cours ARO07–MSSD
Random Models of Dynamical Systems
Introduction to SDE’s
Numerical schemes
for stochastic differential equations

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Stochastic differential equations

Euler scheme: strong error estimate

Euler scheme: weak error estimate
consider the simpler equation

\[ X(t) = X(0) + \int_0^t b(X(s)) \, ds + \int_0^t \sigma(X(s)) \, dB(s) \]

with a \( m \)-dimensional Brownian motion \( B = (B(t), \, t \geq 0) \), and time-independent coefficients:

- a \( d \)-dimensional drift vector \( b(x) \) defined on \( \mathbb{R}^d \)
- a \( d \times m \) diffusion matrix \( \sigma(x) \) defined on \( \mathbb{R}^d \)

**global Lipschitz** condition: there exists a positive constant \( L > 0 \) such that for any \( x, x' \in \mathbb{R}^d \)

\[
\| b(x) - b(x') \| \leq L \| x - x' \| \quad \text{and} \quad \| \sigma(x) - \sigma(x') \| \leq L \| x - x' \|
\]

**linear growth** condition (simple consequence in this case): there exists a positive constant \( K > 0 \) such that for any \( x \in \mathbb{R}^d \)

\[
| b(x) | \leq K (1 + |x|) \quad \text{and} \quad \| \sigma(x) \| \leq K (1 + |x|)
\]
Strong vs. weak error

objective: associated with a uniform subdivision $0 = t_0 < \cdots < t_k < \cdots$ (with constant time–step $h = t_k - t_{k-1}$), design a numerical scheme $\bar{X}_k$ that approximates the solution $X(t_k)$, and provide an approximate continuous–time process $\bar{X}(t)$ (to be made precise later on)

**Definition** the numerical scheme is *strongly* convergent of order $\alpha > 0$ if for any $0 \leq t \leq T$

$$\left\{ \mathbb{E}|X(t) - \bar{X}(t)|^2 \right\}^{1/2} \leq C(T) h^\alpha$$

**Definition** [approximation of moments] the numerical scheme is *weakly* convergent of order $\beta > 0$ if for any regular enough real–valued function $f$ and for any $0 \leq t \leq T$

$$|\mathbb{E}[f(X(t))] - \mathbb{E}[f(\bar{X}(t))]| \leq C(f, T) h^\beta$$
**Remark** if a numerical scheme is strongly convergent of order $\alpha > 0$, then it is also weakly convergent of the same order $\alpha > 0$ (for a Lipschitz continuous function $f$)

indeed: if

$$|f(x) - f(x')| \leq L |x - x'|$$

for any $x, x' \in \mathbb{R}^d$, then

$$|\mathbb{E}[f(X(t))] - \mathbb{E}[f(\tilde{X}(t))]| = |\mathbb{E}[f(X(t)) - f(\tilde{X}(t))]|$$

$$\leq \mathbb{E}|f(X(t)) - f(\tilde{X}(t))|$$

$$\leq L \mathbb{E}|X(t) - \tilde{X}(t)|$$

$$\leq L \{\mathbb{E}|X(t) - \tilde{X}(t)|^2\}^{1/2}$$

$$\leq L C(T) h^\alpha$$
Euler scheme

special important case: Euler scheme

same initial condition $\bar{X}_0 = X(0)$ for $k = 0$, and for any $k \geq 1$

$$\bar{X}_k = \bar{X}_{k-1} + b(\bar{X}_{k-1})(t_k - t_{k-1})$$

$$+ \sigma(\bar{X}_{k-1})(B(t_k) - B(t_{k-1}))$$

and continuous–time approximation interpolating points $\tilde{X}_k$ at time instants $t_k$

$$\tilde{X}(t) = \bar{X}_{k-1} + b(\bar{X}_{k-1})(t - t_{k-1})$$

$$+ \sigma(\bar{X}_{k-1})(B(t) - B(t_{k-1}))$$

for any time $t_{k-1} \leq t \leq t_k$ between two discretization times
Euler approximation seen as an Itô process, with frozen coefficients on each interval of the subdivision: indeed, for any $t_{k-1} \leq t \leq t_k$

