$

Using the second representation of the identity operator given in (28) in front of the terms with L^{-1} and L^{-2} except for the term $\lambda_0 L^{-2}u''(t_n)$ in (27) we obtain

$$\begin{aligned} E_n(h) &= \big(S-\lambda_0\big)L^{-2}u''(t_n) + h\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}\Big(I + \mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A}\big(\frac{h}{2}L - I\big)\Big)g(t_n) \\ &\quad + \frac{h^2}{2}\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}\Big(I - \mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A}\Big)g'(t_n) + h^3\mathcal{K}_3L^{-1}g(t_n) + h^3\mathcal{K}_3L^{-2}g'(t_n) \\ &\quad + h^4\mathcal{K}_3L^{-1}g'(t_n). \end{aligned} \tag{35}$$

Another way of writing (32) is

$$I + \mathrm{e}^{\frac{h}{2}B}\mathrm{e}^{\frac{h}{2}A}\Big(\frac{h}{2}L - I\Big) = \frac{h^2}{4}\mathcal{K}_2, \tag{36}$$

see again (19). Thus, inserting the relations (32) and (36) into (35) we obtain

$$\begin{aligned} E_n(h) &= (S-\lambda_0)L^{-2}u''(t_n) + \frac{h^3}{4}\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}\mathcal{K}_2 g(t_n) - \frac{h^3}{2}SLg'(t_n) \\ &\quad + \frac{h^4}{8}\mathrm{e}^{\frac{h}{2}A}\mathrm{e}^{\frac{h}{2}B}\mathcal{K}_2 g'(t_n) + h^3\mathcal{K}_3L^{-1}g(t_n) + h^3\mathcal{K}_3L^{-2}g'(t_n) \\ &\quad + h^4\mathcal{K}_3L^{-1}g'(t_n). \end{aligned} \tag{37}$$

From the domain invariance of $\mathcal{D}(L)$ under S and [**6**], we know that for $0 < \gamma < 1$

$$\|L^{-1+\gamma}(S-\lambda_0)L^{-2}u''(t_n)\| \le Ch^{2+\nu}\|L^{\gamma+\nu-1}u''(t_n)\|.$$

Now, using the domain invariance of $\mathcal{D}(L^{1-\gamma})$, $0 < \gamma < 1$ under the semigroups in (37), as well as the bound given in (30), we obtain for $0 \le \nu \le 1$ the following estimate

$$\|L^{-1+\gamma}E_n(h)\| \le Ch^{2+\nu}\Big(\|L^{\gamma+\nu-1}u''(t_n)\| + \|L^{\gamma+\nu}g(t_n)\| + \|L^{\gamma}g'(t_n)\|\Big). \tag{38}$$

Thus, using the bounds given in (34) and (38) as well as the smoothing property for the exponential Strang splitting, see Lemma 5, we obtain for $0 \le \nu \le 1$

$$\begin{aligned} \|e_{n+1}\| &\le \sum_{k=0}^{n-1}\|S^{n-k}L^{1-\gamma}\|\|L^{-1+\gamma}E_k(h)\| + \|E_n(h)\| + \sum_{k=0}^{n-k}\|S^{n-k}\|\|R_{3,k}(h)\| \\ &\le Ch^{1+\nu}\sum_{k=0}^{n-1}\frac{h}{((n-k)h)^{1-\gamma}}\Big(\|L^{\gamma+\nu-1}u''(t_k)\| + \|L^{\gamma+\nu}g(t_k)\| + \|L^{\gamma}g'(t_k)\|\Big) \\ &\quad + Ch^{1+\nu}\Big(\|L^{\nu-1}u''(t_n)\| + \|L^{\nu}g(t_n)\| + \|g'(t_n)\|\Big) + \sum_{k=0}^{n-k}\|R_{3,k}(h)\|. \end{aligned} \tag{39}$$

By (10) we have $\|R_{3,k}(h)\| \le Ch^3\sup_{t\in[0,T]}\|g''(t)\|$. Hence, interpreting the sum in (39) as a lower Riemann sum leads to the desired estimate (24). □

4. Application to inhomogeneous Dirichlet boundary conditions

The main aim of this section is to understand the disappointing behavior of splitting methods when applied to problems involving inhomogeneous boundary conditions with the aid of the convergence results derived in the previous section. Our model problem in this section thus reads

$$\begin{aligned} &\partial_t w(t,x,y) = \mathcal{L}(\partial_x,\partial_y) w(t,x,y), \quad (x,y)\in\Omega=(0,1)^2, \quad t\in(0,T],\\ &w(0,x,y) = w_0(x,y),\\ &w(t,\cdot,\cdot)\big|_{\partial\Omega} = f(t,\cdot,\cdot), \quad t\in[0,T], \end{aligned} \tag{40}$$

where $\mathcal{L}(\partial_x,\partial_y) = \partial_x(a(x,y)\partial_x) + \partial_y(b(x,y)\partial_y)$ and f, defined on $[0,T]\times\partial\Omega$, is the boundary data.

