INSA Rennes, 4GM–AROM
Random Models of Dynamical Systems
Introduction to SDE's
Written Exam (aka DS)

January 8, 2020

The objective of this problem is to characterize the first hitting time of a sphere centered at the origin, by the d–dimensional Brownian motion and by the two–dimensional Ornstein–Uhlenbeck process (a Gaussian diffusion process).

Let B be a d–dimensional standard Brownian motion, and adapted to a given filtration $\mathcal{F}$. Let $\theta = (\theta(t)\,,\, t \geq 0)$ be a d–dimensional stochastic process in $M^2 = \bigcap_{T \geq 0} M^2([0,T])$, seen as a row–vector, adapted to the same filtration $\mathcal{F}$. Here is a first preliminary result.

(i) **If**

$$\int_0^t \theta(s)\, \theta^*(s)\, ds = t \ ,$$

almost surely for any $t \geq 0$, then the process defined by

$$W(t) = \int_0^t \theta(s)\, dB(s) \ ,$$

for any $t \geq 0$, satisfies

$$\mathbb{E}[\exp\{i\, \lambda\, (W(t) - W(s))\} \mid \mathcal{F}(s)] = \exp\{-\tfrac{1}{2}\, \lambda^2\, (t-s)\} \ ,$$

almost surely, for any scalar λ and for any $0 \leq s \leq t$, hence it is a one–dimensional standard Brownian motion.

[Hint: Write the Itô formula for the process W and for the complex–valued function $f(x) = \exp\{i\, \lambda\, x\}$ where the scalar λ is fixed, between the time instants s and t, with $0 \leq s \leq t$.]

Solution

Clearly, the complex–valued function $f(x) = \exp\{i\, \lambda\, x\}$ is twice continuously differentiable with $f'(x) = i\, \lambda\, f(x)$ and $f''(x) = -\lambda^2\, f(x)$, and writing the Itô formula yields

$$\begin{aligned}
\exp\{i\, \lambda\, W(t)\} \ &= \ \exp\{i\, \lambda\, W(s)\} + i\, \lambda \int_s^t \exp\{i\, \lambda\, W(u)\}\, dW(u) \\
&\quad - \tfrac{1}{2}\, \lambda^2 \int_s^t \exp\{i\, \lambda\, W(u)\}\, \theta(u)\, \theta^*(u)\, du \\
&= \ \exp\{i\, \lambda\, W(s)\} + i\, \lambda \int_s^t \exp\{i\, \lambda\, W(u)\}\, \theta(u)\, dB(u) \\
&\quad - \tfrac{1}{2}\, \lambda^2 \int_s^t \exp\{i\, \lambda\, W(u)\}\, du \ .
\end{aligned}$$

Note that

$$\mathbb{E} \int_0^T |\exp\{i\,\lambda\, W(u)\}|^2\, \theta(u)\, \theta^*(u)\, du = T < \infty \ ,$$

for any $T \geq 0$, so that the stochastic integral is a square–integrable martingale, with zero expectation, hence

$$\mathbb{E}[\int_s^t \exp\{i\,\lambda\, W(u)\}\, \theta(u)\, dB(u) \mid \mathcal{F}(s)] = 0 \ .$$

Therefore

$$\mathbb{E}[\exp\{i\,\lambda\, W(t)\} \mid \mathcal{F}(s)] \ = \ \exp\{i\,\lambda\, W(s)\} - \tfrac{1}{2}\,\lambda^2 \int_s^t \mathbb{E}[\exp\{i\,\lambda\, W(u)\} \mid \mathcal{F}(s)]\, du \ ,$$

so that the function defined by

$$f(t) = \mathbb{E}[\exp\{i\,\lambda\, W(t)\} \mid \mathcal{F}(s)] \ ,$$

satisfies the linear ODE

$$f(t) = f(s) - \tfrac{1}{2}\,\lambda^2 \int_s^t f(u)\, du \ ,$$

the explicit solution of which is

$$f(t) = f(s)\ \exp\{-\tfrac{1}{2}\,\lambda^2\,(t-s)\} \ .$$

In other words

$$\mathbb{E}[\exp\{i\,\lambda\, W(t)\} \mid \mathcal{F}(s)] = \exp\{i\,\lambda\, W(s)\}\ \exp\{-\tfrac{1}{2}\,\lambda^2\,(t-s)\} \ ,$$

or equivalently

$$\mathbb{E}[\exp\{i\,\lambda\,(W(t) - W(s)\} \mid \mathcal{F}(s)] = \exp\{-\tfrac{1}{2}\,\lambda^2\,(t-s)\} \ .$$