$$\tilde{X}(t) = \tilde{X}_{k-1} + \int_{t_{k-1}}^{t} b(\tilde{X}(\pi(s)))\, ds + \int_{t_{k-1}}^{t} \sigma(\tilde{X}(\pi(s)))\, dB(s)$$

and more generally for any $t \geq 0$

$$\tilde{X}(t) = \tilde{X}(0) + \int_{0}^{t} b(\tilde{X}(\pi(s)))\, ds + \int_{0}^{t} \sigma(\tilde{X}(\pi(s)))\, dB(s)$$

where

$$\pi(s) = t_{k-1} \quad \text{and} \quad \tilde{X}(\pi(s)) = \tilde{X}_{k-1} \quad \text{if} \quad t_{k-1} \leq s < t_k$$

there exists a positive constant $M(T)$, independent of the time-step $h$, such that

$$\max_{0 \leq t \leq T} \mathbb{E}|\tilde{X}(t)|^2 \leq M(T)$$
Stochastic differential equations

Euler scheme: strong error estimate

Euler scheme: weak error estimate
Euler scheme: strong error estimate

Theorem 1 the Euler scheme is strongly convergent of order $\frac{1}{2}$, i.e.

$$\max_{0 \leq t \leq T} \mathbb{E}|X(t) - \tilde{X}(t)|^2 = O(h)$$
Proof for any time $t_{k-1} \leq t \leq t_k$ between two discretization times, it holds

$$X(t) = X(t_{k-1}) + \int_{t_{k-1}}^{t} b(X(s)) \, ds + \int_{t_{k-1}}^{t} \sigma(X(s)) \, dB(s)$$

and (Euler approximation interpolating points $\bar{X}_k$ at time instants $t_k$)

$$\bar{X}(t) = \bar{X}_{k-1} + b(\bar{X}_{k-1})(t - t_{k-1})$$

$$+ \sigma(\bar{X}_{k-1})(B(t) - B(t_{k-1}))$$

by difference, for any $t_{k-1} \leq t \leq t_k$

$$X(t) - \bar{X}(t) = X(t_{k-1}) - \bar{X}_{k-1} + \int_{t_{k-1}}^{t} [b(X(s)) - b(\bar{X}_{k-1})] \, ds$$

$$+ \int_{t_{k-1}}^{t} [\sigma(X(s)) - \sigma(\bar{X}_{k-1})] \, dB(s)$$
using the Itô formula yields

\[ |X(t) - \bar{X}(t)|^2 = |X(t_{k-1}) - \bar{X}_{k-1}|^2 \]

\[ + 2 \int_{t_{k-1}}^{t} (X(s) - \bar{X}(s))^* [b(X(s)) - b(\bar{X}_{k-1})] \, ds \]

\[ + 2 \int_{t_{k-1}}^{t} (X(s) - \bar{X}(s))^* [\sigma(X(s)) - \sigma(\bar{X}_{k-1})] \, dB(s) \]

\[ + \int_{t_{k-1}}^{t} \|\sigma(X(s)) - \sigma(\bar{X}_{k-1})\|^2 \, ds \]
using the bound \(2 u^* v \leq |u|^2 + |v|^2\), and taking expectation (assuming the stochastic integral is a (true, square–integrable) martingale), yields

\[
\mathbb{E}|X(t) - \bar{X}(t)|^2 \leq \mathbb{E}|X(t_k-1) - \bar{X}_{k-1}|^2
\]

\[
+ \mathbb{E} \int_{t_{k-1}}^{t} |X(s) - \bar{X}(s)|^2 \, ds
\]

\[
+ \mathbb{E} \int_{t_{k-1}}^{t} |b(X(s)) - b(\bar{X}_{k-1})|^2 \, ds
\]

\[
+ \mathbb{E} \int_{t_{k-1}}^{t} \|\sigma(X(s)) - \sigma(\bar{X}_{k-1})\|^2 \, ds
\]
note that

\[ |b(X(s)) - b(\bar{X}_{k-1})| \]

\[ \leq |b(X(s)) - b(X(t_{k-1}))| + |b(X(t_{k-1})) - b(\bar{X}_{k-1})| \]