In Section 4.1 we will discuss the case where f is defined on the whole domain $[0,T]\times\Omega$ and, in particular, is smooth in space and time, whereas in Section 4.2 we focus on the more common situation where f is only known as a boundary function.

4.1. Boundary data given as the restriction of a smooth function. — Here we assume that the boundary function f is the restriction of a smooth function F defined on $[0,T]\times\Omega$, i.e.

$$f(t,\cdot,\cdot) = F(t,\cdot,\cdot)\big|_{\partial\Omega} \quad \text{for all}\, t\in[0,T]. \tag{41}$$

The ansatz is to rewrite (40) as an inhomogeneous evolution equation in $L^2(\Omega)$ with homogeneous Dirichlet boundary conditions. Employing the transformation $U(t,x,y) = w(t,x,y) - F(t,x,y)$, we see that U is the solution of the problem

$$\begin{aligned} &\partial_t U(t,x,y) = \mathcal{L}(\partial_x,\partial_y) U(t,x,y) + \psi(t,x,y), \quad (x,y)\in\Omega=(0,1)^2, \quad t\in(0,T],\\ &U(0,x,y) = U_0(x,y),\\ &U(t,\cdot,\cdot)\big|_{\partial\Omega} = 0 \end{aligned} \tag{42}$$

with $\psi(t,x,y) = \mathcal{L}(\partial_x,\partial_y)F(t,x,y) - \partial_t F(t,x,y)$ and $U_0(x,y) = w_0(x,y) - F(0,x,y)$. This equation can be formulated as an abstract evolution equation in $L^2(\Omega)$ with $u(t) = U(t,\cdot,\cdot)$ and $g(t) = \psi(t,\cdot,\cdot)$, i.e.,

$$\begin{aligned} &u'(t) = Lu(t) + g(t), \quad t\in(0,T],\\ &u(0) = u_0, \end{aligned} \tag{43}$$

where now the problem is equipped with homogeneous Dirichlet boundary conditions, i.e., $\mathcal{D}(L) = H^2(\Omega)\cap H_0^1(\Omega)$. Thus, we can apply the convergence results for the exponential Lie and Strang splitting stated in Section 3.

We are in particular interested how the compatibility conditions on the initial value u_0 and the inhomogeneity g of the reformulated inhomogeneous problem (43) relate to the compatibility assumptions on w_0 and the boundary data f of the original inhomogeneous boundary value problem (40).

First-order convergence of the exponential Lie splitting requires the regularity assumptions (13) on the initial value u_0 and inhomogeneity g for some $\gamma > 0$, see Theorem 6. Expressed in terms of the original data in (40) (using (41)) assumption (13) reads

$$\begin{aligned} &\mathcal{L}(\partial_x, \partial_y)F(t,\cdot,\cdot) - \partial_t F(t,\cdot,\cdot) \in \mathcal{C}([0,T];\mathcal{D}(L^\gamma)) \cap \mathcal{C}^1([0,T];L^2(\Omega)), \\ &w_0 - F(0,\cdot,\cdot) \in \mathcal{D}(L^{1+\gamma}). \end{aligned} \tag{44}$$

Thus, for first-order convergence of the exponential Lie splitting method, the initial value of the original problem (40) needs to satisfy for some $\gamma > 0$

$$w_0 \in \left\{\varphi \in H^{2+2\gamma}(\Omega)\,;\, \varphi|_{\partial\Omega} = f(0)\right\}. \tag{45}$$

