INSA Rennes, 4GM–AROM

Random Models of Dynamical Systems
Introduction to SDE's

Written Exam (aka DS)

January 8, 2019

Stochastic integral in intrinsic clock = Brownian motion

Let B be a one–dimensional standard Brownian motion, with $B(0) = 0$, and adapted to a given filtration $\mathcal{F}$, and consider the stochastic process X defined by

$$X(t) = \int_0^t \phi(u)\, dB(u) \ ,$$

for any $t \geq 0$, where ϕ belongs to M^2_{loc}, i.e.

$$A(t) = \int_0^t |\phi(u)|^2\, du < \infty \ ,$$

almost surely, for any $t \geq 0$.

(i) **Write the Itô formula for the process X and for the complex–valued function $f(x) = \exp\{i\,\lambda\, x\}$ where the scalar λ is fixed, between the time instants s and t, with $0 \leq s \leq t$.**

For any $t \geq 0$, define

$$\tau(t) = \inf\{s \geq 0 \ : \ \int_0^s |\phi(u)|^2\, du \geq t\} \ ,$$

if such time exists, and $\tau(t) = \infty$ otherwise.

(ii) **For any $t \geq 0$, show that the random variable $\tau(t)$ is a stopping time, and that the equivalent definition**

$$\tau(t) = \inf\{s \geq 0 \ : \ \int_0^s |\phi(u)|^2\, du = t\} \ .$$

holds, hence $A(\tau(t)) = t$.

Show that $\tau(t) \uparrow \infty$ almost surely as $t \uparrow \infty$.

From now on, it is assumed that

$$\int_0^\infty |\phi(u)|^2 \, du = \infty \ ,$$

i.e. $A(t) \uparrow \infty$ almost surely as $t \uparrow \infty$, so that $\tau(t) < \infty$ for any $t < \infty$.

(iii) **Show (a simple graphic could help to prove (a) and (b)) that**

(a) the mapping $t \mapsto \tau(t)$ is non–decreasing and left–continuous,

(b) for any $t, s \geq 0$

$$\{A(t) \geq s\} = \{\tau(s) \leq t\} \ ,$$

(c) for any nonnegative Borel measurable function f and for any $t \geq 0$

$$\int_0^t f(\tau(s)) \, ds = \int_0^{\tau(t)} f(s) \, dA(s) \ .$$

[Hint: just prove (c) for any function f of the form of an indicator function, defined by $f(s) = 1_{(0 \leq s \leq L)}$ for any $s \geq 0$ and for some $L > 0$ (the same result for an arbitrary nonnegative Borel function would follow by a monotone class argument).]

The mapping τ is called the *intrinsic clock* (or *intrinsic time*) for the stochastic process X, and the time–changed stochastic process Z is defined by

$$Z(t) = X(\tau(t)) = \int_0^{\tau(t)} \phi(u) \, dB(u) \ ,$$

for any $t \geq 0$.

(iv) **Using the representation obtained in question (i) and using the result obtained in question (iii-c), show that**

$$\begin{aligned} \exp\{i \, \lambda \, Z(t)\} \ &= \ \exp\{i \, \lambda \, Z(s)\} + i \, \lambda \int_{\tau(s)}^{\tau(t)} \exp\{i \, \lambda \, X(u)\} \, \phi(u) \, dB(u) \\ &\quad - \tfrac{1}{2} \, \lambda^2 \int_s^t \exp\{i \, \lambda \, Z(u)\} \, du \ , \end{aligned}$$

for any $0 \leq s \leq t$.

Introduce the σ–algebra $\mathcal{A}(t) = \mathcal{F}(\tau(t))$, i.e. $A \in \mathcal{A}(t)$ iff for any $u \geq 0$

$$A \cap \{\tau(t) \leq u\} \in \mathcal{F}(u) \ .$$

If the stochastic process M, defined by the stochastic integral

$$M(t) = \int_0^t \exp\{i \, \lambda \, X(u)\} \, \phi(u) \, dB(u) \ ,$$

for any $t \geq 0$, would be a martingale, and if the stopping time $\tau(t)$ would be almost surely bounded, then the optional sampling theorem would yield

$$\mathbb{E}[M(\tau(t)) - M(\tau(s)) \mid \mathcal{A}(s)] = \mathbb{E}[\int_{\tau(s)}^{\tau(t)} \exp\{i\,\lambda\, X(u)\}\, \phi(u)\, dB(u) \mid \mathcal{A}(s)] = 0 \ ,$$

for any $0 \leq s \leq t$. The purpose of the next question is to show that the same identity holds in the more general case considered here, where ϕ does belong to $M^2_{\rm loc}$ only.

