The objective of this problem is to characterize the first hitting time of a sphere centered at the origin, by the $d$–dimensional Brownian motion and by the two–dimensional Ornstein–Uhlenbeck process (a Gaussian diffusion process).

Let $B$ be a $d$–dimensional standard Brownian motion, and adapted to a given filtration $\mathcal{F}$. Let $\theta = (\theta(t), t \geq 0)$ be a $d$–dimensional stochastic process in $M^2 = \bigcap_{T \geq 0} M^2([0, T])$, seen as a row–vector, adapted to the same filtration $\mathcal{F}$. Here is a first preliminary result.

(i) If
\[ \int_0^t \theta(s) \theta^*(s) \, ds = t, \]
almost surely for any $t \geq 0$, then the process defined by
\[ W(t) = \int_0^t \theta(s) \, dB(s), \]
for any $t \geq 0$, satisfies
\[ \mathbb{E}[\exp\{i \lambda (W(t) - W(s))\} \mid \mathcal{F}(s)] = \exp\{-\frac{1}{2} \lambda^2 (t - s)\}, \]
almost surely, for any scalar $\lambda$ and for any $0 \leq s \leq t$, hence it is a one–dimensional standard Brownian motion.

[Hint: Write the Itô formula for the process $W$ and for the complex–valued function $f(x) = \exp\{i \lambda x\}$ where the scalar $\lambda$ is fixed, between the time instants $s$ and $t$, with $0 \leq s \leq t$.]

---

**Solution**

Clearly, the complex–valued function $f(x) = \exp\{i \lambda x\}$ is twice continuously differentiable with $f'(x) = i \lambda f(x)$ and $f''(x) = -\lambda^2 f(x)$, and writing the Itô formula yields
\[
\exp\{i \lambda W(t)\} = \exp\{i \lambda W(s)\} + i \lambda \int_s^t \exp\{i \lambda W(u)\} \, dW(u) \\
- \frac{1}{2} \lambda^2 \int_s^t \exp\{i \lambda W(u)\} \theta(u) \theta^*(u) \, du \\
= \exp\{i \lambda W(s)\} + i \lambda \int_s^t \exp\{i \lambda W(u)\} \theta(u) \, dB(u) \\
- \frac{1}{2} \lambda^2 \int_s^t \exp\{i \lambda W(u)\} \, du.
\]
Note that
\[ \mathbb{E} \int_0^T \left| \exp\{i\lambda W(u)\} \right|^2 \theta(u) \theta^*(u) \, du = T < \infty , \]
for any \( T \geq 0 \), so that the stochastic integral is a square–integrable martingale, with zero expectation, hence
\[ \mathbb{E}\left[ \int_s^t \exp\{i\lambda W(u)\} \theta(u) \, dB(u) \mid \mathcal{F}(s) \right] = 0 . \]
Therefore
\[ \mathbb{E}\left[ \exp\{i\lambda W(t)\} \mid \mathcal{F}(s) \right] = \exp\{i\lambda W(s)\} - \frac{1}{2} \lambda^2 \int_s^t \mathbb{E}\left[ \exp\{i\lambda W(u)\} \mid \mathcal{F}(s) \right] \, du , \]
so that the function defined by
\[ f(t) = \mathbb{E}\left[ \exp\{i\lambda W(t)\} \mid \mathcal{F}(s) \right] , \]
satisfies the linear ODE
\[ f(t) = f(s) - \frac{1}{2} \lambda^2 \int_s^t f(u) \, du , \]
the explicit solution of which is
\[ f(t) = f(s) \exp\{-\frac{1}{2} \lambda^2 (t - s)\} . \]
In other words
\[ \mathbb{E}\left[ \exp\{i\lambda W(t)\} \mid \mathcal{F}(s) \right] = \exp\{i\lambda W(s)\} \exp\{-\frac{1}{2} \lambda^2 (t - s)\} , \]
or equivalently
\[ \mathbb{E}\left[ \exp\{i\lambda (W(t) - W(s)) \mid \mathcal{F}(s) \right] = \exp\{-\frac{1}{2} \lambda^2 (t - s)\} . \]
This shows that the increment \( (W(t) - W(s)) \) is independent of \( \mathcal{F}(s) \), normally distributed with zero mean and variance \( (t - s) \). This holds for any \( 0 \leq s \leq t \), hence the process \( W \) is a one–dimensional standard Brownian motion, adapted to the filtration \( \mathcal{F} \).

