Reachability games with relaxed energy constraints

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We study games with reachability objectives under energy constraints. We first prove that under strict energy constraints (either only lower-bound constraint or interval constraint), those games are LOGSPACE-equivalent to energy games with the same energy constraints but without reachability objective (i.e., for infinite runs). We then consider two kinds of relaxations of the upper-bound constraints (while keeping the lower-bound constraint strict): in the first one, called *weak upper bound*, the upper bound is *absorbing*, in the sense that it allows receiving more energy when the upper bound is already reached, but the extra energy will not be stored; in the second one, we allow for *temporary violations* of the upper bound, imposing limits on the number or on the amount of violations.

We prove that when considering weak upper bound, reachability objectives require memory, but can still be solved in polynomial-time for one-player arenas; we prove that they are in coNP in the two-player setting. Allowing for bounded violations makes the problem PSPACE-complete for one-player arenas and EXPTIME-complete for two players.

1 Introduction

Games on weighted graphs. Weighted games are a common way to formally address questions related to consumption, production and storage of resources: the arena of such game are two-player turn-based games in which transitions carry positive or negative integers, representing the accumulation or consumption of resource. Various objectives have been considered for such arenas, such as optimizing the total or average amount of resources that have been collected along the play, or maintaining the total amount within given bounds. The latter kind of objectives, usually referred to as *energy objectives* [8, 3], has been widely studied in the untimed setting [9, 11, 13, 18, 22, 24, 28, 6, 4, 14], and to a lesser extent in the timed setting [2, 5]. As their name indicates, energy objectives can be used to model the evolution of the available energy in an autonomous system: besides achieving its tasks, the system has to take care of recharging batteries regularly enough so as to never run out of power. Energy objectives were also used to model moulding machines: such machines inject molten plastic into a mould, using pressure obtained by storing liquid in a tank [7]; the level of liquid has to be controlled in such a way that enough pressure is always available, but excessive pressure in the tank would reduce the service life of the valve.

Energy games impose strict constraints on the total amount of energy at all stages of the play. Two kinds of constraints have been mainly considered in the literature: lower-bound constraints (a.k.a. L-energy constraints) impose a strict lower bound (usually zero), but impose no upper bound; on the other hand, lower- and upper-bound constraints (a.k.a. LU-energy constraints) require that the energy level always remain within a bounded interval $[L;U]$. Finding strategies that realize L-energy objectives along infinite runs is in PTIME in the one-player setting, and in $\mathsf{NP} \cap \mathsf{coNP}$ for two players; for LU-energy objectives, it is respectively PSPACE-complete and EXPTIME-complete [3]. Some works have also considered the existence of an initial energy level for which a winning strategy exist [11].

In this paper, we focus on weighted games combining energy objectives together with reachability objectives. Our first result is the (expected) proof that L-energy games with or without reachability objectives are interreducible; the same holds for LU-energy games. We then focus on relaxations of the

Submitted to:
Gandalf 2019

energy constraints, in two different directions. In both cases, the lower bound remains unchanged, as it corresponds to running out of energy, which we always want to avoid. We thus only relax the upper-bound constraint. The first direction concerns *weak upper bounds*, already introduced in [3]: in that setting, hitting the upper bound is allowed, but there is no overload, i.e. trying to exceed the upper bound will simply maintain the energy level at this maximal level. Yet, a strict lower bound is still imposed. We name these objectives LW-energy objectives. This could be used as a (simplified) model for batteries. When considered alone, LW-energy objectives are not much different from L-energy objectives, in the sense that the aim is to find a reachable *positive loop*. LW-energy games are in PTIME for one-player games, and in NP $\cap$ coNP for two players. When combining LW-energy and reachability objectives, the situation changes: different loops may have different effects on the energy level, and we have to keep track of the final energy level reached when iterating those loops.

The second way of relaxing upper bounds, which we call *soft upper bound*, allows a limited number (or amount) of violations of the soft bound (possibly within an additional strong upper-bound): when modeling a pressure tank, the lower-bound constraint is strict (pressure should always be available) but the upper bound is soft (excessive pressure may be temporarily allowed if needed). We consider different kinds of limits (on the number or amount of violations), and prove that deciding whether Player 1 has a strategy to keep violations below a given bound is PSPACE-complete for one-player arenas, and EXPTIME-complete for two-player ones. We also provide algorithms to optimize violations of the soft upper bound under a given strict upper bound.