This shows that the increment $(W(t) - W(s))$ is independent of $\mathcal{F}(s)$, normally distributed with zero mean and variance $(t-s)$. This holds for any $0 \leq s \leq t$, hence the process W is a one–dimensional standard Brownian motion, adapted to the filtration $\mathcal{F}$.

□

Bessel processes

(ii) **Write the Itô formula for the d–dimensional standard Brownian motion B and for the real–valued function $f(x) = |x|^2 = x^*\, x$.**

Solution

Clearly

$$f'(x) = 2\, x^* \qquad \text{and} \qquad f''(x) = 2\, I \ ,$$

hence

$$\begin{aligned} |B(t)|^2 \ &= \ |B(0)|^2 + 2 \int_0^t B^*(s)\, dB(s) + \tfrac{1}{2} \int_0^t \text{trace}(2I)\, ds \\ &= \ |B(0)|^2 + d\, t + 2 \int_0^t B^*(s)\, dB(s) \ , \end{aligned}$$

since $\mathrm{trace}(I) = d$ for the $d \times d$ identity matrix I.

□

(iii) **Using the result proved at question (i), show that the squared Bessel process defined by $V(t) = |B(t)|^2$ for any $t \geq 0$, satisfies the SDE**

$$V(t) = V(0) + d\,t + 2 \int_0^t \sqrt{V(s)}\, dW(s) \ ,$$

for any $t \geq 0$, and for some one–dimensional standard Brownian motion W.

Solution

Note that

$$\int_0^t B^*(s)\, dB(s) = \int_0^t B^*(s)\, 1_{(B(s) \neq 0)}\, dB(s) = \int_0^t |B(s)| \, \frac{B^*(s)}{|B(s)|}\, 1_{(B(s) \neq 0)}\, dB(s) \ .$$

Introducing

$$\theta(s) = \frac{B^*(s)}{|B(s)|}\, 1_{(B(s) \neq 0)} \ ,$$

it holds

$$\theta(s)\, \theta^*(s) = \frac{B^*(s)}{|B(s)|}\, \frac{B(s)}{|B(s)|}\, 1_{(B(s) \neq 0)} = 1_{(B(s) \neq 0)} \ ,$$

hence

$$\int_0^t \theta(s)\, \theta^*(s)\, ds = \int_0^t 1_{(B(s) \neq 0)}\, ds = t - \int_0^t 1_{(B(s) = 0)}\, ds \ .$$

Finally, using the Fubini theorem yields

$$\mathbb{E} \int_0^t 1_{(B(s) = 0)}\, ds = \int_0^t \mathbb{P}[B(s) = 0]\, ds = 0 \ ,$$

hence

$$\int_0^t 1_{(B(s) = 0)}\, ds = 0 \ ,$$

almost surely, since a nonnegative random variable with zero expectation is almost surely equal to zero. Therefore

$$\int_0^t \theta(s)\, \theta^*(s)\, ds = t \ ,$$

almost surely for any $t \geq 0$, and it follows from the result proved at question (i) that the process defined by

$$W(t) = \int_0^t \theta(s)\, dB(s) \ ,$$

for any $t \geq 0$, is a one–dimensional standard Brownian motion, and

$$\int_0^t B^*(s)\, dB(s) = \int_0^t |B(s)|\, \theta(s)\, dB(s) = \int_0^t |B(s)|\, dW(s) \ .$$

□

It can be shown that there exists a unique solution (even though the diffusion coefficient is only Hölder continuous) to this one–dimensional SDE. The boundary $\{0\}$ is never reached, and if the initial condition is $V(0) = 0$ then the boundary $\{0\}$ is left immediately and never returned to.