\[ \leq L \left[ |X(s) - X(t_{k-1})| + |X(t_{k-1}) - \bar{X}_{k-1}| \right] \]

and similarly

\[ \|\sigma(X(s)) - \sigma(\bar{X}_{k-1})\| \]

\[ \leq L \left[ |X(s) - X(t_{k-1})| + |X(t_{k-1}) - \bar{X}_{k-1}| \right] \]

with two different contributions to the error

- discretization error at previous iteration
- modulus of continuity of the solution
therefore

\[ \mathbb{E}|X(t) - \tilde{X}(t)|^2 \leq (1 + 4L^2(t - t_{k-1})) \mathbb{E}|X(t_{k-1}) - \tilde{X}_{k-1}|^2 \]

\[ + 4L^2 \mathbb{E} \int_{t_{k-1}}^{t} |X(s) - X(t_{k-1})|^2 \, ds \]

\[ + \mathbb{E} \int_{t_{k-1}}^{t} |X(s) - \tilde{X}(s)|^2 \, ds \]

note that the modulus of continuity for the solution satisfies

\[ \mathbb{E}|X(s) - X(t_{k-1})|^2 \leq C(s - t_{k-1}) \]

hence

\[ \mathbb{E}|X(t) - \tilde{X}(t)|^2 \leq (1 + 4L^2h) \mathbb{E}|X(t_{k-1}) - \tilde{X}_{k-1}|^2 + 4L^2C h^2 \]

\[ + \mathbb{E} \int_{t_{k-1}}^{t} |X(s) - \tilde{X}(s)|^2 \, ds \]
the Gronwall lemma yields

\[ \mathbb{E}|X(t) - \tilde{X}(t)|^2 \leq \]

\[ \leq [(1 + 4L^2h) \mathbb{E}|X(t_{k-1}) - \tilde{X}_{k-1}|^2 + 4L^2Ch^2] \exp\{t - t_{k-1}\} \]

introducing

\[ \varepsilon_k = \max_{t_{k-1} \leq t \leq t_k} \mathbb{E}|X(t) - \tilde{X}(t)|^2 \]

it holds

\[ \varepsilon_k \leq (1 + 4L^2h) \exp\{h\} \varepsilon_{k-1} + 4L^2Ch^2 \exp\{h\} \]

and by induction

\[ \varepsilon_k \leq \frac{4L^2Ch^2 \exp\{h\}}{(1 + 4L^2h) \exp\{h\} - 1} [(1 + 4L^2h) \exp\{h\}]^k \]

note that

\[ \frac{4L^2Ch^2 \exp\{h\}}{(1 + 4L^2h) \exp\{h\} - 1} = \frac{4L^2Ch^2}{4L^2h + (1 - \exp\{-h\})} = O(h) \]
for any $k = 1 \cdots [T/h]$, the following bound holds

$$[(1 + 4 L^2 h) \exp\{h\}]^k \leq [(1 + 4 L^2 h) \exp\{h\}]^{[T/h]} \leq \exp\{(1 + 4 L^2) T\}$$

therefore

$$\max_{0 \leq t \leq T} \mathbb{E}|X(t) - \tilde{X}(t)|^2 = \max_{k=1\cdots[T/h]} \varepsilon_k$$

$$\leq \frac{4 L^2 C h^2}{4 L^2 h + (1 - \exp\{-h\})} \exp\{(1 + 4 L^2) T\} = O(h) \quad \square$$
Stochastic differential equations

Euler scheme: strong error estimate

Euler scheme: weak error estimate
Euler scheme: weak error estimate

let \( T \geq 0 \) be fixed (as in the PDE) and consider specifically a uniform subdivision of the form \( 0 = t_0 < \cdots < t_k < \cdots < t_n = T \) of the interval \([0, T]\) (with constant time–step \( h = T/n \))

Theorem 2 under some additional technical assumptions (on the coefficients of the SDE and on the test function) the Euler scheme is weakly convergent of order 1, i.e.