Related work. Quantitative games have been the focus of numerous research articles since the 1970s, with various kinds of objectives, such as ultimately optimizing the total payoff, mean-payoff [16, 29], or discounted sum [29, 1]. Energy objectives, which are a kind of safety objectives on the total payoff, were introduced in [8] and rediscovered in [3]. Several works have extended those works by combining quantitative conditions together, e.g. multi-dimensional energy conditions [18, 25] or conjunctions of energy- and mean-payoff objectives [11]. Combinations with qualitative objectives (e.g. reachability [10] or parity objectives [9, 12]) were also considered. Similar objectives have been considered in slightly different settings e.g. Vector Addition Systems with States [27] and one-counter machines [20, 21].

2 Preliminaries

Definition 1 *A two-player arena is a* 3*-tuple* $G = (Q_1, Q_2, E)$ *where* $Q = Q_1 \uplus Q_2$ *is a set of states,* $E \subseteq Q \times \mathbb{Z} \times Q$ *is a set of weighted edges. For* $q \in Q$*, we let* $qE = \{(q, w, q') \in E \mid w \in \mathbb{Z},\ q' \in Q\}$*, which we assume is non-empty for any* $q \in Q$*. A one-player arena is a two-player arena where* $Q_2 = \emptyset$*.*

Consider a state $q_0 \in Q$. A *finite path* in an arena G from an initial state q_0 is an finite sequence of edges $\pi = (e_i)_{0 \leq i < n}$ such that for every $0 \leq i < n$, writing $e_i = (q_i, w_i, q'_i)$, it holds $q'_i = q_{i+1}$. Fix a path $\pi = (e_i)_{0 \leq i < n}$. Using the notations above, we write $|\pi|$ for its size n of π, $\hat{\pi}_i$ for the i-th state q_i of π (with the convention that $q_n = q'_{n-1}$), and $\mathsf{first}(\pi) = \hat{\pi}_0$ for its first state and $\mathsf{last}(\pi) = \hat{\pi}_n$ for its last state. The empty path is a special finite path from q_0; its length is zero, and q_0 is both its first and last state. Given two finite paths $\pi = (e_i)_{0 \leq i < n}$ and $\pi' = (e'_j)_{0 \leq j \leq n'}$ such that $\mathsf{last}(\pi_1) = \mathsf{first}(\pi_2$, the concatenation $\pi_1 \cdot \pi_2$ is the finite path $(f_k)_{0 \leq k < n+n'}$ such that $f_k = e_k$ if $0 \leq k < n$ and $f_k = e'_{k-n}$ if $n \leq k < n + n'$. For $0 \leq k \leq n$, the k-th prefix of π is the finite path $\pi_{<k} = (e_i)_{0 \leq i < k}$. We write $\mathsf{FPaths}(G, q_0)$ for the set of finite paths in G issued from q_0 (we may omit to mention G in this notation when it is clear from the context). Infinite paths are defined analogously; we write $\mathsf{Paths}(G, q_0)$ for the set of infinite paths from q_0.

A *strategy* for Player 1 (resp. Player 2) from q_0 is a function $\sigma\colon \mathsf{FPaths}(q_0) \to E$ associating with any finite path π with $\mathsf{last}(\pi) \in Q_1$ (resp. $\mathsf{last}(\pi) \in Q_2$) an edge originating from $\mathsf{last}(\pi)$. A strategy is said *memoryless* when $\sigma(\pi) = \sigma(\pi')$ as soon as $\mathsf{last}(\pi) = \mathsf{last}(\pi')$.

A finite path $\pi = (e_i)_{0\leq i<n}$ *conforms* to a strategy σ of Player 1 (resp. of Player 2) from q_0 if $\mathsf{first}(\pi) = q_0$ and for every $0 \leq k < n$, either $e_k = \sigma(\pi_{<k})$, or $\mathsf{last}(\pi_{<k}) \in Q_2$ (resp. $\mathsf{last}(\pi_{<k}) \in Q_1$). This is extended to infinite paths in the natural way. Given a strategy σ of Player 1 (resp. of Player 2) from q_0, the outcomes of σ is the set of all paths π issued from q_0 that conform to σ.

