cours ARO07–MSSD

Random Models of Dynamical Systems
Introduction to SDE's

TD 3 : Stochastic differential equations and diffusion processes

November 24, 2021

Exercise 1 [Brownian motion on the circle] Let $B = (B(t)\,,\, 0 \le t \le T)$ be a one–dimensional standard Brownian motion defined on the interval $[0,T]$, with $B(0) = 0$. Consider the two–dimensional (bilinear) SDE

$$X(t) = X(0) - \int_0^t F\,X(s)\,ds + \int_0^t R\,X(s)\,dB(s)\ ,$$

with initial condition $X(0) = (0,1)$, and with the 2×2 matrices

$$F = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ .$$

(i) **Check that this SDE has a unique solution.**

(ii) **Write the Itô formula for the real–valued function $f(x) = |x|^2$ defined on $\mathbb{R}^2$. Conclude that the solution satisfies the invariant: $|X(t)|^2 = 1$ almost surely, for any $0 \le t \le T$.**

Exercise 2 [Stationary Gaussian diffusion] Let $B = (B(t)\,,\, 0 \le t \le T)$ be a two–dimensional standard Brownian motion defined on the interval $[0,T]$, with $B(0) = 0$. Consider the two–dimensional (linear) SDE

$$X(t) = X(0) + \int_0^t (-c\,I + R)\,X(s)\,ds + \sigma\,B(t)\ ,$$

with two real numbers $c > 0$ and σ, and with the 2×2 matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ .$$

It is further assumed that the initial condition $X(0)$ has zero mean $\mathbb{E}[X(0)] = 0$ and finite variance $\mathbb{E}[X(0)\,X^*(0)] = \Sigma$.

(i) **Check that this SDE has a unique solution. Show that the solution satisfies $\mathbb{E}[X(t))] = 0$ for any $0 \le t \le T$.**

(ii) **Write the Itô formula for the matrix–valued function $f(x) = x\,x^*$ defined on $\mathbb{R}^2$, and give the differential equation satisfied by the covariance matrix $\Sigma(t) = \mathbb{E}[X(t)\,X^*(t)]$.**

[Hint: consider the real–valued process $u^*\,X(t)$, where u is an arbitrary two–dimensional vector, and write the Itô formula for the real–valued function $f(r) = r^2$ defined on $\mathbb{R}$.]

(iii) **Under which condition on c and σ^2, and on the variance Σ at initial time $t = 0$, is the solution stationary (in the following weak sense: $\mathbb{E}[X(t)\,X^*(t)] = \Sigma$ for any $0 \le t \le T$)? What is the expression of $\mathbb{E}|X(t)|^2$ in this case?**

Exercise 3 [Wright–Fisher diffusion approximation] Consider the following simplified model for the reproduction of individuals through the transmissions of alleles (alternative types of the same gene). Consider here the case of one gene, with two alleles A and a. The population size N is assumed finite and constant at each generation. At generation k, each individual inherits the allele of its parent, a randomly (uniformly) selected (with replacement) individual present in the population at generation $(k-1)$. Define the random variable Y_k^N to be the number of allele of type A present at generation k.

(i) **Show that the random variable Y_k^N takes values in $\{0, 1, \cdots, N\}$, and that the sequence $(Y_k^N\,,\, k \ge 0)$ forms a Markov chain with transition probability matrix**

$$\pi_{i,j}^N = \mathbb{P}[Y_k^N = j \mid Y_{k-1}^N = i] = \binom{N}{j} \left(\frac{i}{N}\right)^j \left(1 - \frac{i}{N}\right)^{N-j} ,$$

for any $i, j \in \{0, 1, \cdots, N\}$.

(ii) **Check that**

$$\mathbb{E}[Y_k^N \mid Y_{k-1}^N = i] = i \qquad \textbf{and} \qquad \mathbb{E}[(Y_k^N - Y_{k-1}^N)^2 \mid Y_{k-1}^N = i] = i\,(1 - \frac{i}{N}) \ .$$

Thinking more in term of frequencies (i.e. proportions) rather than in terms of number of individuals, introduce the normalized random variable $X_k^N = Y_k^N/N$.

(iii) **Show that the random variable X_k^N takes values in $\{0, 1/N, \cdots, 1-1/N, 1\} \subset [0, 1]$, and check that**

$$\mathbb{E}[X_k^N \mid X_{k-1}^N = p] = p \ ,$$

and

$$\mathbb{E}[(X_k^N - X_{k-1}^N)^2 \mid X_{k-1}^N = p] = \frac{1}{N}\,p\,(1-p) \ ,$$

for any $p \in \{0, 1/N, \cdots, 1 - 1/N, 1\}$.