\[ \square \]

**Bessel processes**

(ii) **Write the Itô formula for the \( d \)--dimensional standard Brownian motion \( B \) and for the real–valued function \( f(x) = |x|^2 = x^* x \).**

**Solution**

Clearly
\[ f'(x) = 2x^* \quad \text{and} \quad f''(x) = 2I , \]
hence
\begin{align*}
|B(t)|^2 &= |B(0)|^2 + 2 \int_0^t B^*(s) \, dB(s) + \frac{1}{2} \int_0^t \text{trace}(2I) \, ds \\
&= |B(0)|^2 + dt + 2 \int_0^t B^*(s) \, dB(s) ,
\end{align*}

since trace($I$) = $d$ for the $d \times d$ identity matrix $I$.

(iii) Using the result proved at question (i), show that the squared Bessel process defined by $V(t) = |B(t)|^2$ for any $t \geq 0$, satisfies the SDE

$$V(t) = V(0) + dt + 2 \int_0^t \sqrt{V(s)} \, dW(s),$$

for any $t \geq 0$, and for some one–dimensional standard Brownian motion $W$.

---

**Solution**

Note that

$$\int_0^t B^*(s) \, dB(s) = \int_0^t B^*(s) \, 1(B(s) \neq 0) \, dB(s) = \int_0^t |B(s)| \frac{B^*(s)}{|B(s)|} \, 1(B(s) \neq 0) \, dB(s).$$

Introducing

$$\theta(s) = \frac{B^*(s)}{|B(s)|} \, 1(B(s) \neq 0),$$

it holds

$$\theta(s) \theta^*(s) = \frac{B^*(s)}{|B(s)|} \frac{B(s)}{|B(s)|} \, 1(B(s) \neq 0) = 1(B(s) \neq 0),$$

hence

$$\int_0^t \theta(s) \theta^*(s) \, ds = \int_0^t 1(B(s) \neq 0) \, ds = t - \int_0^t 1(B(s) = 0) \, ds.$$

Finally, using the Fubini theorem yields

$$\mathbb{E} \int_0^t 1(B(s) = 0) \, ds = \int_0^t \mathbb{P}[B(s) = 0] \, ds = 0,$$

hence

$$\int_0^t 1(B(s) = 0) \, ds = 0,$$

almost surely, since a nonnegative random variable with zero expectation is almost surely equal to zero. Therefore

$$\int_0^t \theta(s) \theta^*(s) \, ds = t,$$

almost surely for any $t \geq 0$, and it follows from the result proved at question (i) that the process defined by

$$W(t) = \int_0^t \theta(s) \, dB(s),$$

for any $t \geq 0$, is a one–dimensional standard Brownian motion, and

$$\int_0^t B^*(s) \, dB(s) = \int_0^t |B(s)| \theta(s) \, dB(s) = \int_0^t |B(s)| \, dW(s).$$
It can be shown that there exists a unique solution (even though the diffusion coefficient is only Hölder continuous) to this one-dimensional SDE. The boundary \( \{0\} \) is never reached, and if the initial condition is \( V(0) = 0 \) then the boundary \( \{0\} \) is left immediately and never returned to.