A game is a triple (G, q_{init}, O) where G is a two-player arena, q_{init} is an initial state in Q, and $O \subseteq \mathsf{Paths}(G, q_{init})$ is a set of infinite paths (for Player 1), also called *objective*. A strategy for Player 1 from q_{init} is winning in (G, q_{init}, H) if its infinite outcomes all belong to O.

Given a set $R \subseteq Q$ of states, the reachability objectives defined by R is the set of all paths containing some state in R, while the safety objectives defined by R is the set of all infinite paths never visiting any state in R. In this paper, we also focus on *energy objectives* [8, 3], which we now define.

Definition 2 *Fix a finite-state arena $G = (Q_1, Q_2, E)$. Let $L \in \mathbb{Z}$. The L-energy arena associated with G is the infinite arena $G_{\mathsf{L}} = (C_1, C_2, T)$ where $C_1 = \{q_{err}\} \cup Q_1 \times [L;+\infty)$ and $C_2 = Q_2 \times [L;+\infty)$ are sets of* configurations*, and $T \subseteq C_1 \times \mathbb{Z} \times C_2$ such that*

- *for any (q,l) and (q',l') in $Q \times [L;+\infty)$ and any $w \in \mathbb{Z}$, we have $((q,l), w, (q',l')) \in T$ if, and only if, $(q,w,q') \in E$ and $l' = l + w$; We also impose a loop $(q_{err}, 0, q_{err}) \in T$.*
- *for any $(q,l) \in Q \times [L;+\infty)$, we have $((q,l), w, q_{err}) \in T$ if, and only if, there is a transition $(q,w,q') \in E$ such that $l + w < L$*

Similarly, given $L \in \mathbb{Z}$ and $U \in \mathbb{Z}$, the LU-energy arena associated with G is the finite-state arena $G_{\mathsf{LU}} = (C_1, C_2, T)$ where $C_1 = \{q_{err}\} \cup Q_1 \times [L;U]$ and $C_2 = Q_2 \times [L;U]$, and $T \subseteq C_1 \times \mathbb{Z} \times C_2$ such that

- *for any (q,l) and (q',l') in $Q \times [L;U]$ and any $w \in \mathbb{Z}$, we have $((q,l), w, (q',l')) \in T$ if, and only if, $(q,w,q') \in E$ and $l' = l + w$; We also impose a loop $(q_{err}, 0, q_{err}) \in T$.*
- *for any $(q,l) \in Q \times [L;U]$, we have $((q,l), w, q_{err}) \in T$ if, and only if, there is a transition $(q,w,q') \in E$ such that $l + w < L$ or $l + w > U$;*

Finally, given $L \in \mathbb{Z}$ and $W \in \mathbb{Z}$, the LW-energy arena associated with G is the finite-state arena $G_{\mathsf{LW}} = (C_1, C_2, T)$ where $C_1 = \{q_{err}\} \cup Q_1 \times [L;W]$ and $C_2 = Q_2 \times [L;W]$, and $T \subseteq C_1 \times \mathbb{Z} \times C_2$ such that

- *for any (q,l) and (q',l') in $Q \times [L;W]$ and any $w \in \mathbb{Z}$, we have $((q,l), w, (q',l')) \in T$ if, and only if, $(q,w,q') \in E$ and $l' = \min(W, l + w)$; We also impose a loop $(q_{err}, 0, q_{err}) \in T$.*
- *for any $(q,l) \in Q \times [L;W]$, we have $((q,l), w, q_{err}) \in T$ if, and only if, there is a transition $(q,w,q') \in E$ such that $l + w < L$;*

An L-run (resp. LU-run, LW-run) ρ in G from q with initial energy level l is a path in G_{L} (resp G_{LU}, G_{LW}) from (q,l) never visiting q_{err}. With such a run $\rho = (t_i)_i$ in G, writing $t_i = ((q_i,l_i), w_i, (q'_i,l'_i))$, we associate the path $\pi = (e_i)_i$ such that $e_i = (q_i, w_i, q'_i)$. We define $\hat{\rho}_i = (q_i,l_i)$, corresponding to the i-th configuration along ρ, and $\widetilde{\rho}_i = l_i$, which we name the energy level in that configuration.