\[
| \mathbb{E}[f(X(T))] - \mathbb{E}[f(\bar{X}(T))] | = O(h)
\]

even more

\[
\mathbb{E}[f(X(T))] - \mathbb{E}[f(\bar{X}(T))] = C(f, T) h + O(h^2)
\]
Corollary 3 [Romberg-Richardson extrapolation] let $\bar{X}^h$ and $\bar{X}^{1/2}_h$ be the Euler approximation with time–step $h$ and $\frac{1}{2} h$ respectively, and define a further approximation as

$$f^{h,\frac{1}{2}h}(T) = 2 \mathbb{E}[f(\bar{X}^{\frac{1}{2}h}(T))] - \mathbb{E}[f(\bar{X}^h(T))]$$

then

$$| \mathbb{E}[f(X(T))] - f^{h,\frac{1}{2}h}(T) | = O(h^2)$$

Proof indeed

$$\mathbb{E}[f(X(T))] - f^{h,\frac{1}{2}h}(T) = 2 ( \mathbb{E}[f(X(T))] - \mathbb{E}[f(\bar{X}^{\frac{1}{2}h}(T))] )$$

$$- ( \mathbb{E}[f(X(T))] - \mathbb{E}[f(\bar{X}^h(T))] )$$

$$= 2 [C(f, T) \frac{1}{2} h + O(h^2)] - [C(f, T) h + O(h^2)] = O(h^2) \quad \square$$
Corollary 4 [Monte Carlo approximation] let \((\tilde{X}^{h,i}, i = 1 \cdots N)\) be \(N\) independent realizations of the same Euler scheme with step–size \(h\), and consider the empirical mean

\[
\hat{f}^{h,N}(T) = \frac{1}{N} \sum_{i=1}^{N} f(\tilde{X}^{h,i}(T))
\]

as a (random) practical approximation, then (bias\(^2 +\) variance)

\[
\mathbb{E} \left| \hat{f}^{h,N}(T) - \mathbb{E}[f(X(T))] \right|^2 = C^2(f, T) h^2 + \frac{\text{var}(f(\tilde{X}^h(T)))}{N} + O(h^3)
\]

Proof clearly

\[
\mathbb{E}[\hat{f}^{h,N}(T)] = \mathbb{E}[f(\tilde{X}^h(T))]
\]

hence

\[
\mathbb{E}[\hat{f}^{h,N}(T)] - \mathbb{E}[f(X(T))] = C(f, T) h + O(h^2)
\]

and

\[
\mathbb{E} \left| \hat{f}^{h,N}(T) - \mathbb{E}[f(\tilde{X}^h(T))] \right|^2 = \frac{\text{var}(f(\tilde{X}^h(T)))}{N}
\]
with the solution of the SDE is associated the second–order partial differential operator

\[ L = \sum_{i=1}^{d} b_i(\cdot) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} \]

let \( u(t, x) \) be the unique (and 'regular enough') solution of the PDE (running backward from \( T \) to 0)

\[ \frac{\partial u}{\partial t}(t, x) + Lu(t, x) = 0 \quad \text{for any } (t, x) \text{ in } [0, T) \times \mathbb{R}^d \]

\[ u(T, x) = f(x) \quad \text{for any } x \text{ in } \mathbb{R}^d \]
Theorem 5* if the drift $b$ and the diffusion matrix $a = \sigma \sigma^*$ have $C^\infty$ regularity, with bounded derivatives of any order, if the test–function $f$ has $C^\infty$ regularity, and at most polynomial growth, then there exists a unique solution $u(t, x)$ to the PDE and this solution has also $C^\infty$ regularity and at most polynomial growth.

Remark this PDE is just instrumental in the proof, i.e. the numerical scheme does not use the solution $u(t, x)$ explicitly.
Proof of Theorem 2  recall that the Itô formula yields

\[ u(t, X(t)) = u(s, X(s)) + \int_s^t \left[ \frac{\partial u}{\partial t}(r, X(r)) + L u(r, X(r)) \right] \, dr \]

\[ + \int_s^t u'(r, X(r)) \sigma(X(r)) \, dB(r) \]

and under the assumptions, the stochastic integral is a (true, square–integrable) martingale, hence

\[ \mathbb{E}[u(t, X(t))] = \mathbb{E}[u(s, X(s))] \]