Consider the hitting time

\[
T_a = \inf\{t \geq 0 : V(t) \geq a\} = \inf\{t \geq 0 : |B(t)| \geq \sqrt{a}\},
\]

and the related Laplace transform

\[
u(x) = \mathbb{E}_{0,x}[\exp\{-\lambda T_a\}],
\]

defined for any \( 0 \leq x \leq a \) and for any \( \lambda > 0 \). Clearly, the function \( \nu(x) \) is finite near \( x = 0 \), and takes the value 1 at \( x = a \).

(iv) Give the expression of the second–order partial differential operator associated with the squared Bessel process.

Let \( \nu = \frac{1}{2} (d - 2) \). Prove that the Laplace transform \( \nu(x) \) satisfies the ODE

\[
2 x \nu''(x) + 2 (\nu + 1) \nu'(x) - \lambda \nu(x) = 0,
\]

for any \( 0 < x < a \), and give the two boundary conditions satisfied near \( x = 0 \) and at \( x = a \).

Solution

The second–order partial differential operator associated with the squared Bessel process is

\[
L = 2 x \frac{d^2}{dx^2} + 2 (\nu + 1) \frac{d}{dx},
\]

with \( \nu = \frac{1}{2} (d - 2) \) so that \( d = 2 (\nu + 1) \), hence the PDE

\[
L \nu(x) - \lambda \nu(x) = 0,
\]

satisfied by the Laplace transform \( \nu(x) \) reduces to the ODE

\[
2 x \nu''(x) + 2 (\nu + 1) \nu'(x) - \lambda \nu(x) = 0,
\]

for any \( 0 < x < a \), with the two constraints that

- it should be finite near \( x = 0 \), i.e. \( \nu(0+) \) is finite,
- it should take the value 1 at \( x = a \), i.e. \( \nu(a) = 1 \).

The explicit solution to this boundary–value problem involves modified Bessel functions: some facts about these special functions are collected in the Appendix.
Check that if the function $w(x)$ is a particular solution of (3), then the function defined by $u(x) = (\sqrt{x})^{-\nu} w(\sqrt{2\lambda x})$ for any $x \geq 0$ is a particular solution of (1).

[Hint: Lengthy calculations, skip this.]

**Solution**

If the function $w(x)$ solves the ODE

$$x^2 \, w''(x) + x \, w'(x) - (x^2 + \nu^2) \, w(x) = 0$$

then upon an obvious change of variable it holds

$$2\lambda \, x \, w''(\sqrt{2\lambda x}) + \sqrt{2\lambda x} \, w'(\sqrt{2\lambda x}) - (2\lambda \, x + \nu^2) \, w(\sqrt{2\lambda x}) = 0 \quad (\star)$$

for any $x \geq 0$. Using the chain rule, the function $w_0(x) = w(\sqrt{2\lambda x})$ satisfies

$$w_0'(x) = \sqrt{2\lambda} \, \frac{1}{2\sqrt{x}} \, w'(\sqrt{2\lambda x})$$

and

$$w_0''(x) = -\sqrt{2\lambda} \, \frac{1}{4x\sqrt{x}} \, w'(\sqrt{2\lambda x}) + 2\lambda \, \frac{1}{4x} \, w''(\sqrt{2\lambda x})$$

hence $u(x) = (\sqrt{x})^{-\nu} \, w_0(x)$ satisfies

$$u'(x) = -\frac{1}{2} \nu \, (\sqrt{x})^{-\nu-2} \, w_0(x) + (\sqrt{x})^{-\nu} \, w_0'(x)$$

$$= -\frac{1}{2} \nu \, (\sqrt{x})^{-\nu-2} \, w(\sqrt{2\lambda x}) + \frac{1}{2} \sqrt{2\lambda} \, (\sqrt{x})^{-\nu-1} \, w'(\sqrt{2\lambda x})$$

and using the Leibniz rule yields

$$u''(x) = \frac{1}{2} \nu \, (\frac{1}{2} \nu + 1) \, (\sqrt{x})^{-\nu-4} \, w_0(x) - 2 \frac{1}{2} \nu \, (\sqrt{x})^{-\nu-2} \, w_0'(x) + (\sqrt{x})^{-\nu} \, w_0''(x)$$