(v) **Show that the stopped process $M^{\tau(t_2)}$, defined by $M^{\tau(t_2)}(t) = M(t \wedge \tau(t_2))$ for any $t \geq 0$, is a square–integrable martingale, and that**

$$\mathbb{E}[\int_{\tau(t_1)}^{\tau(t_2)} \exp\{i\,\lambda\, X(u)\}\, \phi(u)\, dB(u) \mid \mathcal{A}(t_1)] = 0 \ .$$

for any $0 \leq t_1 \leq t_2$.

[Hint: recall that for a uniformly integrable martingale, the optional sampling theorem holds for any almost surely *finite* (and not necessarily *bounded*) stopping times.]

(vi) **Show that the following identity**

$$\mathbb{E}[\exp\{i\,\lambda\,(Z(t) - Z(s))\} \mid \mathcal{A}(s)] = \exp\{-\tfrac{1}{2}\,\lambda^2\,(t-s)\} \ ,$$

holds for the conditional chraracteristic function.

Conclude that the process $Z = (Z(t)\,,\, t \geq 0)$ is a standard Brownian motion w.r.t. the filtration $\mathcal{A} = (\mathcal{A}(t)\,,\, t \geq 0)$.

(vii) **Show that**

$$\frac{\displaystyle\int_0^T \phi(u)\, dB(u)}{\displaystyle\int_0^T |\phi(u)|^2\, du} \longrightarrow 0$$

almost surely as $T \uparrow \infty$.

[Hint: use the law of large numbers for Brownian motion.]

Sequential maximum likelihood estimation

Consider the following statistical model: there exist a parametric family $(\mathbb{P}_\theta\,,\, \theta \in \mathbb{R})$ of probability measures and a one–dimensional stochastic process X, such that under $\mathbb{P}_\theta$ it holds

$$dX(t) = \theta\, b(X(t))\, dt + dW_\theta(t) \ ,$$

where W_θ is a standard Brownian motion, and where the drift function b satisfies the *global Lipschitz* and *linear growth* conditions.

It is assumed that the maximum likelihood estimator of the parameter θ based on the observation of $(X(t)\,,\, 0 \leq t \leq T)$ in the time interval $[0,T]$ is given by the following expression

$$\widehat{\theta}(T) = \frac{\displaystyle\int_0^T b(X(t))\,dX(t)}{\displaystyle\int_0^T |b(X(t))|^2\,dt} \ .$$

Let θ_0 denote the (unknown) true value of the parameter, and let $\mathbb{P}_0 = \mathbb{P}_{\theta_0}$ denote the corresponding probability measure.

(viii) **Show that under $\mathbb{P}_0$ the maximum likelihood estimator satisfies**

$$\widehat{\theta}(T) = \theta_0 + \frac{\displaystyle\int_0^T b(X(t))\,dW_0(t)}{\displaystyle\int_0^T |b(X(t))|^2\,dt} \ ,$$

where W_0 is a standard Brownian motion.

Note that this expression cannot be used in practice, since neither is $(W_0(t)\,,\, 0 \leq t \leq T)$ observed (available), nor is θ_0 known. The purpose of this expression is rather to analyze the behaviour of the estimator $\widehat{\theta}(T)$, for instance its asymptotic behaviour as $T \uparrow \infty$.

(ix) **Show that under $\mathbb{P}_0$ the maximum likelihood estimator is strongly consistent, i.e. $\widehat{\theta}(T) \to \theta_0$ almost surely as $T \uparrow \infty$.**

Actually, studying the ratio of two random variables is not so easy, and it is more convenient to study the time–changed estimator

$$\overline{\theta}(H) = \widehat{\theta}(\tau(H)) \qquad \text{where} \qquad \tau(H) = \inf\{T \geq 0 \, : \, \int_0^T |b(X(t))|^2\,dt = H\} \ .$$

(x) **Show that under $\mathbb{P}_0$ the time–changed maximum likelihood estimator satisfies**

$$\overline{\theta}(H) = \theta_0 + \frac{1}{H} \int_0^{\tau(H)} b(X(t))\,dW_0(t) \ .$$

The benefit of considering the time–changed maximum likelihood estimator is that the denominator is now deterministic, and the problem reduces to studying a stochastic integral under its intrinsic clock.

(xi) **Using the results obtained in the first part, show that under $\mathbb{P}_0$ the time–changed maximum likelihood estimator**

- **is strongly consistent, i.e. $\overline{\theta}(H) \to \theta_0$ almost surely as $H \uparrow \infty$,**
- **is unbiased (i.e. has a mean equal to the true value θ_0),**
- **has a (nonasymptotic) variance equal to $1/H$,**
- **is normally distributed, with mean θ_0 and variance $1/H$.**