note that

\[ \mathbb{E}[f(X(T))] = \mathbb{E}[u(T, X(T))] = \mathbb{E}[u(0, X(0))] = \mathbb{E}[u(0, \bar{X}_0)] \]

and \( f(\bar{X}(T)) = u(T, \bar{X}_n) \) (initial condition at time \( T = t_n \)), hence

\[ \mathbb{E}[f(\bar{X}(T))] - \mathbb{E}[f(X(T))] = \mathbb{E}[u(T, \bar{X}_n) - u(0, \bar{X}_0)] \]

\[ = \sum_{k=1}^n \mathbb{E}[u(t_k, \bar{X}_k) - u(t_{k-1}, \bar{X}_{k-1})] \]
with the Euler approximation

\[
\tilde{X}(t) = \tilde{X}_{k-1} + b(\tilde{X}_{k-1})(t - t_{k-1})
\]

\[+ \sigma(\tilde{X}_{k-1})(B(t) - B(t_{k-1}))\]

valid for \( t_{k-1} \leq t \leq t_k \), is associated the second–order partial differential operator with frozen coefficients

\[
L_k = \sum_{i=1}^{d} b_i(\tilde{X}_{k-1}) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(\tilde{X}_{k-1}) \frac{\partial^2}{\partial x_i \partial x_j}
\]

note that

\[
L_k \phi(x) = \sum_{i=1}^{d} b_i(\tilde{X}_{k-1}) \frac{\partial \phi}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(\tilde{X}_{k-1}) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x)
\]

and

\[
L \phi(x) = \sum_{i=1}^{d} b_i(x) \frac{\partial \phi}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x)
\]

do coincide when \( x = \tilde{X}_{k-1} \)
using the Itô formula for the Euler approximation and for the time–dependent function \( u(t, x) \) yields

\[
\begin{align*}
\text{Euler scheme: strong error estimate} \\
\text{Euler scheme: weak error estimate}
\end{align*}
\]

\[
u(t, \bar{X}(t)) - u(t_{k-1}, \bar{X}_{k-1}) =
\int_{t_{k-1}}^{t} \left[ \frac{\partial u}{\partial t}(s, \bar{X}(s)) + L_k u(s, \bar{X}(s)) \right] ds + \int_{t_{k-1}}^{t} u'(s, \bar{X}(s)) \sigma(\bar{X}_{k-1}) dB(s)
\]

\[
= \int_{t_{k-1}}^{t} \left[ \frac{\partial u}{\partial t}(s, \bar{X}(s)) + L u(s, \bar{X}(s)) \right] ds + \int_{t_{k-1}}^{t} u'(s, \bar{X}(s)) \sigma(\bar{X}_{k-1}) dB(s)
\]

\[
+ \int_{t_{k-1}}^{t} \left[ L_k u(s, \bar{X}(s)) - L u(s, \bar{X}(s)) \right] ds
\]

\[
= \int_{t_{k-1}}^{t} \left[ L_k u(s, \bar{X}(s)) - L u(s, \bar{X}(s)) \right] ds + \int_{t_{k-1}}^{t} u'(s, \bar{X}(s)) \sigma(\bar{X}_{k-1}) dB(s)
\]

since

\[
\frac{\partial u}{\partial t}(s, y) + L u(s, y) = 0
\]

for any \( y \in \mathbb{R}^d \), and the identity holds in particular for \( y = \bar{X}(s) \).
taking $t = t_k$ and taking expectation (assuming that the stochastic integral has zero expectation) yields

$$
\mathbb{E}[u(t_k, \bar{X}_k) - u(t_{k-1}, \bar{X}_{k-1})]
$$

$$
= \mathbb{E} \int_{t_{k-1}}^{t_k} [L_k u(s, \bar{X}(s)) - L u(s, \bar{X}(s))] \, ds
$$

$$
= \mathbb{E} \int_{t_{k-1}}^{t_k} u'(s, \bar{X}(s)) [b(\bar{X}_{k-1}) - b(\bar{X}(s))] \, ds
$$

