Approximate travelling wave solutions to the 2D Euler equation on the torus

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Abstract

We consider the two-dimensional Euler equation with periodic boundary conditions. We construct approximate solutions of this equation made of localized travelling profiles with compact support propagating over a stationary state depending on only one variable. The direction or propagation is orthogonal to this variable, and the support is concentrated around flat points of the stationary state. Under regularity assumptions, we prove that the approximation error can be made exponentially small with respect to the width of the support of the travelling wave. We illustrate this result by numerical simulations.

1 Introduction

We consider the two-dimensional Euler equation written in terms of vorticity

$$\partial_t \omega + u \cdot \nabla \omega = 0,$$

where $\omega(t, x, y) \in \mathbb{R}$, $\nabla = (\partial_x, \partial_y)^T$ with $(x, y) \in \mathbb{T}^2$ the two-dimensional torus $(\mathbb{R}/2\pi\mathbb{Z})^2$. The divergence free velocity field u is given by the formula

$$u = J\nabla\psi \quad \text{with} \quad \psi = (-\Delta)^{-1}\omega, \quad \text{where} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the canonical symplectic matrix. Here $(-\Delta)^{-1}$ is the inverse of the Laplace operator on functions with average 0 on $\mathbb{T}^2$. We can rewrite this equation as

$$\begin{cases} \partial_t \omega + \{\psi, \omega\} = 0, \\ -\Delta\psi = \omega, \end{cases} \tag{1.1}$$

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with the 2D Poisson bracket for functions on $\mathbb{T}^2$:

$$\{f,g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g).$$

For two functionals $H(\omega)$ and $G(\omega)$, we set

$$\{H,G\}_\omega = \int_{\mathbb{T}^2} \frac{\delta H}{\delta \omega}\left\{\frac{\delta G}{\delta \omega},\omega\right\} = -\{G,H\}_\omega.$$

The Euler equation (1.1) is a Hamiltonian PDE associated with this non canonical Poisson structure, and with Hamiltonian

$$E(\omega) = \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}\frac{1}{2}\|u\|^2 = \frac{1}{2(2\pi)^2}\int_{\mathbb{T}^2}\omega(-\Delta)^{-1}\omega = \|\omega\|^2_{H^{-1}},$$

which is quadratic in ω. In other words, we can write (1.1) as

$$\partial_t\omega + \left\{\frac{\delta E}{\delta \omega},\omega\right\} = 0,$$

and from the definition of the Poisson structure, we observe that $E(\omega(t)) = E(\omega(0))$ for all time (preservation of the energy). Moreover, the flow is volume preserving in the sense that for all smooth functions $h : \mathbb{R} \mapsto \mathbb{R}$, we have

$$\forall\, t \geq 0 \qquad \int_{\mathbb{T}^2} h(\omega(t,x,y))\mathrm{d}x\mathrm{d}y = \int_{\mathbb{T}^2} h(\omega(0,x,y))\mathrm{d}x\mathrm{d}y, \tag{1.2}$$

which expresses the preservation of the Casimirs of the Poisson structure.

The equation (1.1) possesses many stationary states. For all functions $F : \mathbb{R} \mapsto \mathbb{R}$ and $\psi^0 : \mathbb{T}^2 \to \mathbb{R}$ satisfying

$$\Delta\psi^0 = F(\psi^0),$$

then the couple of functions $\omega(t,x,y) = F(\psi^0(x,y))$ and $\psi(t,x,y) = \psi^0(x,y)$ solve (1.1). Another class of stationary states are given by functions depending only on one variable (shear flows): for any smooth $V(y)$ periodic in y, the couple $\omega(t,x,y) = V''(y)$ and $\psi(t,x,y) = -V(y)$ is solution of the 2D Euler equation.

The goal of this paper is to construct approximate travelling solutions of (1.1) based on such stationary states. More precisely, we will construct families of functions of the form

$$\omega^\varepsilon(t,x,y) = V''(y) + \Omega^\varepsilon(x - x_0 - ct, y - y_0) \tag{1.3}$$

for ε sufficiently small, where the functions $\Omega^\varepsilon(x,y)$ are *profiles* exponentially decaying with respect to $(|x|+|y|)/\varepsilon$, and with compact support. Then, under the assumption that V is smooth and that y_0 is such that V is locally linear around y_0 - so that V'' is locally zero - we can prove that (1.3) is solution of (1.1) for all time up to an error of order ε^N for all N. If moreover V has a Gevrey regularity, we can optimize the truncation error and obtain an exponentially small error with respect to ε.