Consider the hitting time

$$T_a = \inf\{t \geq 0 \,:\, V(t) \geq a\} = \inf\{t \geq 0 \,:\, |B(t)| \geq \sqrt{a}\} \ ,$$

and the related Laplace transform

$$u(x) = \mathbb{E}_{0,x}[\exp\{-\lambda\, T_a\}] \ ,$$

defined for any $0 \leq x \leq a$ and for any $\lambda > 0$. Clearly, the function $u(x)$ is finite near $x = 0$, and takes the value 1 at $x = a$.

(iv) **Give the expression of the second–order partial differential operator associated with the squared Bessel process.**
Let $\nu = \frac{1}{2}\,(d-2)$. Prove that the Laplace transform $u(x)$ satisfies the ODE

$$2\,x\,u''(x) + 2\,(\nu+1)\,u'(x) - \lambda\,u(x) = 0 \ , \tag{1}$$

for any $0 < x < a$, and give the two boundary conditions satisfied near $x = 0$ and at $x = a$.

Solution

The second–order partial differential operator associated with the squared Bessel process is

$$L = 2\,x\,\frac{d^2}{dx^2} + 2\,(\nu+1)\,\frac{d}{dx} \ ,$$

with $\nu = \frac{1}{2}\,(d-2)$ so that $d = 2\,(\nu+1)$, hence the PDE

$$L\,u(x) - \lambda\,u(x) = 0 \ ,$$

satisfied by the Laplace transform $u(x)$ reduces to the ODE

$$2\,x\,u''(x) + 2\,(\nu+1)\,u'(x) - \lambda\,u(x) = 0 \ ,$$

for any $0 < x < a$, with the two constraints that

- it should be finite near $x = 0$, i.e. $u(0+)$ is finite,
- it should take the value 1 at $x = a$, i.e. $u(a) = 1$.

□

The explicit solution to this boundary–value problem involves modified Bessel functions: some facts about these special functions are collected in the Appendix.

(v) **Check that if the function $w(x)$ is a particular solution of (3), then the function defined by $u(x) = (\sqrt{x})^{-\nu}\, w(\sqrt{2\lambda x})$ for any $x \geq 0$ is a particular solution of (1).**

[Hint: Lengthy calculations, skip this.]

Solution

If the function $w(x)$ solves the ODE

$$x^2\, w''(x) + x\, w'(x) - (x^2 + \nu^2)\, w(x) = 0$$

then upon an obvious change of variable it holds

$$2\lambda x\, w''(\sqrt{2\lambda x}) + \sqrt{2\lambda x}\, w'(\sqrt{2\lambda x}) - (2\lambda\, x + \nu^2)\, w(\sqrt{2\lambda x}) = 0 \qquad (\star)$$

for any $x \geq 0$. Using the chain rule, the function $w_0(x) = w(\sqrt{2\lambda x})$ satisfies

$$w_0'(x) = \sqrt{2\lambda}\, \frac{1}{2\sqrt{x}}\, w'(\sqrt{2\lambda x})$$

and

$$w_0''(x) = -\sqrt{2\lambda}\, \frac{1}{4x\sqrt{x}}\, w'(\sqrt{2\lambda x}) + 2\lambda\, \frac{1}{4x}\, w''(\sqrt{2\lambda x})$$

hence $u(x) = (\sqrt{x})^{-\nu}\, w_0(x)$ satisfies

$$\begin{aligned} u'(x) &= -\tfrac{1}{2}\, \nu\, (\sqrt{x})^{-\nu-2}\, w_0(x) + (\sqrt{x})^{-\nu}\, w_0'(x) \\ &= -\tfrac{1}{2}\, \nu\, (\sqrt{x})^{-\nu-2}\, w(\sqrt{2\lambda x}) + \tfrac{1}{2}\, \sqrt{2\lambda}\, (\sqrt{x})^{-\nu-1}\, w'(\sqrt{2\lambda x}) \end{aligned}$$