$$
+ \frac{1}{2} \mathbb{E} \int_{t_{k-1}}^{t_k} \text{trace}[u''(s, \bar{X}(s)) [a(\bar{X}_{k-1}) - a(\bar{X}(s))] ] \, ds
$$

the next step is to write the Itô formula for the time–dependent function

$$
v(s, x) = u'(s, x) (b(\bar{X}_{k-1}) - b(x))
$$

$$
+ \frac{1}{2} \text{trace}[u''(s, x) (a(\bar{X}_{k-1}) - a(x))] 
$$

this requires some regularity
note that \( v(t_{k-1}, \bar{X}_{k-1}) = 0 \), hence

\[
v(s, \bar{X}(s)) = \int_{t_{k-1}}^{s} \left[ \frac{\partial v}{\partial t}(r, \bar{X}(r)) + L_k v(r, \bar{X}(r)) \right] dr \]

\[
+ \int_{t_{k-1}}^{s} v'(r, \bar{X}(r)) \sigma(\bar{X}_{k-1}) dB(r)
\]

then

\[
\int_{t_{k-1}}^{t_k} v(s, \bar{X}(s)) ds = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} \left[ \frac{\partial v}{\partial t}(r, \bar{X}(r)) + L_k v(r, \bar{X}(r)) \right] dr \, ds
\]

\[
+ \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} v'(r, \bar{X}(r)) \sigma(\bar{X}_{k-1}) dB(r) \, ds
\]

taking expectation (assuming that the stochastic integral has zero expectation) yields

\[
\mathbb{E} \int_{t_{k-1}}^{t_k} v(s, \bar{X}(s)) ds = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} \mathbb{E}\left[ \frac{\partial v}{\partial t}(r, \bar{X}(r)) + L_k v(r, \bar{X}(r)) \right] dr \, ds
\]

\[
= O((t - t_{k-1})^2)
\]
under some regularity assumptions on

- the coefficients (drift vector and diffusion matrix) of the stochastic
differential equation
- the solution of the partial differential equation

the following estimate holds

\[ | \mathbb{E}[u(t_k, \bar{X}_{k-1}) - u(t_{k-1}, \bar{X}_{k-1})] | \leq C (t_k - t_{k-1})^2 \]

where the constant \( C \) does not depend on the time–step, hence

\[
\begin{align*}
| \mathbb{E}[f(\bar{X}(T))] - \mathbb{E}[f(X(T))] | & \leq \sum_{k=1}^{n} | \mathbb{E}[u(t_k, \bar{X}_k) - u(t_{k-1}, \bar{X}_{k-1})] | \\
& \leq C \sum_{k=1}^{n} (t_k - t_{k-1})^2 \
& \leq C T h
\end{align*}
\]
Remark if the test–function $f$ is not regular, for instance some indicator function, then assumptions of the theorem are not satisfied provided the drift $b$ and the diffusion matrix $a$ have the same regularity and growth condition as in the theorem, and if the following \textit{uniform ellipticity} (non–degeneracy) condition holds

$$v^* a(x) v \geq \lambda |v|^2$$

for any $x \in \mathbb{R}^d$ and any $d$–dimensional vector $v$, then the properties (weak convergence of order 1 and expansion of the error) remain true

the proof relies on Malliavin calculus (or stochastic calculus of variations) and is far beyond the scope of this course
Proof of (second part of) Theorem 2 recall that

\[ \mathbb{E}[u(t_k, \tilde{X}_k) - u(t_{k-1}, \tilde{X}_{k-1})] = \mathbb{E} \int_{t_{k-1}}^{t_k} \nu(s, \tilde{X}(s)) \, ds \]

\[ \begin{align*} 
&= \mathbb{E} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t} \left[ \frac{\partial \nu}{\partial t}(s, \tilde{X}(s)) + L_k \nu(s, \tilde{X}(s)) \right] ds \, dt \\
&= \mathbb{E} \int_{t_{k-1}}^{t_k} \left[ \int_{t_{k-1}}^{t} Z(s) \, ds \right] dt 
\end{align*} \]

where

\[ Z(s) = w(s, \tilde{X}(s)) \quad \text{and} \quad w(s, x) = \frac{\partial \nu}{\partial t}(s, x) + L_k \nu(s, x) \]