Strikingly, this construction holds when the velocity c is given by $c = V'(y_0)$. In other words, the speed of the profile depends on the local shape of the surface defined by the stationary state $V(y)$. Moreover, we can take $\|\Omega^\varepsilon\|_{L^\infty} \simeq \varepsilon^{-\alpha}$ where α is any real number.

Note that as $\omega^\varepsilon(t,x,y)$ is a travelling wave with constant velocity, it automatically satisfies the preservation law (1.2), and we can easily prove that the energy is preserved up to exponentially small terms for all times.

The method of proof is based of an asymptotic matching between the profiles and the shear flow $V(y)$ as in [10]. Such technic has already been used in the context of Euler equations, see for instance [2, 3] and the references therein.

We conclude this introduction by remarking that the construction can be easily extended to a finite number of travelling profiles if $V''(y)$ possesses several flat points, by just adding exponentially decreasing profile with non interacting supports:

$$\omega^\varepsilon(t,x,y) = V''(y) + \sum_{k=0}^{K} \Omega_k^\varepsilon(x - x_k - c_k t, y - y_k) \tag{1.4}$$

where the points y_k are flat points of $V''(y)$. Note that this can be done at the same level $y_1 = y_0$ provided $x_1 \neq x_0$ and ε small enough to ensure the non interaction of the supports of the profiles.

Therefore, the function (1.4) is a quasi-periodic function satisfying (1.1) up to exponentially small terms. The question of the existence of quasi-periodic exact solutions for the 2D Euler equation, possibly close to the functions constructed above, remains an open problem.

We conclude this paper by a numerical illustration of the previous results.

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2 Ansatz

Let (ω,ψ) a solution of (1.1), and $V(y)$ a smooth periodic function. Let us set

$$\xi(t,x,y) = \omega(t,x,y) - V''(y), \quad \text{and} \quad \eta(t,x,y) = \psi(t,x,y) + V(y).$$

These functions satisfy

$$\begin{cases} \partial_t \xi + V' \partial_x \xi + V''' \partial_x \eta + \{\eta, \xi\} = 0, \\ -\Delta \eta = \xi. \end{cases} \tag{2.1}$$

Let $\varepsilon > 0$ and $(x_0, y_0) \in \mathbb{T}^2$. We search a *formal* solution of the previous equation under the form

$$\xi^\varepsilon(t,x,y) = \varepsilon^{-\alpha} \Omega \left(\frac{x - x_0 - ct}{\varepsilon}, \frac{y - y_0}{\varepsilon} \right) \tag{2.2}$$

and

$$\eta^\varepsilon(t,x,y) = \varepsilon^{2-\alpha} \Psi \left(\frac{x - x_0 - ct}{\varepsilon}, \frac{y - y_0}{\varepsilon} \right), \tag{2.3}$$

where Ω and Ψ are profiles decreasing away from $(0,0)$. Note that α is here arbitrary and can be positive or negative. We assume that y_0 is such that

$$V''(y_0) = \cdots = V^{(n)}(y_0) = 0, \quad \forall\, n \geq 2. \tag{2.4}$$

It means that V'' is flat in the variable y in the vicinity of y_0. We perform the scaling

$$X = \frac{x - x_0}{\varepsilon}, \quad Y = \frac{y - y_0}{\varepsilon}, \quad \text{and} \quad T = \frac{t}{\varepsilon}, \tag{2.5}$$

and plug the expressions (2.2) and (2.3) into (2.1). This yields

$$\begin{cases} -c \partial_X \Omega + V'(y_0 + \varepsilon Y) \partial_X \Omega + \varepsilon^2 V'''(y_0 + \varepsilon Y) \partial_X \Psi + \varepsilon^{1-\alpha} \{\Psi, \Omega\} = 0, \\ -\Delta_{X,Y} \Psi = \Omega, \end{cases} \tag{2.6}$$

where the Poisson bracket and the Laplace operator express in terms of X and Y. Note that here we consider $(X,Y) \in \mathbb{R}^2$. This is a *profile* equation as for boundary layers in singular perturbation theory, see for instance [10].