and using the Leibniz rule yields

$$\begin{aligned} u''(x) &= \tfrac{1}{2}\, \nu\, (\tfrac{1}{2}\, \nu + 1)\, (\sqrt{x})^{-\nu-4}\, w_0(x) - 2\, \tfrac{1}{2}\, \nu\, (\sqrt{x})^{-\nu-2}\, w_0'(x) + (\sqrt{x})^{-\nu}\, w_0''(x) \\ &= \tfrac{1}{2}\, \nu\, (\tfrac{1}{2}\, \nu + 1)\, (\sqrt{x})^{-\nu-4}\, w(\sqrt{2\lambda x}) - \tfrac{1}{2}\, \nu\, \sqrt{2\lambda}\, (\sqrt{x})^{-\nu-3}\, w'(\sqrt{2\lambda x}) \\ &\quad -\tfrac{1}{4}\, \sqrt{2\lambda}\, (\sqrt{x})^{-\nu-3}\, w'(\sqrt{2\lambda x}) + \tfrac{1}{2}\, \lambda\, (\sqrt{x})^{-\nu-2}\, w''(\sqrt{2\lambda x}) \\ &= \tfrac{1}{2}\, \nu\, (\tfrac{1}{2}\, \nu + 1)\, (\sqrt{x})^{-\nu-4}\, w(\sqrt{2\lambda x}) - \tfrac{1}{2}\, (\nu + \tfrac{1}{2})\, \sqrt{2\lambda}\, (\sqrt{x})^{-\nu-3}\, w'(\sqrt{2\lambda x}) \\ &\quad +\tfrac{1}{2}\, \lambda\, (\sqrt{x})^{-\nu-2}\, w''(\sqrt{2\lambda x}) \end{aligned}$$

Therefore

$$2\,x\,u''(x) + 2\,(\nu+1)\,u'(x) - \lambda\,u(x)$$

$$= 2\,x\,[\tfrac{1}{2}\,\nu\,(\tfrac{1}{2}\,\nu+1)\,(\sqrt{x})^{-\nu-4}\,w(\sqrt{2\lambda x}) - \tfrac{1}{2}\,(\nu+\tfrac{1}{2})\,\sqrt{2\lambda}\,(\sqrt{x})^{-\nu-3}\,w'(\sqrt{2\lambda x})$$

$$+ \tfrac{1}{2}\,\lambda\,(\sqrt{x})^{-\nu-2}\,w''(\sqrt{2\lambda x})]$$

$$+ 2\,(\nu+1)\,[-\tfrac{1}{2}\,\nu\,(\sqrt{x})^{-\nu-2}\,w(\sqrt{2\lambda x}) + \tfrac{1}{2}\,\sqrt{2\lambda}\,(\sqrt{x})^{-\nu-1}\,w'(\sqrt{2\lambda x})]$$

$$- \lambda\,(\sqrt{x})^{-\nu}\,w(\sqrt{2\lambda x})$$

$$= \lambda\,x\,(\sqrt{x})^{-\nu-2}\,w''(\sqrt{2\lambda x})$$

$$+ [-(\nu+\tfrac{1}{2}) + (\nu+1)]\,\sqrt{2\lambda}\,(\sqrt{x})^{-\nu-1}\,w'(\sqrt{2\lambda x})$$

$$+ [\nu\,(\tfrac{1}{2}\,\nu+1) - \nu\,(\nu+1)]\,(\sqrt{x})^{-\nu-2}\,w(\sqrt{2\lambda x}) - \lambda\,(\sqrt{x})^{-\nu}\,w(\sqrt{2\lambda x})$$

$$= \tfrac{1}{2}\,(\sqrt{x})^{-\nu-2}\,[2\lambda\,x\,w''(\sqrt{2\lambda x}) + \sqrt{2\lambda\,x}\,w'(\sqrt{2\lambda x}) - (2\lambda\,x+\nu^2)\,w(\sqrt{2\lambda x})]$$

and it follows from $(\star)$ that

$$2\,x\,u''(x) + 2\,(\nu+1)\,u'(x) - \lambda\,u(x) = 0$$

□

Clearly, the functions defined by $u_1(x) = (\sqrt{x})^{-\nu}\,I_\nu(\sqrt{2\lambda x})$ and by $u_2(x) = (\sqrt{x})^{-\nu}\,K_\nu(\sqrt{2\lambda x})$ for any $x \geq 0$ are two particular solutions of (1). These two particular solutions are linearly independent, so that any solution of (3) is a linear combination

$$u(x) = c_1\,u_1(x) + c_2\,u_2(x) = c_1\,(\sqrt{x})^{-\nu}\,I_\nu(\sqrt{2\lambda x}) + c_2\,(\sqrt{x})^{-\nu}\,K_\nu(\sqrt{2\lambda x})\ .$$

