4GM INSA Rennes, cours ARO07–MSSD

Random Models of Dynamical Systems
Introduction to SDE's

Written Exam (aka DS)

January 3, 2022

Exercise 1: Refined discretization of a Brownian trajectory

Let $B(t)$ be a d–dimensional standard Brownian motion. The purpose of this exercise is to characterise the conditional probability distribution of the random vector $B(s)$ given the pair $(B(t_{i-1}), B(t_i))$, with $t_{i-1} \leq s \leq t_i$.

(i) **Show that the characteristic function of the conditional probability distribution of the random vector $B(s)$ given the pair $(B(t_{i-1}), B(t_i))$ satisfies the identity**

$$\mathbb{E}[\, \exp\{i\, u^*\, B(s)\} \mid B(t_{i-1}), B(t_i)\,]$$

$$= \exp\{i\, u^*\, B(t_{i-1})\}\; \mathbb{E}[\, \exp\{i\, u^*\, (B(s) - B(t_{i-1}))\} \mid B(t_i) - B(t_{i-1})\,] \; .$$

Recall the following result about conditioning in Gaussian random vectors. Let (X, Y) be a Gaussian random vector, with mean vector and covariance matrix

$$\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} Q_X & Q_{XY} \\ Q_{YX} & Q_Y \end{pmatrix} \; ,$$

respectively. If the covariance matrix Q_Y is invertible, then the conditional probability distribution of X given Y is a Gaussian probability distribution, with mean vector and covariance matrix given by

$$\mu_{X|Y} = \mu_X + Q_{XY}\, Q_Y^{-1}\, (Y - \mu_Y) \qquad \text{and} \qquad Q_{X|Y} = Q_X - Q_{XY}\, Q_Y^{-1}\, Q_{YX} \; ,$$

respectively.

(ii) **Show that the $2d$–dimensional random vector $(B(s) - B(t_{i-1}), B(t_i) - B(t_{i-1}))$ is Gaussian, with zero mean vector and a covariance matrix to be specified.**

Conclude that, conditionally with respect to the increment $B(t_i) - B(t_{i-1})$, the increment $B(s) - B(t_{i-1})$ is a Gaussian random vector, with a mean vector and a covariance matrix to be specified.

(iii) **Conclude that, conditionally with respect to the pair $(B(t_{i-1}), B(t_i))$, the random variable $B(s)$ is Gaussian, with mean vector**

$$B_{\text{lin}} = \frac{t_i - s}{t_i - t_{i-1}}\, B(t_{i-1}) + \frac{s - t_{i-1}}{t_i - t_{i-1}}\, B(t_i) \ ,$$

and covariance matrix

$$\frac{(t_i - s)\,(s - t_{i-1})}{t_i - t_{i-1}}\, I \ .$$

Exercise 2: Brownian motion on the circle

Let $B(t)$ be a one–dimensional standard Brownian motion, and consider the two–dimensional (bilinear) SDE

$$X(t) = X(0) - \tfrac{1}{2} \int_0^t X(s)\, ds + \int_0^t R\, X(s)\, dB(s) \ ,$$

with initial condition $X(0) = (1, 0)$, and with the 2×2 matrix

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

(i) **Check that this SDE has a unique solution.**

(ii) **Write the Itô formula for the Itô process $X(t)$ and for the real–valued function $f(x) = |x|^2$ defined on $\mathbb{R}^2$. Conclude that the solution satisfies the invariant: $|X(t)|^2 = 1$ for any $t \geq 0$.**

Thinking in terms of polar coordinates, it follows from question (ii) that the process $X(t)$ takes its values on the unit circle, i.e. the radial component of the process $X(t)$ is constant, and it remains to describe the angular component of the process $X(t)$. Write

$$X(t) = \begin{pmatrix} \cos\theta(t) \\ \\ \sin\theta(t) \end{pmatrix} \ ,$$

and assume that the angular component $\theta(t)$ has the representation

$$\theta(t) = \theta(0) + \int_0^t \psi(s)\, ds + \int_0^t \phi(s)\, dB(s) \ ,$$

as a one–dimensional Itô process. For instance, the initial condition $X(0) = (1, 0)$ would imply that $\theta(0) = 0$ modulo 2π.

(iii) **Get a representation of the process $X_1(t) = \cos\theta(t)$ as an Itô process.**

[Hint: Write the Itô formula for the Itô process $\theta(t)$ and for the real–valued function $f_1(\theta) = \cos\theta$.]

(iv) **Similarly, get a representation of the process $X_2(t) = \sin\theta(t)$ as an Itô process.**

[Hint: Write the Itô formula for the Itô process $\theta(t)$ and for the real–valued function $f_2(\theta) = \sin\theta$.]

(v) **Using the results obtained at questions (iii) and (iv), show that the process $X(t)$ has the representation**

$$X(t) = X(0) + \int_0^t [R\,X(s)\,\psi(s) - \tfrac{1}{2}\,X(s)\,\phi^2(s)]\,ds + \int_0^t R\,X(s)\,\phi(s)\,dB(s)\ ,$$

as an Itô process.

(vi) **Comparing the two representations of the process $X(t)$ as an an Itô process — the representation provided by the SDE, and the representation obtained at question (v) — show that the coefficients $\psi(s)$ and $\phi(s)$ are uniquely defined as $\psi(s) = 0$ and $\phi(s) = 1$ for any $s \geq 0$.**

Conclude that the angular component $\theta(t)$ is a Brownian motion.

[Hint: The decomposition of a stochastic process as an Itô process is unique.]