Now under the assumption (2.4), we have that for all N, $V'(y_0 + \varepsilon Y) = V'(y_0) + \mathcal{O}(\varepsilon^N Y^N)$ and similarly $V'''(y_0 + \varepsilon Y) = \mathcal{O}(\varepsilon^N Y^N)$. Hence at least from the formal point of view, we can solve (2.6) up to high powers of ε if we take

$$c = V'(y_0),$$

and if Ω and Ψ are solutions of the equation

$$-\Delta_{X,Y} \Psi = \Omega, \quad \text{and} \quad \{\Psi, \Omega\} = 0 \quad \text{in} \quad \mathbb{R}^2, \tag{2.7}$$

where $\Delta_{X,Y} = \partial_X^2 + \partial_Y^2$ is the Laplace operator in coordinates $(X,Y) \in \mathbb{R}^2$. This equation means that (Ψ, Ω) is a stationary state of the Euler equation on $\mathbb{R}^2$. To match the asymptotic profile with the stationary state $V(y)$, we naturally seek for exponentially decreasing solutions.

We are lead to make the following assumption:

Hypothesis 2.1 *The functions $\Omega(X,Y)$ and $\Psi(X,Y)$ satisfy (2.7) and are exponentially decreasing in the sense that there exists constants μ and β such that*

$$\forall\,(X,Y)\in\mathbb{R}^2,\quad \forall\,m=0,1,\quad |\nabla^m\Omega(X,Y)|+|\nabla^m\Psi(X,Y)|\leq \mu e^{-\beta(|Y|+|X|)}. \tag{2.8}$$

Example 2.2 *Let g be a smooth function from $\mathbb{R}$ to itself. If (Ψ,Ω) are solutions of the problem*

$$-\Delta_{X,Y}\Psi = g(\Psi) =: \Omega, \tag{2.9}$$

then it is automatically a solution of (2.7). For example we can take the soliton equation

$$\Delta\Psi+\Psi^3-\Psi \quad in \quad \mathbb{R}^2,$$

and the associated ground state with radial symmetry, which yields exponentially decaying solutions in the previous sense, see [9, 11]. We also refer to [1, 8, 4, 5] and the references therein for other examples of stationary states of the two-dimensional Euler equation on $\mathbb{R}^2$.

3 Approximate solution

Let a,b two real numbers such that $0<a<b<\pi/2$, and let $\chi(x,y)$ be a smooth cut-off function satisfying

$$\chi(x,y)=0 \quad \text{for} \quad |x|+|y|>b, \quad \text{and} \quad \chi(x,y)=1 \quad \text{for} \quad |x|+|y|<a. \tag{3.1}$$

Theorem 3.1 *Assume that V satisfies (2.4) for some $y_0\in\mathbb{T}^2$, and let (Ψ,Ω) a couple of functions satisfying Hypothesis 2.1. For $x_0\in\mathbb{T}^2$, $\alpha\in\mathbb{R}$, and $\varepsilon>0$, we define*

$$\omega^\varepsilon(t,x,y)=V''(y)+\varepsilon^{-\alpha}\Omega\left(\frac{x-x_0-ct}{\varepsilon},\frac{y-y_0}{\varepsilon}\right)\chi(x-x_0-ct,y-y_0) \tag{3.2}$$

and

$$\psi^\varepsilon(t,x,y)=-V(y)+\varepsilon^{2-\alpha}\Psi\left(\frac{x-x_0-ct}{\varepsilon},\frac{y-y_0}{\varepsilon}\right)\chi(x-x_0-ct,y-y_0), \tag{3.3}$$

where $c=V'(y_0)$. Then for all N, there exists constants ε_0 and C_N such that for all $\varepsilon<\varepsilon_0$, the couple $(\omega^\varepsilon,\psi^\varepsilon)$ satisfies

$$\forall\,t\in\mathbb{R},\quad \|\partial_t\omega^\varepsilon+\{\psi^\varepsilon,\omega^\varepsilon\}\|_{L^\infty}+\|\Delta\psi^\varepsilon+\omega^\varepsilon\|_{L^\infty}\leq C_N{\varepsilon}^N. \tag{3.4}$$