$$= \frac{1}{2} \nu \, (\frac{1}{2} \nu + 1) \, (\sqrt{x})^{-\nu-4} \, w(\sqrt{2\lambda x}) - \frac{1}{2} \nu \sqrt{2\lambda} \, (\sqrt{x})^{-\nu-3} \, w'(\sqrt{2\lambda x})$$

$$- \frac{1}{4} \sqrt{2\lambda} \, (\sqrt{x})^{-\nu-3} \, w'(\sqrt{2\lambda x}) + \frac{1}{2} \lambda \, (\sqrt{x})^{-\nu-2} \, w''(\sqrt{2\lambda x})$$

$$= \frac{1}{2} \nu \, (\frac{1}{2} \nu + 1) \, (\sqrt{x})^{-\nu-4} \, w(\sqrt{2\lambda x}) - \frac{1}{2} \left( \nu + \frac{1}{2} \right) \sqrt{2\lambda} \, (\sqrt{x})^{-\nu-3} \, w'(\sqrt{2\lambda x})$$

$$+ \frac{1}{2} \lambda \, (\sqrt{x})^{-\nu-2} \, w''(\sqrt{2\lambda x})$$

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Therefore

\[ 2x u''(x) + 2 (\nu + 1) u'(x) - \lambda u(x) \]

\[ = 2x \left[ \frac{1}{2} \nu \left( \frac{1}{2} \nu + 1 \right) (\sqrt{x})^{-\nu-4} w(\sqrt{2\lambda x}) - \frac{1}{2} (\nu + \frac{1}{2}) \sqrt{2\lambda} (\sqrt{x})^{-\nu-3} w'(\sqrt{2\lambda x}) \right. \]

\[ + \frac{1}{2} \lambda (\sqrt{x})^{-\nu-2} w''(\sqrt{2\lambda x}) \]

\[ + 2 (\nu + 1) \left[ -\frac{1}{2} \nu (\sqrt{x})^{-\nu-2} w(\sqrt{2\lambda x}) + \frac{1}{2} \sqrt{2\lambda} (\sqrt{x})^{-\nu-1} w'(\sqrt{2\lambda x}) \right. \]

\[ - \lambda (\sqrt{x})^{-\nu} w(\sqrt{2\lambda x}) \]

\[ = \lambda x (\sqrt{x})^{-\nu-2} w''(\sqrt{2\lambda x}) \]

\[ + [-(\nu + \frac{1}{2}) + (\nu + 1)] \sqrt{2\lambda} (\sqrt{x})^{-\nu-1} w'(\sqrt{2\lambda x}) \]

\[ + \left[ \nu \left( \frac{1}{2} \nu + 1 \right) - \nu (\nu + 1) \right] (\sqrt{x})^{-\nu-2} w(\sqrt{2\lambda x}) - \lambda (\sqrt{x})^{-\nu} w(\sqrt{2\lambda x}) \]

\[ = \frac{1}{2} (\sqrt{x})^{-\nu-2} \left[ 2\lambda x w''(\sqrt{2\lambda x}) + \sqrt{2\lambda x} w'(\sqrt{2\lambda x}) - (2\lambda x + \nu^2) w(\sqrt{2\lambda x}) \right] \]

and it follows from (\star) that

\[ 2x u''(x) + 2 (\nu + 1) u'(x) - \lambda u(x) = 0 \]

\[ \square \]

Clearly, the functions defined by \( u_1(x) = (\sqrt{x})^{-\nu} I_{\nu}(\sqrt{2\lambda x}) \) and by \( u_2(x) = (\sqrt{x})^{-\nu} K_{\nu}(\sqrt{2\lambda x}) \) for any \( x \geq 0 \) are two particular solutions of (1). These two particular solutions are linearly independent, so that any solution of (3) is a linear combination

\[ u(x) = c_1 u_1(x) + c_2 u_2(x) = c_1 (\sqrt{x})^{-\nu} I_{\nu}(\sqrt{2\lambda x}) + c_2 (\sqrt{x})^{-\nu} K_{\nu}(\sqrt{2\lambda x}) . \]