(vi) **Show that the Laplace transform is given by**

$$u(x) = (\frac{\sqrt{a}}{\sqrt{x}})^\nu\ \frac{I_\nu(\sqrt{2\lambda x})}{I_\nu(\sqrt{2\lambda a})}\ ,$$

and in particular

$$u(0) = \frac{(\frac{1}{2}\,\sqrt{2\lambda})^\nu}{\Gamma(\nu+1)}\ \frac{(\sqrt{a})^\nu}{I_\nu(\sqrt{2\lambda a})}\ ,$$

at $x = 0$.

Solution

The two particular solutions $u_1(x)$ and $u_2(x)$ differ by their asymptotic behaviour near $x = 0$. Indeed

$$I_\nu(\sqrt{2\lambda x}) \sim \frac{(\frac{1}{2}\sqrt{2\lambda x})^\nu}{\Gamma(\nu+1)} = \frac{(\frac{1}{2}\sqrt{2\lambda})^\nu}{\Gamma(\nu+1)} (\sqrt{x})^\nu ,$$

hence

$$u_1(x) = (\sqrt{x})^{-\nu} I_\nu(\sqrt{2\lambda x}) \sim \frac{(\frac{1}{2}\sqrt{2\lambda})^\nu}{\Gamma(\nu+1)} ,$$

meets the constraint that the solution should be finite near $x = 0$. On the other hand, if $\nu > 0$ then

$$K_\nu(\sqrt{2\lambda x}) \sim \tfrac{1}{2} \frac{\Gamma(\nu)}{(\frac{1}{2}\sqrt{2\lambda x})^\nu} = \tfrac{1}{2} \frac{\Gamma(\nu)}{(\frac{1}{2}\sqrt{2\lambda})^\nu} (\sqrt{x})^{-\nu} ,$$

hence

$$u_2(x) = (\sqrt{x})^{-\nu} K_\nu(\sqrt{2\lambda x}) \sim \tfrac{1}{2} \frac{\Gamma(\nu)}{(\frac{1}{2}\sqrt{2\lambda})^\nu} x^{-\nu} ,$$

does not meet the constraint that the solution should be finite near $x = 0$, and similarly if $\nu = 0$ then

$$u_2(x) = K_0(\sqrt{2\lambda x}) \sim -\log \sqrt{2\lambda x} ,$$

does not meet the constraint that the solution should be finite near $x = 0$. Therefore, a solution of (1) that meets the constraint that it should be finite near $x = 0$ is necessarily of the form

$$u(x) = c_1 (\sqrt{x})^{-\nu} I_\nu(\sqrt{2\lambda x}) ,$$

and the integration constant c_1 is determined by the constraint that the solution should take the value 1 at $x = a$, hence

$$c_1 = \frac{1}{(\sqrt{a})^{-\nu} I_\nu(\sqrt{2\lambda a})} ,$$

and

$$u(x) = \frac{(\sqrt{x})^{-\nu} I_\nu(\sqrt{2\lambda x})}{(\sqrt{a})^{-\nu} I_\nu(\sqrt{2\lambda a})} = (\frac{\sqrt{a}}{\sqrt{x}})^\nu \frac{I_\nu(\sqrt{2\lambda x})}{I_\nu(\sqrt{2\lambda a})} ,$$

and in particular

$$u(0) = \frac{(\frac{1}{2}\sqrt{2\lambda})^\nu}{\Gamma(\nu+1)} \frac{(\sqrt{a})^\nu}{I_\nu(\sqrt{2\lambda a})} ,$$

at $x = 0$.