Similarly, a path π is said L-feasible (resp. LU-feasible, LW-feasible) from initial energy level L if there exists an L-run (resp. LU-run, LW-run) from $(\mathsf{first}(\pi), L)$ whose associated path is π. Notice that if such a run exists, it is unique.

The L-energy (resp. LU-energy, LW-energy) objective is the set of infinite paths that are L-feasible (resp. LU-feasible, LW-feasible) (from initial energy level L). Similarly, given a target set $R \subseteq Q$, the L-energy- (resp. LU-energy-, LW-energy-) reachability objective is the set of L-feasible (resp. LU-feasible, LW-feasible) paths visiting R.

Remark 3 *Taking L as the initial energy level results in no loss of generality, since any energy level can be obtained by adding an initial transition from (q_0, L).*

In many cases, strong upper bounds are too strict, as many system do not break as soon as their maximal energy level is reached. Imposing a *weak* upper bound is a way to relax these constraints. We introduce another way to relax energy constraints, by allowing for (limited) violations of the upper bound: given two strict bounds L and U in $\mathbb{Z}$, a soft upper bound S, a threshold $V \in \mathbb{Z}$, and an LU-run ρ, the set of violations along ρ is the set $\mathsf{V}(\rho) = \{i \in [0;|\rho|] \mid \widetilde{\rho}_i > S\}$ of positions along ρ where the energy level exceeds the soft upper bound S. There are many ways to quantify violations along a run. We consider three of them in this paper, namely the total number of violations, the maximal number of consecutive violations, and the sum of the violations. We thus define the following three quantities: $\#\mathsf{V}(\rho) = |\mathsf{V}(\rho)|$, $\overline{\#}\mathsf{V}(\rho) = \max\{i - j \mid \forall k \in [i, j].\ k \in \mathsf{V}(\rho)\}$, and $\Sigma\mathsf{V}(\rho) = \sum_{i \in \mathsf{V}(\rho)} (\widetilde{\rho}_i - U)$.

Figure 1 shows the evolution of $\#\mathsf{V}$ along a winning run in an $\mathsf{LSU}^{\#}$-energy game. One can notice that with a strong upper bound of 3, state q_t would not be reachable. On the other hand, if the strong upper bound is set to 6, then then there exists a run from q_0 to q_T, but that requires 3 violations of soft upper bound $S = 3$ (and the total amount of violations is 6).

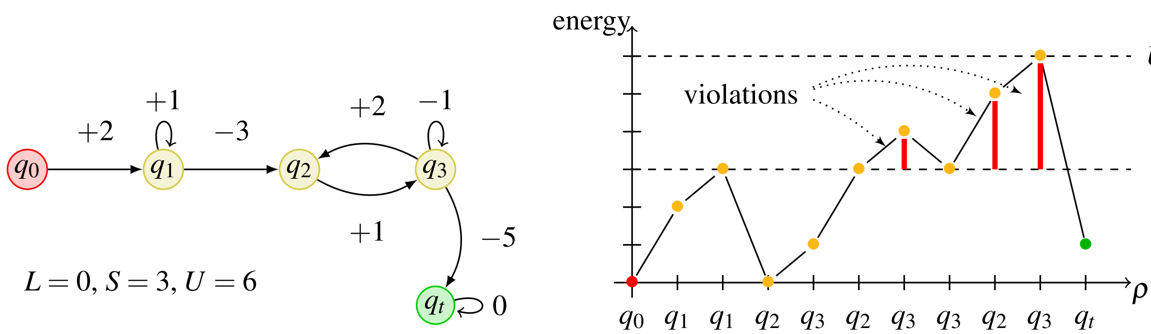

Figure 1: Energy level and $\#\mathsf{V}$ along a winning run in a $\mathsf{LSU}^{\#}$-energy reachability game.