Exercise 3: Bilinear SDE (general case)

Let $B(t)$ be a p–dimensional standard Brownian motion, i.e. the components $B_1(t), \cdots, B_p(t)$ are p independent one–dimensional standard Brownian motions.

(i) **Let the coefficients $\psi(s)$ and $\phi(s) = (\phi_1(s), \cdots, \phi_p(s))$ be a scalar and a p–dimensional row vector, respectively. Show that the one–dimensional process $Z(t)$ is positive and has the representation**

$$Z(t) = Z(0) + \int_0^t Z(s)\,\psi(s)\,ds + \int_0^t Z(s)\,\phi(s)\,dB(s)\ ,$$

as an Itô process, <u>if and only if</u> it has the explicit expression

$$Z(t) = Z(0)\ \exp\{\int_0^t [\psi(s) - \tfrac{1}{2}\,|\phi(s)|^2]\,ds + \int_0^t \phi(s)\,dB(s)\}\ .$$

Both the representation as an Itô process and the explicit expression will be denoted by $\mathcal{E}(\psi, \phi)$.

[Hint: For the 'only if' part, write the Itô formula for the Itô process $Z(t)$ and for the real-valued function $g(z) = \log z$ defined on $(0, \infty)$. For the 'if' part, write the Itô formula for the Itô process $Y(t) = \log Z(t)$ and for the real–valued function $f(y) = \exp\{y\}$.]

This preliminary result will not be used before question (iv). Consider the d–dimensional SDE

$$X(t) = X(0) + \int_0^t A_0\, X(s)\, ds + \sum_{i=1}^p \int_0^t A_i\, X(s)\, dB_i(s) \ ,$$

where $A_0, A_1, \cdots, A_p$ are $d \times d$ matrices. Equivalently, introducing the $d \times p$ matrix

$$A \cdot x = (A_1\, x \cdots A_p\, x) \ ,$$

defined for any $x \in \mathbb{R}^d$, the p columns of which are d–dimensional column vectors, it holds

$$X(t) = X(0) + \int_0^t A_0\, X(s)\, ds + \int_0^t (A \cdot X(s))\, dB(s) \ ,$$

in coordinate–free notation. Note that A is a *linear* mapping defined on $\mathbb{R}^d$ and taking values in the space of $d \times p$ matrices, so that $A \cdot x$ is a $d \times p$ matrix for any $x \in \mathbb{R}^d$, however A alone is not a matrix.

(ii) **Check that this SDE has a unique solution.**

Thinking in terms of spherical coordinates, introduce the two processes

$$\xi(t) = \frac{X(t)}{|X(t)|} \qquad \text{and} \qquad \rho(t) = \log |X(t)| \ ,$$

a spherical component taking values in the $(d-1)$–dimensional unit sphere $S^{d-1} \subset \mathbb{R}^d$ and a log–radial component taking values in the real line $\mathbb{R}$, respectively.

(iii) **Get a representation of the process $Z(t) = |X(t)|^2$ as an Itô process.**

[Hint: Write the Itô formula for the Itô process $X(t)$ and for the real–valued function $f(x) = |x|^2$ defined on $\mathbb{R}^d$.]

The next question relies on the preliminary result obtained at question (i).

(iv) **Check that the process $Z(t) = |X(t)|^2$ has the representation $\mathcal{E}(\psi^{(2)}, \phi^{(2)})$ as an Itô process with coefficients**

$$\psi^{(2)}(s) = 2\, \xi^*(s)\, A_0\, \xi(s) + |A_1\, \xi(s)|^2 + \cdots + |A_p\, \xi(s)|^2 \ ,$$

and

$$\phi^{(2)}(s) = 2\, (\xi^*(s)\, A_1\, \xi(s), \cdots, \xi^*(s)\, A_p\, \xi(s)) \ ,$$

a scalar and a p–dimensional row vector, respectively.

Get a representation of the log–radial component $\rho(t)$ as an Itô process, with coefficients that depend on the spherical component only, and not on the radial component.

Check that the process $|X(t)|$ has the representation $\mathcal{E}(\psi^{(1)}, \phi^{(1)})$ as an Itô process with coefficients

$$\begin{aligned}\psi^{(1)}(s) \;&=\; \tfrac{1}{2}\,\psi^{(2)}(s) - \tfrac{1}{8}\,|\phi^{(2)}(s)|^2 \\ &=\; \xi^*(s)\,A_0\,\xi(s) + \tfrac{1}{2}\,|A_1\,\xi(s)|^2 + \cdots + \tfrac{1}{2}\,|A_p\,\xi(s)|^2 \\ &\qquad -[\,\tfrac{1}{2}\,(\xi^*(s)\,A_1\,\xi(s))^2 + \cdots + \tfrac{1}{2}\,(\xi^*(s)\,A_p\,\xi(s))^2\,]\ ,\end{aligned}$$

and

$$\phi^{(1)}(s) = \tfrac{1}{2}\,\phi^{(2)}(s) = (\xi^*(s)\,A_1\,\xi(s), \cdots, \xi^*(s)\,A_p\,\xi(s))\ ,$$

a scalar and a p–dimensional row vector, respectively.

[Hint: Use the preliminary result obtained at question (i) and simple algebra. In other words, stochastic calculus and the Itô formula are not needed here.]

(v) **Show that the spherical component $\xi(t)$ is the solution of an autonomous SDE, with drift and diffusion coefficients that depend on the spherical component only, and not on the radial component.**

[Hint: Write the Itô formula for the Itô process $(V(t), W(t))$ where $V(t) = u^* X(t)$, u is an arbitrary d–dimensional vector and $W(t) = |X(t)|$, and for the real–valued function $f(v, w) = v/w$ defined on $\mathbb{R} \times (0, \infty)$.]