Proof. First, we note that as $b < \pi/2$, the functions ω^ε and ψ^ε define smooth periodic functions in $(x, y) \in \mathbb{T}^2$. We calculate, omitting the argument $(x - x_0 - ct, y - y_0)/\varepsilon$, that

$$\Delta_{x,y}\psi^\varepsilon = -V''(y) + \varepsilon^{-\alpha+2}\big((\Delta_{x,y}\chi)\Psi + \chi\Delta_{x,y}\Psi + 2\nabla_{x,y}\chi \cdot \nabla_{x,y}\Psi)\big),$$

where here $\Delta_{x,y} = \partial_x^2 + \partial_y^2$. Hence we get

$$\Delta_{x,y}\psi^\varepsilon + \omega^\varepsilon = \varepsilon^{-\alpha}\chi(\varepsilon^2\Delta_{x,y}\Psi + \Omega) + R(t, x, y, \varepsilon),$$

where

$$R(t, x, y, \varepsilon) = \varepsilon^{-\alpha+2}\big((\Delta_{x,y}\chi)\Psi + 2\nabla_{x,y}\chi \cdot \nabla_{x,y}\Psi)\big).$$

By construction and with the notation (2.5), we have

$$\begin{aligned}(\varepsilon^2\Delta_{x,y}\Psi + \Omega)\left(\frac{x - x_0 - ct}{\varepsilon}, \frac{y - y_0}{\varepsilon}\right)\chi(x - x_0 - ct, y - y_0)& \\ = (\Delta_{X,Y}\Psi + \Omega)(X - cT, Y)\chi(\varepsilon(X - cT), \varepsilon Y) = 0&,\end{aligned}$$

using (2.9), and where this expression is well defined as χ is with compact support.

Now let us examine the terms in $R(t, x, y, \varepsilon)$. We consider for example the term

$$\varepsilon^{-\alpha+2}\partial_x\chi(x - x_0 - ct, y - y_0)\partial_x\Omega\left(\frac{x - x_0 - ct}{\varepsilon}, \frac{y - y_0}{\varepsilon}\right) \tag{3.5}$$

This expression vanishes unless (t, x, y) satisfies

$$a < |x - x_0 - ct| + |y - y_0| < b,$$

which corresponds to the interval where the gradient of χ is non zero (see (3.1)). For such points, we have

$$\left|\partial_x\Omega\left(\frac{x - x_0 - ct}{\varepsilon}, \frac{y - y_0}{\varepsilon}\right)\right| = \frac{1}{\varepsilon}\left|\partial_X\Omega\left(\frac{x - x_0 - ct}{\varepsilon}, \frac{y - y_0}{\varepsilon}\right)\right| \le \varepsilon^{-1}\mu e^{-\beta a/\varepsilon}, \tag{3.6}$$

owing to (2.8). As χ is smooth with bounded gradient, this shows that the term (3.5) is smaller than $c_N\varepsilon^N$ for a suitable constant N and a sufficiently small ε. The other are treated similarly, and we finally get

$$\|R(x, y, t)\|_{L^\infty} \le C_N\varepsilon^N. \tag{3.7}$$

This shows that for all time t, we have

$$\|\Delta\psi^\varepsilon + \omega^\varepsilon\|_{L^\infty} \le C_N\varepsilon^N.$$

Similarly, we can write

$$\begin{aligned}\partial_t\omega^\varepsilon + \{\psi^\varepsilon, \omega^\varepsilon\} &= \partial_t\omega^\varepsilon + (\partial_x\psi^\varepsilon)(\partial_y\omega^\varepsilon) - (\partial_y\psi^\varepsilon)(\partial_x\omega^\varepsilon) \\ &= \partial_t\xi^\varepsilon + V'\partial_x\xi^\varepsilon + V'''\partial_x\eta^\varepsilon + \{\eta^\varepsilon, \xi^\varepsilon\}\end{aligned}$$

where we have set $\xi^\varepsilon = \omega^\varepsilon - V''(y)$ and $\eta^\varepsilon = \psi^\varepsilon + V(y)$. In this expression, by similar arguments using the exponential decay of Ω and Ψ and the fact that χ is constant near $(0,0)$, we have