(vi) Show that the Laplace transform is given by

\[ u(x) = \left( \frac{\sqrt{a}}{\sqrt{x}} \right)^{\nu} \frac{I_\nu(\sqrt{2\lambda x})}{I_\nu(\sqrt{2\lambda a})} , \]

and in particular

\[ u(0) = \left( \frac{1}{2} \sqrt{2\lambda} \right)^{\nu} \frac{(\sqrt{a})^\nu}{\Gamma(\nu + 1)} \frac{I_\nu(\sqrt{2\lambda a})}{I_\nu(\sqrt{2\lambda a})} , \]

at \( x = 0. \)

\[ \text{Solution} \]
The two particular solutions $u_1(x)$ and $u_2(x)$ differ by their asymptotic behaviour near $x = 0$. Indeed

$$I_\nu(\sqrt{2\lambda x}) \sim \left(\frac{1}{2} \sqrt{2\lambda}\right)^\nu \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1)} (\sqrt{x})^\nu,$$

hence

$$u_1(x) = (\sqrt{x})^{-\nu} I_\nu(\sqrt{2\lambda x}) \sim \left(\frac{1}{2} \sqrt{2\lambda}\right)^\nu \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1)} (\sqrt{x})^\nu,$$

meets the constraint that the solution should be finite near $x = 0$. On the other hand, if $\nu > 0$ then

$$K_\nu(\sqrt{2\lambda x}) \sim \frac{1}{2} \frac{\Gamma(\nu)}{(\frac{1}{2} \sqrt{2\lambda})^\nu} = \frac{1}{2} \frac{\Gamma(\nu)}{(\frac{1}{2} \sqrt{2\lambda})^\nu} (\sqrt{x})^{-\nu},$$

hence

$$u_2(x) = (\sqrt{x})^{-\nu} K_\nu(\sqrt{2\lambda x}) \sim \left(\frac{1}{2} \sqrt{2\lambda}\right)^\nu (\sqrt{x})^{-\nu},$$

does not meet the constraint that the solution should be finite near $x = 0$, and similarly if $\nu = 0$ then

$$u_2(x) = K_0(\sqrt{2\lambda x}) \sim -\log \sqrt{2\lambda x},$$

does not meet the constraint that the solution should be finite near $x = 0$. Therefore, a solution of (1) that meets the constraint that it should be finite near $x = 0$ is necessarily of the form

$$u(x) = c_1 (\sqrt{x})^{-\nu} I_\nu(\sqrt{2\lambda x}),$$

and the integration constant $c_1$ is determined by the constraint that the solution should take the value 1 at $x = a$, hence

$$c_1 = \frac{1}{(\sqrt{a})^{-\nu} I_\nu(\sqrt{2\lambda a})},$$

and

$$u(x) = \left(\frac{\sqrt{a}}{(\sqrt{a})^{-\nu} I_\nu(\sqrt{2\lambda a})}\right) (\sqrt{a})^\nu I_\nu(\sqrt{2\lambda x}) \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 1)} I_\nu(\sqrt{2\lambda a}),$$

and in particular

$$u(0) = \left(\frac{1}{2} \sqrt{2\lambda}\right)^\nu \frac{(\sqrt{a})^\nu}{\Gamma(\nu + 1)} I_\nu(\sqrt{2\lambda a}),$$

at $x = 0$. 

\[\square\]

**Squared radial Ornstein–Uhlenbeck process**

Let $B = (B(t), t \geq 0)$ be a two–dimensional standard Brownian motion, with $B(0) = 0$. Consider the two–dimensional (linear) SDE already considered in the MATLAB practical session

$$X(t) = X(0) + \int_0^t (-c I + R) X(s) ds + B(t),$$

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with \( c > 0 \) and with the \( 2 \times 2 \) matrices
\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

It is further assumed that the initial condition \( X(0) \) has zero mean \( \mathbb{E}[X(0)] = 0 \) and finite variance \( \mathbb{E}[X(0) X^*(0)] \).