Upon using Theorem 8, a convergence rate of order $1+\nu$, $0 \le \nu \le 1$, of the exponential Strang splitting method can be achieved under the additional regularity assumptions (23) for some $\gamma > 0$. In terms of the solution w in the original problem (40) (and using (41)) the additional assumptions read

$$\begin{aligned} &\mathcal{L}(\partial_x, \partial_y)F(t,\cdot,\cdot) - \partial_t F(t,\cdot,\cdot) \in \mathcal{C}([0,T];\mathcal{D}(L^{\gamma+\nu})) \cap \mathcal{C}^1([0,T];\mathcal{D}(L^\gamma)) \cap \mathcal{C}^2([0,T];L^2(\Omega)), \\ &w_0 - F(0,\cdot,\cdot) \in \mathcal{D}(L^{1+\nu+\gamma}). \end{aligned} \tag{46}$$

As γ can be chosen arbitrarily small and $\mathcal{D}(L^\delta)$ is free from boundary conditions for $\delta < \frac{1}{4}$ we obtain convergence of order $1.25 - 2\varepsilon$ under the additional smoothness and compatibility assumptions on the initial value w_0

$$w_0 \in \left\{\varphi \in H^{2.5}(\Omega)\,;\, \varphi|_{\partial\Omega} = f(0)\right\} \tag{47}$$

by choosing $\nu = 0.25 - 2\varepsilon$ and $\gamma = \varepsilon$ in (46).

Note that in order to have full-order convergence of the Strang splitting we need to choose $\nu = 1$ in (46). This leads in particular to the following assumption on F

$$\mathcal{L}(\partial_x, \partial_y)F(t,\cdot,\cdot)|_{\partial\Omega} = \partial_t F(t,\cdot,\cdot)|_{\partial\Omega} \quad \text{for all } t. \tag{48}$$

Corollary 11. — *The exponential Lie and Strang splitting applied to* (40) *are convergent of order* 1*, respectively,* $1.25 - \varepsilon$ *(for every* $\varepsilon > 0$*) under the natural assumptions* (44) *and* (46)*, respectively. Full second-order convergence of the exponential Strang splitting method is obtained if the extension* F *of the boundary data* f *(see* (41)*) in addition satisfies* (48)*.*

4.2. General boundary data. — Here the ansatz is to rewrite the original problem (40) as an abstract integral equation. First we formulate the problem in $L^2(\Omega)$ as follows

$$\begin{aligned} &L^{-1}y'(t) = y(t) - \mathcal{G}f(t), \\ &y(0) = y_0 = u(0), \end{aligned} \tag{49}$$

where $\mathcal{G}f(t)$ denotes the Dirichlet map, i.e., the solution of the elliptic problem

$$\begin{aligned} &\mathcal{L}(\partial_x, \partial_y)v = 0 \text{ in } \Omega, \\ &v|_{\partial\Omega} = f(t), \end{aligned}$$

see [**10**] for further details. Note that $\mathcal{G}f(t) \notin \mathcal{D}(L) = H^2(\Omega) \cap H_0^1(\Omega)$ unless $f = 0$ and thus in particular $y'(t) \notin L^2(\Omega)$. Hence, the problem needs to be analyzed carefully.

In order to state the order of convergence of the splitting methods applied to (49), we need to reformulate the equation as an integral equation, i.e.,

$$y(t) = \mathrm{e}^{tL} y_0 - \int_0^t L^{1-\nu} \mathrm{e}^{(t-s)L} L^\nu \mathcal{G}f(s)\,\mathrm{d}s. \tag{50}$$

Note that the above formulation is valid for $0 < \nu < \frac{1}{4}$ in $L^2(\Omega)$ if $y_0 \in L^2(\Omega)$ and $\mathcal{G}f(\cdot) \in \mathcal{C}([0,T]; H^{2\nu}(\Omega))$, as $\mathcal{D}(L^\nu)$ is free from boundary conditions for $\nu < \frac{1}{4}$. By choosing $\nu < \frac{1}{4}$ we have

$$\begin{aligned}\left\| \int_0^t \mathrm{e}^{(t-s)L} L\mathcal{G}f(s)\,\mathrm{d}s \right\| &\le \int_0^t \|L^{1-\nu}\mathrm{e}^{(t-s)L} L^\nu \mathcal{G}f(s)\|\,\mathrm{d}s \\ &\le \sup_{s\in[0,t]} \|L^\nu \mathcal{G}f(s)\| \int_0^t (t-s)^{\nu-1}\,\mathrm{d}s \le C.\end{aligned}$$

In the following (see Corollary 13 below) we will show that even if the Dirichlet map is arbitrarily smooth, the exponential Lie and Strang splitting approximating the solution (50) are only expected to be convergent of order $0.25 - \varepsilon$ for every $\varepsilon > 0$, see Figure 3 (left) for the sharpness of this result.