and using integration by parts yields

\[ \int_{t_{k-1}}^{t_k} \left[ \int_{t_{k-1}}^{t} Z(s) \, ds \right] dt = \int_{t_{k-1}}^{t_k} (t_k - t) Z(t) \, dt \]
one step further, note that

\[ \int_{t_{k-1}}^{t_k} (t_k - t) Z(t) \, dt - \frac{1}{2} (t_k - t_{k-1}) \int_{t_{k-1}}^{t_k} Z(t) \, dt \]

\[ = \int_{t_{k-1}}^{t_k} \left( \frac{1}{2} (t_{k-1} + t_k) - t \right) Z(t) \, dt \]

\[ = \int_{t_{k-1}}^{t_k} \left( \frac{1}{2} (t_{k-1} + t_k) - t \right) (Z(t) - Z(t_{k-1})) \, dt \]

using the Itô formula for the Euler approximation and for the time–dependent function \( w(s, x) \) yields

\[ Z(t) - Z(t_{k-1}) = w(t, \tilde{X}(t)) - w(t_{k-1}, \tilde{X}_{k-1}) \]

\[ = \int_{t_{k-1}}^{t} \left[ \frac{\partial w}{\partial t}(s, \tilde{X}(s)) + L_k w(s, \tilde{X}(s)) \right] ds \]

\[ + \int_{t_{k-1}}^{t} w'(s, \tilde{X}(s)) \sigma(\tilde{X}_{k-1}) \, dB(s) \]
\[
\int_{t_{k-1}}^{t_k} \left( \frac{1}{2} (t_{k-1} + t_k) - t \right) (Z(t) - Z(t_{k-1})) \, dt
\]

\[
= \int_{t_{k-1}}^{t_k} \left( \frac{1}{2} (t_{k-1} + t_k) - t \right) \int_{t_{k-1}}^{t} \left[ \frac{\partial W}{\partial t}(s, \bar{X}(s)) + L_k w(s, \bar{X}(s)) \right] \, ds \, dt
\]

\[
+ \int_{t_{k-1}}^{t_k} \left( \frac{1}{2} (t_{k-1} + t_k) - t \right) \int_{t_{k-1}}^{t} w'(s, \bar{X}(s)) \sigma(\bar{X}_{k-1}) \, dB(s) \, dt
\]

taking expectation (assuming that the stochastic integral has zero expectation) yields

\[
\mathbb{E} \int_{t_{k-1}}^{t_k} v(s, \bar{X}(s)) \, ds - \frac{1}{2} (t_k - t_{k-1}) \mathbb{E} \int_{t_{k-1}}^{t_k} w(s, \bar{X}(s)) \, ds
\]

\[
= \int_{t_{k-1}}^{t_k} \left( \frac{1}{2} (t_{k-1} + t_k) - t \right) \int_{t_{k-1}}^{t} \mathbb{E} \left[ \frac{\partial W}{\partial t}(s, \bar{X}(s)) + L_k w(s, \bar{X}(s)) \right] \, ds \, dt
\]
using integration by parts yields

$$
\int_{t_{k-1}}^{t_k} \left( \frac{1}{2} (t_{k-1} + t_k) - t \right) \left[ \int_{t_{k-1}}^{t} g(s) \, ds \right] \, dt
$$

$$
= -\frac{1}{2} \int_{t_{k-1}}^{t_k} (t_k - t)(t - t_{k-1}) \, g(t) \, dt
$$

hence

$$
\mathbb{E}[u(t_k, \bar{X}_k) - u(t_{k-1}, \bar{X}_{k-1})] - \frac{1}{2} (t_k - t_{k-1}) \mathbb{E} \int_{t_{k-1}}^{t_k} w(s, \bar{X}(s)) \, ds
$$

$$
= -\frac{1}{2} \int_{t_{k-1}}^{t_k} (t_k - t)(t - t_{k-1}) \mathbb{E} \left[ \frac{\partial w}{\partial t}(t, \bar{X}(t)) + L_k w(t, \bar{X}(t)) \right] \, dt
$$