This SDE has a unique solution \( X = (X(t), t \geq 0) \), also known as the Ornstein–Uhlenbeck process, with zero mean \( \mathbb{E}[X(t)] = 0 \) and finite variance \( \mathbb{E}[X(t) X^*(t)] \) for any \( t \geq 0 \).

(vii) Write the Itô formula for the two–dimensional Itô process \( X \) and for the real–valued function \( f(x) = |x|^2 = x^* x \).

\\[ \text{Solution} \]

Recall that
\[
f'(x) = 2 x^* \quad \text{and} \quad f''(x) = 2 I,
\]
hence
\[
|X(t)|^2 = |X(0)|^2 + 2 \int_0^t X^*(s) \, dX(s) + \frac{1}{2} \int_0^t \text{trace}(2I) \, ds
\]
\[
= |X(0)|^2 + 2t + 2 \int_0^t X^*(s) ((-cI + R) X(s) \, ds + dB(s)) ,
\]
since \( \text{trace}(I) = 2 \) for the \( 2 \times 2 \) identity matrix \( I \). Note that \( x^* R x = x^* R^* x = -x^* R x \), since the matrix \( R \) is antisymmetric, hence \( x^* R x = 0 \) for any \( x \in \mathbb{R}^2 \), and
\[
|X(t)|^2 = |X(0)|^2 + 2t - 2c \int_0^t |X(s)|^2 ds + 2 \int_0^t X^*(s) dB(s) .
\]

(viii) Using the result proved at question (i), show that the squared radial Ornstein–Uhlenbeck process defined by \( V(t) = |X(t)|^2 \) for any \( t \geq 0 \), satisfies the SDE
\[
V(t) = V(0) + 2t - 2c \int_0^t V(s) \, ds + 2 \int_0^t \sqrt{V(s)} \, dW(s) ,
\]
for any \( t \geq 0 \), and for some one–dimensional standard Brownian motion \( W \).

\\[ \text{Solution} \]

Proceeding as in the answer to question (iii), note that
\[
\int_0^t X^*(s) dB(s) = \int_0^t X^*(s) 1(X(s) \neq 0) dB(s) = \int_0^t \frac{X^*(s)}{|X(s)|} 1(X(s) \neq 0) dB(s) .
\]
Introducing
\[ \theta(s) = \frac{X^*(s)}{|X(s)|} 1(X(s) \neq 0), \]
it holds
\[ \theta(s) \theta^*(s) = \frac{X^*(s)}{|X(s)|} \frac{X(s)}{|X(s)|} 1(X(s) \neq 0) = 1(X(s) \neq 0), \]
hence
\[ \int_0^t \theta(s) \theta^*(s) \, ds = \int_0^t 1(X(s) \neq 0) \, ds = t - \int_0^t 1(X(s) = 0) \, ds. \]
Finally, using the Fubini theorem yields
\[ \mathbb{E} \int_0^t 1(X(s) = 0) \, ds = \int_0^t \mathbb{P}[X(s) = 0] \, ds = 0, \]
hence
\[ \int_0^t 1(X(s) = 0) \, ds = 0, \]
almost surely, since a nonnegative random variable with zero expectation is almost surely equal to zero. Therefore
\[ \int_0^t \theta(s) \theta^*(s) \, ds = t, \]
almost surely for any \( t \geq 0 \), and it follows from the result proved at question (i) that the process defined by
\[ W(t) = \int_0^t \theta(s) \, dB(s), \]
for any \( t \geq 0 \), is a one–dimensional standard Brownian motion, and
\[ \int_0^t X^*(s) \, dB(s) = \int_0^t |X(s)| \theta(s) \, dB(s) = \int_0^t |X(s)| \, dW(s). \]

It can be shown that there exists a unique solution (even though the diffusion coefficient is only Hölder continuous) to this one–dimensional SDE. The boundary \( \{0\} \) is never reached, and if the initial condition is \( V(0) = 0 \) then the boundary \( \{0\} \) is left immediately and never returned to.