(iv) **Show that the candidate limit (in distribution, as the population size $N \uparrow \infty$) of the continuous–time process interpolating points X_k^N at time instants $t_k^N = k/N$, is the solution of the SDE**

$$X(t) = X(0) + \int_0^t \sqrt{X(s)\,(1 - X(s))}\, dB(s) \ .$$

It is assumed that there exists a unique solution to this SDE, taking values in the interval $[0, 1]$.

Exercise 4 [Exit time of a one–dimensional diffusion process] Let $B(t)$ be a one–dimensional standard Brownian motion, and consider the SDE

$$X(t) = X(0) + \int_0^t b(X(s))\, ds + \int_0^t \sigma(X(s))\, dB(s) \ ,$$

where the drift and the diffusion coefficients satisfy the global Lipschitz condition and the linear growth condition. Let L denote the associated second–order differential operator. Let $a < c$ and consider the two hitting times

$$T_a = \inf\{t \geq 0\,,\, X(t) = a\} \qquad \text{and} \qquad T_c = \inf\{t \geq 0\,,\, X(t) = c\} \ ,$$

of a and c respectively, and let

$$T_{a,c} = T_a \wedge T_c = \inf\{t \geq 0 \,:\, X(t) \notin (a, c)\} \ ,$$

denote the exit time from the open interval (a, c). Assume that there exist two bounded functions f and g, twice differentiable with bounded first and second derivatives, such that

$$L\, f(x) = 0 \qquad \text{for any } a \leq x \leq c,$$

up to two (multiplicative and additive) arbitrary normalizing constants, and such that

$$L\, g(x) = -1 \qquad \text{for any } a \leq x \leq c,$$

with conditions $g(a) = g(c) = 0$, respectively.

(i) **Show that**

$$\mathbb{E}_{0,x}[T_a \wedge T_c] = g(x) < \infty \qquad \textbf{and} \qquad \mathbb{P}_{0,x}[T_a < T_c] = \frac{f(c) - f(x)}{f(c) - f(a)} \ ,$$

for any starting point $x \in (a, c)$.

(ii) **Apply these general results to the special case of the (limiting SDE in the) Wright–Fisher genetic model.**

Exercise 5 [Exact time–discretization] Consider the stochastic differential equation

$$dX(t) = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} X(t)\, dt + \sigma \begin{pmatrix} 0 \\ I \end{pmatrix} dB(t) \qquad \text{where} \qquad X(t) = \begin{pmatrix} r(t) \\ v(t) \end{pmatrix} ,$$

or equivalently

$$dr(t) = v(t)\, dt \qquad \text{and} \qquad dv(t) = \sigma\, dB(t) ,$$

in integral form, where $(B(t)\,,\, t \geq 0)$ is a two–dimensional standard Brownian motion.

(i) **Show that**

$$r(t) = r(s) + v(s)\,(t-s) + \sigma \int_s^t (t-u)\, dB(u) .$$

[Hint: show and use the identity

$$\int_s^t (B(u) - B(s))\, du = \int_s^t (t-u)\, dB(u) .]$$

(ii) **Show that the conditional probability distribution of $X(t)$ given $X(s)$ is Gaussian, with mean vector and covariance matrix**

$$\begin{pmatrix} r(s) + (t-s)\, v(s) \\ v(s) \end{pmatrix} \qquad \textbf{and} \qquad \sigma^2 \begin{pmatrix} \frac{1}{3}\,(t-s)^3\, I & \frac{1}{2}\,(t-s)^2\, I \\ \frac{1}{2}\,(t-s)^2\, I & (t-s)\, I \end{pmatrix}$$

respectively.

Exercise 6 [Ornstein–Uhlenbeck process] Let $B(t)$ be a one–dimensional standard Brownian motion, and for any positive real $\beta > 0$ and any real γ, consider the one–dimensional SDE

$$X(t) = X(0) - \beta \int_0^t X(s)\, ds + \gamma\, B(t) ,$$

where the initial condition $X(0)$ is square–integrable and independent of the Brownian motion.

(i) **Check that there exists a unique solution to this SDE.**

(ii) **Show that the solution is given explicitly as**

$$X(t) = \exp\{-\beta\, t\}\; X(0) + \gamma \int_0^t \exp\{-\beta\,(t-s)\}\; dB(s) .$$

[Hint: use the variation of the constant method for the process $Y(t) = X(t) - \gamma\, B(t)$.]

(iii) **Give the expression of the mean, the variance and the correlation coefficient, defined as**

$$m(t) = \mathbb{E}[X(t)] \qquad \textbf{and} \qquad \sigma^2(t) = \mathbb{E}|X(t) - m(t)|^2$$

and

$$\rho(t,h) = \mathbb{E}[(X(t+h) - m(t+h))\,(X(t) - m(t))] \ ,$$

respectively. Show that for a special choice of γ in terms of $\beta > 0$ and $\sigma^2(0)$, the variance and the correlation coefficient do not depend on $t \geq 0$.