$$
= O((t_k - t_{k-1})^3)
$$
under some further regularity assumptions on
  • the coefficients (drift vector and diffusion matrix) of the stochastic
differential equation
  • the solution of the partial differential equation
the following estimate holds

$$| \mathbb{E}[u(t_k, \tilde{X}_{k-1}) - u(t_{k-1}, \tilde{X}_{k-1})] - \frac{1}{2} (t_k - t_{k-1}) \int_{t_{k-1}}^{t_k} \mathbb{E}[w(s, \tilde{X}(s))] \, ds | $$

$$\leq C (t_k - t_{k-1})^3$$

where the constant $C$ does not depend on the time–step, therefore

$$| \mathbb{E}[f(\tilde{X}(T))] - \mathbb{E}[f(X(T))] - \frac{1}{2} h \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \mathbb{E}[w(s, \tilde{X}(s))] \, ds | $$

$$\leq \sum_{k=1}^{n} | \mathbb{E}[u(t_k, \tilde{X}_k) - u(t_{k-1}, \tilde{X}_{k-1})] - \frac{1}{2} h \int_{t_{k-1}}^{t_k} \mathbb{E}[w(s, \tilde{X}(s))] \, ds |$$

$$\leq C \sum_{k=1}^{n} (t_k - t_{k-1})^3 \leq C T h^2$$
Illustration #1

Brownian motion on the circle

\[ X(t) = X(0) - \int_0^t F X(s) \, ds + \int_0^t R X(s) \, dB(s) \]

with initial condition \( X(0) = (0, 1) \), and with \( 2 \times 2 \) matrices

\[
F = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

time–invariant: \( |X(t)| = 1 \) for any \( t \geq 0 \)
objective: approximate time–invariant \( \mathbb{E}|X(t)|^2 = 1 \) for any \( t \geq 0 \), using

- Euler approximation with time–step \( h \) or \( \frac{1}{2} h \)
- Monte Carlo approximation with \( N \) samples

i.e. coarse grid approximation

\[
\hat{f}^{h,N}(t) = \frac{1}{N} \sum_{i=1}^{N} |\bar{X}^{h,i}(t)|^2
\]

fine grid approximation

\[
\hat{f}^{\frac{1}{2}h,N}(t) = \frac{1}{N} \sum_{i=1}^{N} |\bar{X}^{\frac{1}{2}h,i}(t)|^2
\]

and Romberg–Richardson extrapolation

\[
\hat{f}^{h,\frac{1}{2}h,N}(t) = 2 \hat{f}^{\frac{1}{2}h,N}(t) - \hat{f}^{h,N}(t)
\]
Stochastic differential equations

Euler scheme: strong error estimate

Euler scheme: weak error estimate
Illustration #2

two–dimensional stationary Gaussian diffusion

\[
X(t) = X(0) + \int_0^t (-\frac{1}{2} I + R) X(s) \, ds + B(t)
\]

with initial condition \(X(0) \sim \mathcal{N}(0, I)\), and with \(2 \times 2\) matrix

\[
R = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

time–invariant (in distribution): \(X(t) \sim \mathcal{N}(0, I)\), in particular
\[
\mathbb{E}[X(t) X^*(t)] = I, \text{ for any } t \geq 0
\]
objective: approximate time–invariant $\mathbb{E}|X(t)|^2 = 2$ for any $t \geq 0$, using

- Euler approximation with time–step $h$ or $\frac{1}{2}h$
- Monte Carlo approximation with $N$ samples

i.e. coarse grid approximation

$$\hat{f}^{h,N}(t) = \frac{1}{N} \sum_{i=1}^{N} |\bar{X}^{h,i}(t)|^2$$

fine grid approximation

$$\hat{f}^{\frac{1}{2}h,N}(t) = \frac{1}{N} \sum_{i=1}^{N} |\bar{X}^{\frac{1}{2}h,i}(t)|^2$$

and Romberg–Richardson extrapolation

$$\hat{f}^{h,\frac{1}{2}h,N}(t) = 2 \hat{f}^{\frac{1}{2}h,N}(t) - \hat{f}^{h,N}(t)$$
Stochastic differential equations

Euler scheme: strong error estimate

Euler scheme: weak error estimate