Consider the hitting time
\[ T_a = \inf\{t \geq 0 : V(t) \geq a\} = \inf\{t \geq 0 : |X(t)| \geq \sqrt{a}\}, \]
and the related Laplace transform
\[ u(x) = \mathbb{E}_{0,x}[\exp\{-\lambda T_a\}], \]
defined for any \( 0 \leq x \leq a \) and for any \( \lambda > 0 \). Clearly, the function \( u(x) \) is finite near \( x = 0 \), and takes the value 1 at \( x = a \).
(ix) Give the expression of the second–order partial differential operator associated with the squared Ornstein–Uhlenbeck process.

Prove that the Laplace transform $u(x)$ satisfies the ODE

$$2 x u''(x) + (2 - 2 c x) u'(x) - \lambda u(x) = 0 ,$$

for any $0 < x < a$, and give the two boundary conditions satisfied near $x = 0$ and at $x = a$.

Solution

The second–order partial differential operator associated with the squared radial Ornstein–Uhlenbeck process is

$$L = 2 x \frac{d^2}{dx^2} + (2 - 2 c x) \frac{d}{dx} ,$$

hence the PDE

$$L u(x) - \lambda u(x) = 0 ,$$

satisfied by the Laplace transform $u(x)$ reduces to the ODE

$$2 x u''(x) + (2 - 2 c x) u'(x) - \lambda u(x) = 0 ,$$

for any $0 < x < a$, with the two constraints that

- it should be finite near $x = 0$, i.e. $u(0^+) \text{ is finite}$,
- it should take the value 1 at $x = a$, i.e. $u(a) = 1$.

The explicit solution to this boundary–value problem involves confluent hypergeometric functions: some facts about these special functions are collected in the Appendix.

(x) Check that if the function $w(x)$ is a particular solution of (4) with $a = \frac{\lambda}{2c}$ and $b = 1$, then the function defined by $u(x) = w(cx)$ for any $x \geq 0$ is a particular solution of (2).

Solution

If the function $w(x)$ solves the ODE

$$x w''(x) + (1 - x) w'(x) - \frac{\lambda}{2c} w(x) = 0 ,$$

then upon an obvious change of variable it holds

$$c x w''(c x) + (1 - c x) w'(c x) - \frac{\lambda}{2c} w(c x) = 0 ,$$

(**)
for any \( x \geq 0 \). Using the chain rule, the function \( u(x) = w(cx) \) satisfies
\[
\begin{align*}
    u'(x) &= c \, w'(cx) \quad \text{and} \quad u''(x) = c^2 \, w''(cx) .
\end{align*}
\]
Therefore
\[
2 \, x \, u''(x) + (2 - 2 \, c \, x) \, u'(x) - \lambda \, u(x)
= 2 \, x \, c^2 \, w''(cx) + (2 - 2 \, c \, x) \, c \, w'(cx) - \lambda \, w(cx)
= 2 \, c \, [c \, x \, w''(cx) + (1 - c \, x) \, w'(cx) - \frac{\lambda}{2 \, c} \, w(cx)] ,
\]
and it follows from (**) that
\[
2 \, x \, u''(x) + (2 - 2 \, c \, x) \, u'(x) - \lambda \, u(x) = 0 .
\]

Clearly, the functions defined by \( u_1(x) = M(\frac{\lambda}{2c}, 1, c \, x) \) and by \( u_2(x) = U(\frac{\lambda}{2c}, 1, c \, x) \) for any \( x \geq 0 \) are two particular solutions of (2) These two particular solutions are linearly independent, so that any solution of (2) is a linear combination
\[
u(x) = c_1 \, u_1(x) + c_2 \, u_2(x) = c_1 \, M(\frac{\lambda}{2c}, 1, c \, x) + c_2 \, U(\frac{\lambda}{2c}, 1, c \, x) .
\]