$$\{\eta^\varepsilon, \xi^\varepsilon\} = \varepsilon^{-2\alpha}\chi(\partial_X\Psi\partial_Y\Omega - \partial_X\Omega\partial_Y\Psi) + \mathcal{O}(\varepsilon^N) = \mathcal{O}(\varepsilon^N),$$

as the couple (Ψ, Ω) satisfies (2.7).
Hence it remains to consider

$$\begin{aligned}&\partial_t\xi^\varepsilon + V'\partial_x\xi^\varepsilon + V'''\partial_x\eta^\varepsilon\\ &= \varepsilon^{-\alpha}(\partial_x\chi)(-c\Omega + V'\Omega + \varepsilon^2V'''\Psi) + \chi\varepsilon^{-\alpha-1}(-c\partial_X\Omega + V'\partial_X\Omega + \varepsilon^2V'''\partial_X\Psi).\end{aligned}$$

Using again the exponential decay of Ω and Ψ, the term in factor of $\partial_x\chi$ is $\mathcal{O}(\varepsilon^N)$. Considering the second term in the right-hand side of the previous equation, we have

$$\begin{aligned}&\varepsilon^{-\alpha-1}\chi(x - x_0 - ct, y - y_0)V'''(y)\partial_X\Psi\left(\frac{x - x_0 - ct}{\varepsilon}, \frac{y - y_0}{\varepsilon}\right)\\ &\qquad\qquad = \varepsilon^{-\alpha-1}\chi(\varepsilon(X - cT), \varepsilon Y)V'''(y_0 + \varepsilon Y)\partial_X\Psi(X - cT, Y),\end{aligned}$$

using the notation (2.5). For $|\varepsilon Y| > b$, this expression vanishes as χ is identically equal to zero.
Let us consider now the case $|\varepsilon Y| < a$. As V is smooth and satisfies (2.4), then for all m there exists a constant C_m such that for all z such that $|z| < a$, we have using a Taylor expansion, that

$$|V'''(y_0 + z)| \le C_m|z|^m. \tag{3.8}$$

Hence for Y such that $|\varepsilon Y| \le a$ and all X, we have with (2.8)

$$|V'''(y_0 + \varepsilon Y)\partial_X\Psi(X - cT, Y)| \le C_m\mu\varepsilon^m|Y|^m e^{-\beta|Y|} \le \tilde{C}_m\varepsilon^m \tag{3.9}$$

for some constant $\tilde{C}_m$. Applying this formula with $m = N + \alpha + 1$, we get

$$\|\chi\varepsilon^{-\alpha-1}V'''\partial_X\Psi\|_{L^\infty} \le C_N\varepsilon^N$$

for some constant C_N and $\varepsilon < \varepsilon_0$ sufficiently small. Similarly, as $-c + V'(y) = V'(y) - V'(y_0)$ satisfies an estimate similar to (3.8), we can prove that

$$\chi\varepsilon^{-\alpha-1}(-c + V')\partial_X\Omega = \mathcal{O}(\varepsilon^N).$$

Gathering together the previous inequalities yields the result. ∎

Remark 3.2 *By construction, the function ω^ε is a travelling wave at constant velocity. Hence the preservation property (1.2) holds for all functions h. In particular all the L^p norms of ω^ε are preserved for all time. Similarly, using the* *estimate* (3.4)*, the energy*

$$\begin{aligned} E(t) &= \int_{\mathbb{T}^2} \omega^\varepsilon(t,x,y)(-\Delta)^{-1}\omega^\varepsilon(t,x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{T}^2} \omega^\varepsilon(t,x,y)\psi^\varepsilon(t,x,y)\mathrm{d}x\mathrm{d}y + \mathcal{O}(\varepsilon^N) = E(0) + \mathcal{O}(\varepsilon^N) \end{aligned}$$

is almost preserved for all times.