$\square$

Squared radial Ornstein–Uhlenbeck process

Let $B = (B(t), t \geq 0)$ be a two–dimensional standard Brownian motion, with $B(0) = 0$. Consider the two–dimensional (linear) SDE already considered in the MATLAB practical session

$$X(t) = X(0) + \int_0^t (-c\, I + R)\, X(s)\, ds + B(t) ,$$

with $c > 0$ and with the 2×2 matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

It is further assumed that the initial condition $X(0)$ has zero mean $\mathbb{E}[X(0)] = 0$ and finite variance $\mathbb{E}[X(0)\,X^*(0)]$.

This SDE has a unique solution $X = (X(t)\,,\, t \geq 0)$, also known as the Ornstein–Uhlenbeck process, with zero mean $\mathbb{E}[X(t)] = 0$ and finite variance $\mathbb{E}[X(t)\,X^*(t)]$ for any $t \geq 0$.

(vii) **Write the Itô formula for the two–dimensional Itô process X and for the real–valued function $f(x) = |x|^2 = x^*\,x$.**

Solution

Recall that

$$f'(x) = 2\,x^* \qquad \text{and} \qquad f''(x) = 2\,I \ ,$$

hence

$$\begin{aligned} |X(t)|^2 \ &= \ |X(0)|^2 + 2 \int_0^t X^*(s)\,dX(s) + \tfrac{1}{2} \int_0^t \text{trace}(2\,I)\,ds \\ &= \ |X(0)|^2 + 2\,t + 2 \int_0^t X^*(s)\,((-c\,I + R)\,X(s)\,ds + dB(s)) \ , \end{aligned}$$

since $\text{trace}(I) = 2$ for the 2×2 identity matrix I. Note that $x^*\,R\,x = x^*\,R^*\,x = -x^*\,R\,x$, since the matrix R is antisymmetric, hence $x^*\,R\,x = 0$ for any $x \in \mathbb{R}^2$, and

$$|X(t)|^2 \ = \ |X(0)|^2 + 2\,t - 2\,c \int_0^t |X(s)|^2\,ds + 2 \int_0^t X^*(s)\,dB(s) \ .$$

□

(viii) **Using the result proved at question (i), show that the squared radial Ornstein–Uhlenbeck process defined by $V(t) = |X(t)|^2$ for any $t \geq 0$, satisfies the SDE**

$$V(t) = V(0) + 2\,t - 2\,c \int_0^t V(s)\,ds + 2 \int_0^t \sqrt{V(s)}\,dW(s) \ ,$$

for any $t \geq 0$, and for some one–dimensional standard Brownian motion W.

Solution

Proceeding as in the answer to question (iii), note that

$$\int_0^t X^*(s)\,dB(s) = \int_0^t X^*(s)\,1_{(X(s)\,\neq\,0)}\,dB(s) = \int_0^t |X(s)|\,\frac{X^*(s)}{|X(s)|}\,1_{(X(s)\,\neq\,0)}\,dB(s) \ .$$

Introducing

$$\theta(s) = \frac{X^*(s)}{|X(s)|}\, 1_{(X(s) \neq 0)} \ ,$$

it holds

$$\theta(s)\, \theta^*(s) = \frac{X^*(s)}{|X(s)|}\, \frac{X(s)}{|X(s)|}\, 1_{(X(s) \neq 0)} = 1_{(X(s) \neq 0)} \ ,$$

hence

$$\int_0^t \theta(s)\, \theta^*(s)\, ds = \int_0^t 1_{(X(s) \neq 0)}\, ds = t - \int_0^t 1_{(X(s) = 0)}\, ds \ .$$

Finally, using the Fubini theorem yields

$$\mathbb{E} \int_0^t 1_{(X(s) = 0)}\, ds = \int_0^t \mathbb{P}[X(s) = 0]\, ds = 0 \ ,$$

hence

$$\int_0^t 1_{(X(s) = 0)}\, ds = 0 \ ,$$

almost surely, since a nonnegative random variable with zero expectation is almost surely equal to zero. Therefore

$$\int_0^t \theta(s)\, \theta^*(s)\, ds = t \ ,$$

almost surely for any $t \geq 0$, and it follows from the result proved at question (i) that the process defined by

$$W(t) = \int_0^t \theta(s)\, dB(s) \ ,$$

for any $t \geq 0$, is a one–dimensional standard Brownian motion, and

$$\int_0^t X^*(s)\, dB(s) = \int_0^t |X(s)|\, \theta(s)\, dB(s) = \int_0^t |X(s)|\, dW(s) \ .$$

□

It can be shown that there exists a unique solution (even though the diffusion coefficient is only Hölder continuous) to this one–dimensional SDE. The boundary $\{0\}$ is never reached, and if the initial condition is $V(0) = 0$ then the boundary $\{0\}$ is left immediately and never returned to.