Theorem 12. — *Let $0 < \nu < \frac{1}{4}$. Under the assumptions that $y_0 \in \mathcal{D}(L^\nu)$ and the Dirichlet map satisfies $\mathcal{G}f(\cdot) \in \mathcal{C}^1([0,T]; H^{2\nu}(\Omega))$, the exponential Lie splitting* (12) *and exponential Strang splitting* (22) *approximating the solution* (50) *satisfy the convergence bound*

$$\|y(t_n) - y_n\| \le h^\nu C \max_{t\in[0,T]} \left(\|L^{-1+\nu} y'(t)\| + \sum_{k=0}^{1} \|L^\nu \mathcal{G}f^{(k)}(t)\| \right). \tag{51}$$

Proof. — In the following let $0 < \nu < \frac{1}{4}$. We start with the convergence analysis of the Lie splitting method (12) and use its error recurrence (20) given in the proof of Theorem 6. We formally replace g by $L\mathcal{G}f$, i.e., the error recurrence reads

$$e_{n+1} = Se_n + h^2 S(\lambda_1 - \lambda_2) L y'(t_n) - h^2 \mathcal{K}_2 \lambda_0 L^{-1} y'(t_n) + h^2 \mathcal{K}_2 \mathcal{G}f(t_n) - R_{2,n}(h). \tag{52}$$

Next we rewrite the above recurrence as

$$\begin{aligned} e_{n+1} = Se_n &+ h^2 S L^{2-\nu}(\lambda_1 - \lambda_2) L^{-1+\nu} y'(t_n) - h^2 \mathcal{K}_2 L^{-\nu} \lambda_0 L^{-1+\nu} y'(t_n) \\ &+ h^2 \mathcal{K}_2 L^{-\nu} L^\nu \mathcal{G}f(t_n) - R_{2,n}(h). \end{aligned} \tag{53}$$

Solving this recurrence leads to

$$\begin{aligned} \|e_{n+1}\| &\le h^2 \sum_{k=0}^{n-1} \|S^{n-k} L^{1-\gamma}\| \|L^{-1+\gamma} J_k\| + h^2 \|J_n\| + \sum_{k=0}^{n} \|S^{n-k}\| \|R_{2,k}(h)\| \\ &\le Ch \sum_{k=0}^{n-1} \frac{h}{((n-k)h)^{1-\gamma}} \|L^{-1+\gamma} J_k\| + h^2 \|J_n\| + \sum_{k=0}^{n} \|R_{2,k}(h)\|, \end{aligned} \tag{54}$$

where we have set

$$J_k = SL^{2-\nu}(\lambda_1 - \lambda_2)L^{-1+\nu}y'(t_k) - \mathcal{K}_2 L^{-\nu}\lambda_0 L^{-1+\nu}y'(t_k) + \mathcal{K}_2 L^{-\nu}L^{\nu}\mathcal{G}f(t_k).$$

Next we use that

$$\|L^{-1+\xi}SL^{2-\delta}(\lambda_1 - \lambda_2)\| \le Ch^{\delta-\xi-1}, \quad \|L^{-1+\xi}\mathcal{K}_2 L^{-\delta}\| \le Ch^{\delta-\xi-1}, \quad 0 \le \xi \le 1, \tag{55}$$

see Definition 4 (using the domain invariance of $\mathcal{D}(L^\gamma)$ under S for $0 \le \gamma \le 1$, i.e., that $L^{-1+\nu}SL^{1-\nu}$ is bounded) and (29).

Thus, upon choosing $\xi > 0$ arbitrarily small and $\delta < \nu$ with $0 < \nu < \frac{1}{4}$ in (55) for the first term in (54) and $\xi = 1$ and $0 < \delta < 1$ in (55) for the second term in (54), we obtain by interpreting the first sum as a lower Riemann sum that

$$\|e_{n+1}\| \le h^\nu C \max_{t\in[0,T]} \left(\|L^{-1+\nu}y'(t)\| + \|L^\nu \mathcal{G}f(t)\|\right) + \sum_{k=0}^{n} \|R_{2,k}(h)\|. \tag{56}$$