Remark 3.3 *The same result holds true for the function* (1.4)*, under the hypothesis that the supports of the functions Ω_k^ε do not interact. The proof is very similar to the previous one and left to the reader.*

4 Exponential estimates

We now make the supplementary assumption that V has Gevrey regularity around y_0:

Hypothesis 4.1 *The function V satisfies* (2.4)*, and moreover, there exist positive constants M, R and $\delta \geq 1$, such that*

$$\forall\, n \in \mathbb{N}, \quad \forall\, y \quad \textit{such that} \quad |y - y_0| < 2a, \quad |V^{(n)}(y)| \leq M R^n n^{\delta n}. \tag{4.1}$$

A typical example of Gevrey function around y_0 is given by the function $e^{-1/(y-y_0)^2}$ if $y > y_0$ and 0 if $y \leq y_0$.

Theorem 4.2 *Assume that V satisfies Hypothesis 4.1 for some $y_0 \in \mathbb{T}^2$ and $\delta \geq 1$, and let (Ψ, Ω) a couple of functions satisfying Hypothesis 2.1. For $x_0 \in \mathbb{T}^2$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$, let $\omega^\varepsilon(t,x,y)$ and $\psi^\varepsilon(t,x,y)$ the functions defined in* (3.2) *and* (3.3)*. Then there exist constants ε_0, γ and C such that for all $\varepsilon < \varepsilon_0$, the couple $(\omega^\varepsilon, \psi^\varepsilon)$ satisfies*

$$\forall\, t \in \mathbb{R}, \quad \|\partial_t \omega^\varepsilon + \{\psi^\varepsilon, \omega^\varepsilon\}\|_{L^\infty} + \|\Delta\psi^\varepsilon + \omega^\varepsilon\|_{L^\infty} \leq C e^{-\gamma\varepsilon^{-\sigma}}, \tag{4.2}$$

where $\sigma = 1/(\delta + 1)$.

Proof. The method of proof is the same as the proof of Theorem 3.1. First, we note that all the estimates involving derivatives of the cut-off functions are in fact exponentially decreasing with respect to ε, see (3.6), so that (3.7) can be easily refined to

$$\|R(t,x,y)\|_{L^\infty} \leq C e^{-\kappa/\varepsilon}$$

for some constants κ and C, and $\varepsilon < \varepsilon_0$ sufficiently small.

The second source in the error term comes from equation (3.8). Under the assumption (4.1), (3.8) becomes for all $|z| < a$,

$$|V'''(y_0 + z)| = \frac{1}{m!}|z|^m \sup_{|t|\leq a} |V^{(m+3)}(y_0 + t)| \leq \frac{M}{m!}|z|^m R^{m+3}(m+3)^{\delta(m+3)},$$

and hence for $m > m_0$ sufficiently large, we get

$$\forall\, |z| < a, \quad |V'''(y_0 + z)| \leq K|z|^m m^{\delta m}.$$

for some constant K independent on m. Hence the second estimate (3.9) yields now an error of the form

$$K m^{\delta m} \mu \varepsilon^m |Y|^m e^{-\beta|Y|} \leq K\mu\Big(\frac{m^{(\delta+1)}\varepsilon}{e\beta}\Big)^m,$$

for $m > m_0$, owing to the estimate

$$\forall\, x > 0, \quad x^m e^{-\beta x} \leq \Big(\frac{m}{e\beta}\Big)^m.$$

Taking

$$m = \left(\frac{\beta}{\varepsilon}\right)^{\frac{1}{\delta+1}}$$

for $\varepsilon < \varepsilon_0$ sufficiently small to ensure $m > m_0$ then yields an error of the form $Ce^{-\gamma\varepsilon^{-\sigma}}$ with $\sigma = 1/(\delta+1)$, $C = K\mu$ and $\gamma = \beta^{\frac{1}{\delta+1}}$. The other error terms are similar. ∎

5 Numerical illustration

In this last section, we would like to show the validity of the previous analysis by a numerical experiment. In the example below, we consider an initial value made of two localized Gaussians over a stationary state V containing large flat parts, and we show the evolution of localized packets at the expected speed $c = V'(y_0)$. For practical reasons, the following numerical simulations below are made on the equation

$$\partial_t \omega - \{\psi, \omega\} = 0, \quad \text{and} \quad -\Delta\psi = \omega,$$

which can be obtained from (1.1) by a change $t \mapsto -t$. Hence the localized profiles are expected to travel at the speed $-V'(y_0)$ which we will numerically aproximate by $\partial_y \psi(x, y_0)$.