Consider the hitting time

$$T_a = \inf\{t \geq 0 \, : \, V(t) \geq a\} = \inf\{t \geq 0 \, : \, |X(t)| \geq \sqrt{a}\} \ ,$$

and the related Laplace transform

$$u(x) = \mathbb{E}_{0,x}[\exp\{-\lambda\, T_a\}] \ ,$$

defined for any $0 \leq x \leq a$ and for any $\lambda > 0$. Clearly, the function $u(x)$ is finite near $x = 0$, and takes the value 1 at $x = a$.

(ix) **Give the expression of the second–order partial differential operator associated with the squared Ornstein–Uhlenbeck process.**

Prove that the Laplace transform $u(x)$ satisfies the ODE

$$2\,x\,u''(x) + (2 - 2\,c\,x)\,u'(x) - \lambda\,u(x) = 0 \ , \tag{2}$$

for any $0 < x < a$, and give the two boundary conditions satisfied near $x = 0$ and at $x = a$.

Solution

The second–order partial differential operator associated with the squared radial Ornstein–Uhlenbeck process is

$$L = 2\,x\,\frac{d^2}{dx^2} + (2 - 2\,c\,x)\,\frac{d}{dx} \ ,$$

hence the PDE

$$L\,u(x) - \lambda\,u(x) = 0 \ ,$$

satisfied by the Laplace transform $u(x)$ reduces to the ODE

$$2\,x\,u''(x) + (2 - 2\,c\,x)\,u'(x) - \lambda\,u(x) = 0 \ ,$$

for any $0 < x < a$, with the two constraints that

- it should be finite near $x = 0$, i.e. $u(0+)$ is finite,
- it should take the value 1 at $x = a$, i.e. $u(a) = 1$.

□

The explicit solution to this boundary–value problem involves confluent hypergeometric functions: some facts about these special functions are collected in the Appendix.

(x) **Check that if the function $w(x)$ is a particular solution of (4) with $a = \dfrac{\lambda}{2c}$ and $b = 1$, then the function defined by $u(x) = w(c\,x)$ for any $x \geq 0$ is a particular solution of (2).**

Solution

If the function $w(x)$ solves the ODE

$$x\,w''(x) + (1 - x)\,w'(x) - \frac{\lambda}{2c}\,w(x) = 0 \ ,$$

then upon an obvious change of variable it holds

$$c\,x\,w''(c\,x) + (1 - c\,x)\,w'(c\,x) - \frac{\lambda}{2c}\,w(c\,x) = 0 \ , \tag{$\star\star$}$$

for any $x \geq 0$. Using the chain rule, the function $u(x) = w(c\,x)$ satisfies

$$u'(x) = c\,w'(c\,x) \qquad \text{and} \qquad u''(x) = c^2\,w''(c\,x) \ .$$

Therefore

$$\begin{aligned} & 2\,x\,u''(x) + (2 - 2\,c\,x)\,u'(x) - \lambda\,u(x) \\ & = 2\,x\,c^2\,w''(c\,x) + (2 - 2\,c\,x)\,c\,w'(c\,x) - \lambda\,w(c\,x) \\ & = 2\,c\,[c\,x\,w''(c\,x) + (1 - c\,x)\,w'(c\,x) - \frac{\lambda}{2\,c}\,w(c\,x)] \ , \end{aligned}$$

and it follows from ($\star\star$) that

$$2\,x\,u''(x) + (2 - 2\,c\,x)\,u'(x) - \lambda\,u(x) = 0 \ .$$

$\square$

Clearly, the functions defined by $u_1(x) = M(\frac{\lambda}{2c}, 1, c\,x)$ and by $u_2(x) = U(\frac{\lambda}{2c}, 1, c\,x)$ for any $x \geq 0$ are two particular solutions of (2) These two particular solutions are linearly independent, so that any solution of (2) is a linear combination

$$u(x) = c_1\,u_1(x) + c_2\,u_2(x) = c_1\,M(\frac{\lambda}{2c}, 1, c\,x) + c_2\,U(\frac{\lambda}{2c}, 1, c\,x) \ .$$