(xi) Show that the Laplace transform is given by
\[
u(x) = \frac{M(\frac{\lambda}{2c}, 1, c \, x)}{M(\frac{\lambda}{2c}, 1, c \, a)} ,
\]
and in particular
\[
u(0) = \frac{1}{M(\frac{\lambda}{2c}, 1, c \, a)} ,
\]
at \( x = 0 \).

Solution

The two particular solutions \( u_1(x) \) and \( u_2(x) \) differ by their asymptotic behaviour near \( x = 0 \). Indeed
\[
u_1(x) = M(\frac{\lambda}{2c}, 1, c \, x) \sim 1 ,
\]
meets the constraint that the solution should be finite near \( x = 0 \). On the other hand
\[
u_2(x) = U(\frac{\lambda}{2c}, 1, c \, x) \to \infty ,
\]
does not meet the constraint that the solution should be finite near $x = 0$. Therefore, a solution of (2) that meets the constraint that it should be finite near $x = 0$ is necessarily of the form

$$u(x) = c_1 M\left(\frac{\lambda}{2c}, 1, c x\right),$$

and the integration constant $c_1$ is determined by the constraint that the solution should take the value 1 at $x = a$, hence

$$c_1 = \frac{1}{M\left(\frac{\lambda}{2c}, 1, c a\right)},$$

and

$$u(x) = \frac{M\left(\frac{\lambda}{2c}, 1, c x\right)}{M\left(\frac{\lambda}{2c}, 1, c a\right)},$$

and in particular

$$u(0) = \frac{1}{M\left(\frac{\lambda}{2c}, 1, c a\right)},$$

at $x = 0$.

\[\square\]

**APPENDIX: SPECIAL FUNCTIONS**

Here are collected some facts about some special functions of interest, the modified Bessel functions and the confluent hypergeometric functions.

The modified Bessel function of the first kind $I_\nu(x)$ (the exact expression of which is irrelevant here and which is available under MATLAB for instance) is a particular solution of the ODE

$$x^2 u''(x) + x u'(x) - (x^2 + \nu^2) u(x) = 0.$$  \hspace{1cm} (3)

Another particular solution of (3) is the modified Bessel function of the second kind $K_\nu(x)$ (the exact expression of which is irrelevant here as well). These two particular solutions differ by their asymptotic behaviour near $x = 0$. Indeed

$$I_\nu(x) \sim \frac{(\frac{1}{2} x)^\nu}{\Gamma(\nu + 1)} \quad \text{for any } \nu \geq 0,$$

$$K_0(x) \sim -\log x \quad \text{for } \nu = 0,$$

$$K_\nu(x) \sim \frac{1}{2} \left(\frac{x}{2}\right)^\nu \quad \text{for any } \nu > 0.$$

Moreover, these two particular solutions are linearly independent, so that any solution of (3) is a linear combination $c_1 I_\nu(x) + c_2 K_\nu(x)$.  

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The confluent hypergeometric function of the first kind $M(a, b, x)$ (also known as the Kummer function, the exact expression of which is irrelevant here) is a particular solution of the ODE
\[ x w''(x) + (b - x) w'(x) - a w(x) = 0. \] (4)

Another particular solution of (4) is the confluent hypergeometric function of the second kind $U(a, b, x)$ (also known as the Tricomi function, the exact expression of which is irrelevant here as well). These two particular solutions differ by their asymptotic behaviour near $x = 0$. Indeed

\[ M(a, b, x) \sim 1 \quad \text{and} \quad U(a, b, x) \to \infty. \]

Moreover, these two particular solutions are linearly independent, so that any solution of (4) is a linear combination $c_1 M(a, b, x) + c_2 U(a, b, x)$. 