(iv) **Assume further that the initial condition is a Gaussian random variable. Show that the process $X(t)$ is Gaussian.**

Exercise 7 [Kramers–Smoluchowski approximation] Let $B(t)$ be a one–dimensional standard Brownian motion, and let the real–valued drift function $b(x)$ be globally Lipschitz continuous, with at most linear growth. Consider the one–dimensional SDE

$$X_*(t) = q + \int_0^t b(X_*(s))\,ds + \sigma\,B(t) \ , \tag{$\star$}$$

with initial condition $X_*(0) = q$, and for any positive $\mu > 0$, consider the two–dimensional SDE

$$\begin{aligned} X_\mu(t) \ &= \ q + \int_0^t V_\mu(s)\,ds \\ V_\mu(t) \ &= \ p + \frac{1}{\mu}\,[-\int_0^t V_\mu(s)\,ds + \int_0^t b(X_\mu(s))\,ds + \sigma\,B(t)\,] \end{aligned} \tag{$\star\star$}$$

with initial condition $(X_\mu(0), V_\mu(0)) = (q,p)$. The objective is to show that the (first component of the) solution of ($\star\star$) provides a smooth (differentiable) approximation of the solution of ($\star$) uniformly on $[0,T]$, as $\mu \to 0$.

(i) **Show that the drift function in the SDE ($\star$), and the drift function in the SDE ($\star\star$), satisfy the linear growth condition.**

(ii) **Check that there exist a unique solution to the SDE ($\star$), and a unique solution to the SDE ($\star\star$).**

(iii) **Show that**

$$\begin{aligned} V_\mu(s) \ &= \ \exp\{-\frac{1}{\mu}\,s\}\,p + \frac{1}{\mu}\int_0^s \exp\{-\frac{1}{\mu}\,(s-u)\}\,b(X_\mu(u))\,du \\ &\quad + \frac{\sigma}{\mu}\int_0^s \exp\{-\frac{1}{\mu}\,(s-u)\}\,dB(u) \ . \end{aligned}$$

[Hint: use the integration by parts formula.]

(iv) **Show that**

$$X_\mu(t) \;=\; q + \mu\,[1 - \exp\{-\frac{1}{\mu}\,t\}]\;p + \int_0^t [1 - \exp\{-\frac{1}{\mu}\,(t-u)\}]\;b(X_\mu(u))\,du$$
$$+\,\sigma\;\int_0^t [1 - \exp\{-\frac{1}{\mu}\,(t-u)\}]\;dB(u)\;.$$

(v) **Show that**

$$|X_\mu(t) - X_*(t)| \;\le\; \mu\,|p| + L\;\int_0^t |X_\mu(s) - X_*(s)|\;ds$$
$$+\int_0^t \exp\{-\frac{1}{\mu}\,(t-u)\}\;|b(X_*(u))|\,du$$
$$+\,\sigma\;|\int_0^t \exp\{-\frac{1}{\mu}\,(t-u)\}\;dB(u)|\;.$$

The challenge is that the stochastic integral is not a martingale, since the integrand depends on both u and t. Classical tools such as the Doob inequality or the Doob maximal inequality could not help here, and a different approach is needed.

(vi) **Show that**

$$\sup_{0\le t\le T}\;|\int_0^t \exp\{-\frac{1}{\mu}\,(t-s)\}\;dB(s)\;| \to 0\;,$$

almost surely as $\mu \downarrow 0$.

[Hint: show and use the identity

$$\int_0^t \exp\{-\frac{1}{\mu}\,(t-s)\}\;dB(s) = B(t) - \frac{1}{\mu}\;\int_0^t \exp\{-\frac{1}{\mu}\,(t-s)\}\;B(s)\,ds\;.$$

and split the integral in two, an integral from 0 to $(t-\delta)$ and an integral from $(t-\delta)$ to t.]

(vii) **Using the Gronwall lemma, show that**

$$\sup_{0\le t\le T}\;|X_\mu(t) - X_*(t)| \to 0\;,$$

almost surely as $\mu \downarrow 0$.

Exercise 8 [SDE for the Brownian bridge] Consider the process defined by

$$Z'(t) = (1-t)\;\int_0^t \frac{dB(s)}{1-s}\;,$$

for any $0 \le t < 1$.

(i) **Show that $Z'(t) \to 0$ in L^2 as $t \to 1$ (and define $Z'(1) = 0$ by continuity, assuming that the convergence holds also almost surely). Show that Z' has the same distribution as the Brownian bridge.**

(ii) **Show that Z' is the unique solution of the SDE**

$$Z'(t) = -\int_0^t \frac{Z'(s)}{1-s}\, ds + B(t) \ ,$$

for any $0 \leq t < 1$.

[Hint: write the Itô formula for Z' seen as the product of two Itô processes.]