We take the initial condition ($x_0 = \pi/2$, $x_1 = 2\pi/3$ and $y_0 = \pi/4$, $y_1 = 7\pi/4$)

$$\omega(t=0,x,y) = V''(y) + 5\exp\left(-\frac{(x-\pi/2)^2 + (y-\pi/4)^2}{\varepsilon^2}\right) - 5\exp\left(-\frac{(x-2\pi/3)^2 + (y-7\pi/4)^2}{\varepsilon^2}\right),$$

where $V''(y)$ is the function

$$V''(y) = \exp\left(-10(y-2\pi/3)^2\right) - \exp\left(-10(y-4\pi/3)^2\right).$$

We choose $\varepsilon = 0.1$. Figure 1 shows the initial condition.

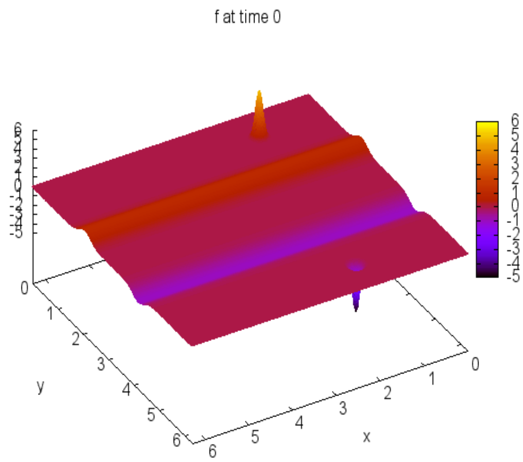

Figure 1: Initial condition $\omega(t=0,x,y)$.

To simulate the solution, we use a semi-Lagrangian scheme (we refer to [6, 7] for the details of the method). The numerical parameters are chosen as follows: we take $N_x = N_y = 512$ grid points in each direction and the time step is $\Delta t = 0.05$.

As expected, the localized profiles start to move in the x-direction, (see Figures 4-5), and remain localized at least for a time $t \leq 20 \simeq \mathcal{O}(\varepsilon^{-1})$. After this time, the structure is lost, which makes sense since the Gaussians are not stable states of the 2D Euler equation on $\mathbb{R}^2$.

We look at the velocity of the profile during this regime. It can be computed numerically, and compared to the value $c = -V'(y_0) = \partial_y \psi(t=0, x=0, y_0)$. This last value is observed on the y-profile of ψ, which is linear in a vicinity of

x_0	time	displacement	velocity
1.5707963267	0		
0.589048622548	5	-0.981747704152	-0.1963495408304
5.91502991808	10	-0.95720400095	-0.19144080019
4.970097752749	15	-0.944932165331	-0.1889864330662
3.9638063558964	20	-1.0062913968526	-0.20125827937052
2.9329712664373	25	-1.0308350894591	-0.20616701789182

Table 1: Numerical values of the center of the profiles, the corresponding time, the displacement with respect to the previous time and the corresponding velocity.

y_0. Figure 2 confirms the linear behaviour around $y_0 = \pi/4$ (we superimpose a linear function of slope -0.19). This value of the velocity can be compared to the velocity of the localized profiles. On Figure 3, x-profiles of ω are plotted for $y = \pi/4$. We can observe the travel of the profile along the periodic x-direction. Each consecutive plot is plotted every $t = 5$ so that the velocity is estimated to -0.19, which is in very good agreement with the previous computed one. More details are given in Table 1. We can observe that the numerical diffusion is quite low since the maximum of the profiles decreases slowly as time advances.

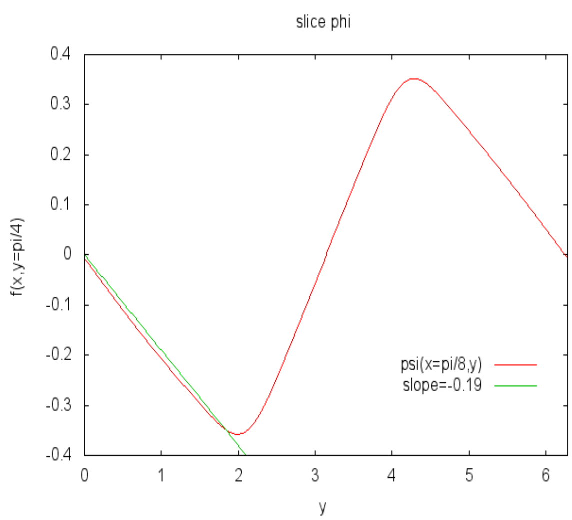

Figure 2: y-profile of ψ at $t = 0$, and the linear function of slope -0.19.