Next we use the representation of $R_{2,k}(h)$ given in (16), where now we formally have $g'(t) = L\mathcal{G}f'(t)$, i.e.,

$$R_{2,k}(h) = \int_0^h \mathrm{e}^{(h-s)L} L^{1-\nu} \left(\int_{t_k}^{t_k+s} L^\nu \mathcal{G}f'(s)\,\mathrm{d}\tau \right) \mathrm{d}s.$$

For this we use the bound

$$\begin{aligned} \|R_{2,k}(h)\| &\le \max_{t\in[t_k,t_k+h]} \|L^\nu \mathcal{G}f'(t)\| \int_0^h \|\mathrm{e}^{(h-s)L}L^{1-\nu}\| s\,\mathrm{d}s \\ &\le \max_{t\in[t_k,t_k+h]} \|L^\nu \mathcal{G}f'(t)\| \int_0^h s(h-s)^{\nu-1}\,\mathrm{d}s \\ &\le h^{1+\nu} C \max_{t\in[t_k,t_k+h]} \|L^\nu \mathcal{G}f'(t)\|. \end{aligned} \tag{57}$$

Thus, plugging the bound (57) into (56) we obtain the final result (51) for the Lie splitting method.

The proof for the exponential Strang splitting follows the line of argumentation, when considering the local error representation (26) and replacing $g(t)$ with $L\mathcal{G}f(t)$. Again, one sees that the bound (51) is optimal in the sense that the first operator $S - \lambda_0$ behaves like $h^{1+\nu}L^\nu$ for $0 \le \nu < \frac{1}{4}$ and we consider in particular situations where the regularity of the solution is restricted by $y(t) \in \mathcal{D}(L^\nu)$ for $\nu < \frac{1}{4}$. □

The above theorem in particular implies an interesting corollary.

Corollary 13. — *Assume that the boundary data are $f \equiv 1$ in* (40)*. Then, although the Dirichlet map $\mathcal{G}f(t) = 1$ lies in $\mathcal{C}^\infty(\overline{\Omega})$ one can only expect convergence order of $0.25 - \varepsilon$ (for every $\varepsilon > 0$) for the exponential Lie and Strang splitting methods when approximating the solution* (50)*.*

Proof. — The corollary follows as $L^{-1}y'(t)$ and $\mathcal{G}f(t)$ lie in $\mathcal{D}(L^\delta)$ only for $\delta < \frac{1}{4}$. Thus, the bound (51) holds for $\nu < \frac{1}{4}$ which implies the assertion. □

Remark 14. — Convergence of order one for the exponential Lie splitting and $1.25 - \varepsilon$ (for any $\varepsilon > 0$) for the exponential Strang splitting method when approximating the solution (50) can be reached in the extrapolation space $(\mathcal{D}(L)^*, \|L^{-1} \cdot \|)$, i.e., when the error is measured in the $\|L^{-1} \cdot \|$ norm. This can be seen by the transformation $v(t) = L^{-1}y(t)$ in (49). For $v(t)$ we obtain the evolution equation

$$\begin{aligned} v'(t) &= Lv(t) - \mathcal{G}f(t), \\ v(0) &= L^{-1}y(0), \end{aligned} \tag{58}$$

which fits into the abstract framework of Section 3. This yields the stated convergence results for the solution $y(t)$ of the original problem (58) with respect to the $\|L^{-1} \cdot \|$ norm.

The sharpness of Corollary 13 and Remark 14 is numerically confirmed, see Figure 3.

5. Numerical experiments

In this section we numerically analyze the order of convergence of the exponential Lie (12) and Strang (22) splitting to underline the theoretical convergence results derived in Section 3. The main interest lies in demonstrating the sharpness of the convergence bound for the exponential Strang splitting for inhomogeneous equations given in Theorem 8, as well as the convergence bounds for the exponential Lie and Strang splitting for general inhomogeneous Dirichlet boundary conditions given in Corollary 13. In the numerical experiments we use standard symmetric finite differences for the spatial discretization of the split differential operators $\mathcal{A}(\partial_x) = \partial_x\big(a(x,y)\partial_x\big)$ and $\mathcal{B}(\partial_y) = \partial_y\big(b(x,y)\partial_b\big)$.

In Examples 15 and 16 presented below, we solve the inhomogeneous model problem (1) with different choices of inhomogeneities ψ. The numerical results confirm the error bound (24). In Example 17 we solve the inhomogeneous boundary problem (50) with the simplest choice of boundary data $f(t) = 1$ so that in particular $\mathcal{G}f(t) = 1 \in \mathcal{C}^\infty(\overline{\Omega})$. The numerical result confirms the severe order reduction stated in Corollary 13.