(xi) **Show that the Laplace transform is given by**

$$u(x) = \frac{M(\frac{\lambda}{2c}, 1, c\,x)}{M(\frac{\lambda}{2c}, 1, c\,a)} \ ,$$

and in particular

$$u(0) = \frac{1}{M(\frac{\lambda}{2c}, 1, c\,a)} \ ,$$

at $x = 0$.

Solution

The two particular solutions $u_1(x)$ and $u_2(x)$ differ by their asymptotic behaviour near $x = 0$. Indeed

$$u_1(x) = M(\frac{\lambda}{2c}, 1, c\,x) \sim 1 \ ,$$

meets the constraint that the solution should be finite near $x = 0$. On the other hand

$$u_2(x) = U(\frac{\lambda}{2c}, 1, c\,x) \to \infty \ ,$$

does not meet the constraint that the solution should be finite near $x = 0$. Therefore, a solution of (2) that meets the constraint that it should be finite near $x = 0$ is necessarily of the form

$$u(x) = c_1\, M(\frac{\lambda}{2c}, 1, c\,x)\ ,$$

and the integration constant c_1 is determined by the constraint that the solution should take the value 1 at $x = a$, hence

$$c_1 = \frac{1}{M(\frac{\lambda}{2c}, 1, c\,a)}\ ,$$

and

$$u(x) = \frac{M(\frac{\lambda}{2c}, 1, c\,x)}{M(\frac{\lambda}{2c}, 1, c\,a)}\ ,$$

and in particular

$$u(0) = \frac{1}{M(\frac{\lambda}{2c}, 1, c\,a)}\ ,$$

at $x = 0$.

□

Appendix: special functions

Here are collected some facts about some special functions of interest, the modified Bessel functions and the confluent hypergeometric functions.

► The modified Bessel function of the first kind $I_\nu(x)$ (the exact expression of which is irrelevant here and which is available under MATLAB for instance) is a particular solution of the ODE

$$x^2\, w''(x) + x\, w'(x) - (x^2 + \nu^2)\, w(x) = 0\ . \tag{3}$$

Another particular solution of (3) is the modified Bessel function of the second kind $K_\nu(x)$ (the exact expression of which is irrelevant here as well). These two particular solutions differ by their asymptotic behaviour near $x = 0$. Indeed

$$I_\nu(x) \sim \frac{(\frac{1}{2}\,x)^\nu}{\Gamma(\nu+1)} \qquad \text{for any } \nu \geq 0,$$

$$K_0(x) \sim -\log x \qquad \text{for } \nu = 0,$$

$$K_\nu(x) \sim \tfrac{1}{2}\,\frac{\Gamma(\nu)}{(\frac{1}{2}\,x)^\nu} \qquad \text{for any } \nu > 0.$$

Moreover, these two particular solutions are linearly independent, so that any solution of (3) is a linear combination $c_1\, I_\nu(x) + c_2\, K_\nu(x)$.

► The confluent hypergeometric function of the first kind $M(a, b, x)$ (also known as the Kummer function, the exact expression of which is irrelevant here) is a particular solution of the ODE

$$x\,w''(x) + (b - x)\,w'(x) - a\,w(x) = 0 \; . \tag{4}$$

Another particular solution of (4) is the confluent hypergeometric function of the second kind $U(a, b, x)$ (also known as the Tricomi function, the exact expression of which is irrelevant here as well). These two particular solutions differ by their asymptotic behaviour near $x = 0$. Indeed

$$M(a, b, x) \sim 1 \qquad \text{and} \qquad U(a, b, x) \to \infty \; .$$

Moreover, these two particular solutions are linearly independent, so that any solution of (4) is a linear combination $c_1\,M(a, b, x) + c_2\,U(a, b, x)$.