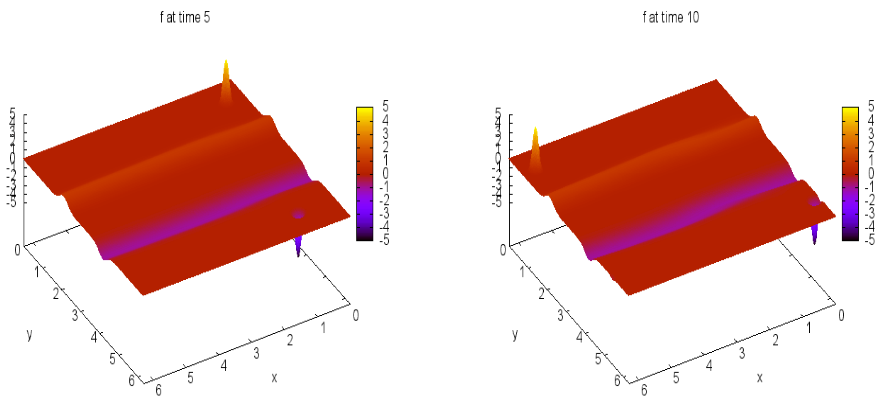

Figure 4: Time evolution of the solution: $\omega(=5,x,y)$ (left) and $\omega(t=10,x,y)$ (right).

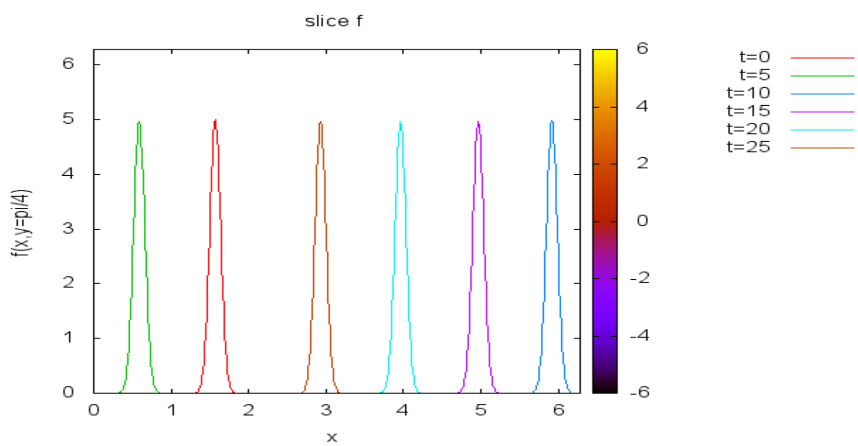

Figure 3: Time evolution of the profile: $\omega(t,x,y=\pi/4)$ as a function of x for different times: $t=0, t=5, t=10, t=15, t=20, t=25$.

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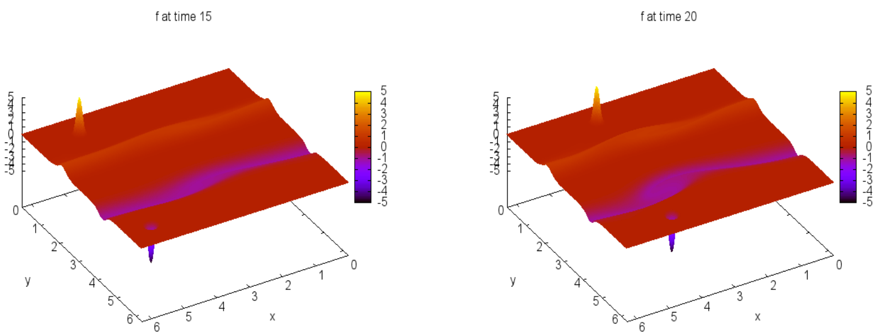

Figure 5: Time evolution of the solution: $\omega(t = 15, x, y)$ (left) and $\omega(t = 20, x, y)$ (right).

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