Example 15 **(Order reduction of the exponential Strang splitting)**
In this experiment we choose the following data in (1):

$$\begin{aligned} &\mathcal{L}(\partial_x, \partial_y) = \partial_x\big((2xy+3)\partial_x\big) + \partial_y\big((2xy^4+1)\partial_y\big), \\ &w_0(x,y) = \mathrm{e}^{8 - \frac{1}{x(1-x)} - \frac{1}{y(1-y)}}, \\ &\psi(t,x,y) = x(1-x)y(1-y) + t\mathrm{e}^{x^3 y}. \end{aligned}$$

Note that the regularity assumptions (13) and (23) are satisfied for $\gamma < \frac{1}{4}$ and $\gamma + \nu < \frac{1}{4}$, respectively. Furthermore note that $\psi(t,\cdot,\cdot)|_{\partial\Omega} \neq 0$ for $t > 0$. To obtain the highest possible convergence order in (24) we chose ν close to a quarter and γ close to zero.

Hence, we expect first-order convergence of the exponential Lie splitting (12) and convergence of order $1.25 - \varepsilon$ for the exponential Strang splitting method (22) applied to (1) formulated in $L^2(\Omega)$. On the contrary, we expect full-order convergence of both methods when formulating

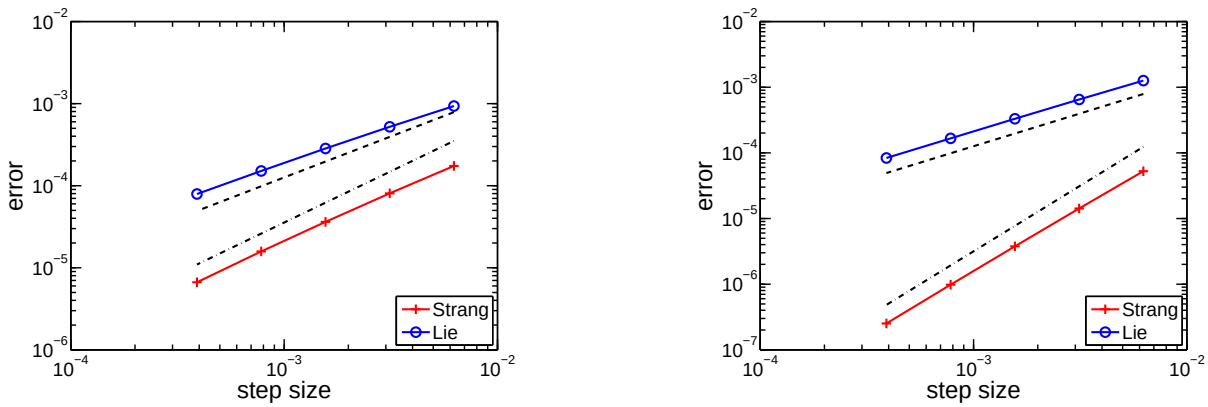

FIGURE 1. Numerical order of the exponential Lie and Strang splitting applied to equation (1) with the data as in Example 15. In the left picture, the errors are measured in a discrete L^2 norm. The slopes of the dashed and dash-dotted lines are 1 and 1.25, respectively. In the right picture, the errors are measured in the discrete $\|L^{-1}\cdot\|$ norm. The slopes of the dashed and dash-dotted lines are 1 and 2, respectively.

the evolution equation (1) in the extrapolation space $(\mathcal{D}(L)^*, \|L^{-1}\cdot\|)$, i.e., when the error is measured with respect to the $\|L^{-1}\cdot\|$ norm, cf. Remark 10. The numerical experiments go in line with our theoretical results, see Figure 1 (left) for the $L^2(\Omega)$ and Figure 1 (right) for the $\mathcal{D}(L)^*$ formulation.

Example 16 **(Full-order convergence of the exponential Strang splitting)**
In this experiment we choose the following data in (1):

$$\begin{aligned}
&\mathcal{L}(\partial_x,\partial_y) = \partial_x\big((2xy+3)\partial_x\big) + \partial_y\big((2xy^4+1)\partial_y\big),\\
&w_0(x,y) = \mathrm{e}^{8-\frac{1}{x(1-x)}-\frac{1}{y(1-y)}},\\
&\psi(t,x,y) = x(1-x)y(1-y)\mathrm{e}^t.
\end{aligned}$$

Note that the regularity assumptions (13) and (23) are satisfied for $\gamma < \frac{1}{4}$ and $\nu = 1$. In particular assumption (23) holds as $g(t) = \psi(t,\cdot,\cdot) \in \mathcal{D}(L^{1+\gamma})$ for $0 \le \gamma < \dfrac{1}{4}$ and all t. Hence, ν can be chosen equal to one in (24).

Thus, we expect first-order convergence of the exponential Lie splitting (12) and second-order convergence of the exponential Strang splitting method (22) applied to (1). The numerical experiment goes in line with our theoretical results, see Figure 2.

Example 17 **(Inhomogeneous Dirichlet boundary conditions)**
Here we model the inhomogeneous boundary problem (50) with the data

$$\mathcal{G}f(t) = 1 \text{ (i.e. } f(t,\cdot,\cdot) = 1) \ , \quad y_0(x,y) = \mathrm{e}^{8-\frac{1}{x(1-x)}-\frac{1}{y(1-y)}}.$$

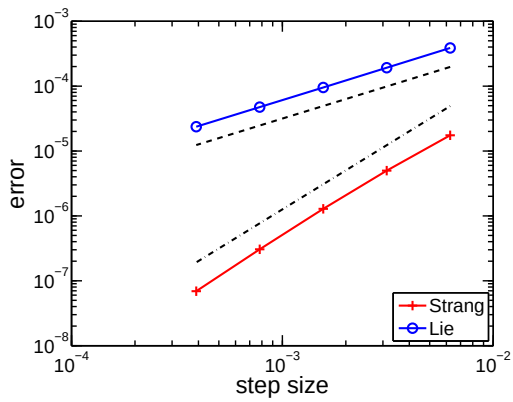

FIGURE 2. Numerical order of the exponential Lie and Strang splitting applied to equation (1) with the data as in Example 16. The error is measured in a discrete L^2 norm. The slopes of the dashed and dash-dotted lines are one and two, respectively.

In the left picture of Figure 3 we clearly see the severe order reduction of the exponential Lie and Strang splitting in $L^2(\Omega)$ whereas convergence order of one and 1.25, respectively, when measured in $\|L^{-1}\cdot\|$. This underlines the sharpness of Corollary 13 and Remark 14.

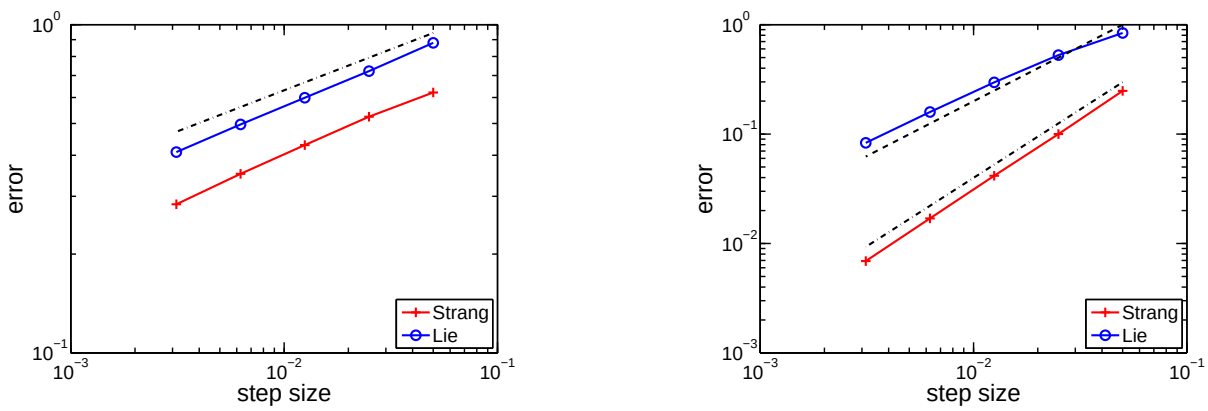

FIGURE 3. Numerical order of the exponential Lie and Strang splitting applied to equation (49) with $\mathcal{G}f(t) = 1$. In the left picture, the errors are measured in a discrete L^2 norm. The slope of the dashed line is one quarter. In the right picture, the errors are measured in the discrete $\|L^{-1}\cdot\|$ norm. The slopes of the dashed and dash-dotted lines are one and 1.25, respectively.

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