Given three values $L \leq S \leq U$, the $\mathsf{LSU}^{\#}$-energy (resp. $\mathsf{LSU}^{\overline{\#}}$-energy, LSU^{Σ}-energy) objective is the set of LU-feasible infinite paths π such that, along their associated runs ρ from (q_{init}, L), the number $\#\mathsf{V}(\rho)$ of violations (resp. maximal number of consecutive violations $\overline{\#}\mathsf{V}(\rho)$, sum $\Sigma\mathsf{V}(\rho)$ of violations) of the soft upper bound S is at most V. Similarly, for a set of states R, the $\mathsf{LSU}^{\#}$-energy (resp. $\mathsf{LSU}^{\overline{\#}}$-energy, LSU^{Σ}-energy) reachability objective is the set of LU-feasible paths π reaching R such that along their associated run from (q_{init}, L), the number $\#\mathsf{V}(\rho)$ of violations (resp. maximal number of consecutive violations $\overline{\#}\mathsf{V}(\rho)$, sum $\Sigma\mathsf{V}(\rho)$ of violations) of the upper bound U is at most V.

We study the complexity of deciding the existence of a winning strategy for the objectives defined above, in both the one- and two-player settings. Further, for $\mathsf{LSU}^{\star}$-energy games, we also address the following problems:

- **bound existence:** Given L, S and V, decide if there exists a value $U \in \mathbb{Z}$ such that Player 1 wins the $\mathsf{LSU}^{\star}$-energy game;
- **minimization:** Given L and S, and a bound $V_{\max}$, find a value $U \in \mathbb{Z}$ such that Player 1 wins the game with the least possible violations less than $V_{\max}$, if any.

Table 1 summarizes known results, and the results obtained in this paper (where $\mathsf{LSU}^{\star}$-energy gathers all three energy constraints with violations). We furthermore show that the minimization problem for

LSU*-energy (reachability) games require algorithms that run in PSPACE in the one-player case, and in EXPTIME in the two-players case.

	Reachability		Infinite runs	
	1 player	2 players	1 player	2 players
L-energy	in PTIME (thm. 4) [10]	in NP ∩ coNP (thm. 4) [10]	PTIME [3]	in NP ∩ coNP [3]
LU-energy	PSPACE-c. (thm. 6)	EXPTIME-c. (thm. 6)	PSPACE-c. [3]	EXPTIME-c. [3]
LW-energy	in PTIME (thm. 21)	in coNP (thm. 25)	in PTIME [3]	in NP ∩ coNP [3]
LSU*-energy	PSPACE-c. (thm. 26)	EXPTIME-c. (thm. 26)	PSPACE-c. (thm. 26)	EXPTIME-c. (thm. 26)
Bound existence	PSPACE-c. (thm. 27)	EXPTIME-c. (thm. 27)	PSPACE-c. (thm. 27)	EXPTIME-c. (thm. 27)

Table 1: Summary of our results

3 Energy reachability games with strict bounds

In this section, we focus on the L-energy-reachability and LU-energy-reachability problems. We first prove that L-energy reachability problems are inter-reducible with L-energy problems, which entails:

Theorem 4 *Two-player L-energy-reachability games are decidable in NP ∩ coNP. The one-player version is in PTIME.*

Remark 5 *It is worth noticing that these results are not a direct consequence of the results of [9] about energy parity games: in that paper, the authors focus on the* existence of an initial energy level *for which Player 1 has a winning strategy with energy-parity objectives (which encompass our energy-reachability objectives). When the answer is positive, they can compute the minimal initial energy level for which a winning strategy exists, but the (deterministic) algorithm runs in exponential time.*

Proof. These results were already proven in [10]: for one-player arenas, the authors develop a PTIME algorithm, while they prove LOGSPACE-equivalence with L-energy games for the two-player setting (the result then follow from [3]). Our proof (detailed in Appendix) uses similar arguments as in the latter proof, but with the same full and direct reductions back and forth both for the one- and two-player cases.

Intuitively, take an L-energy game; Player 1 can win by either growing the energy level arbitrarily high, or by taking cycles of total weight zero. By adding a small[1] positive ε to all weights on transitions, we can assume that only the first case occurs (see Fig. 2); we then allow Player 1 to take a transition (with large negative weight) to a target state: she will be able to reach that state if, and only if, she indeed can maintain the energy level above the lower bound in the original game.

The converse proof is similar: this time, all weights of a given L-energy-reachability game are decreased by a small ε, and the target state is equiped with a self-loop of weight zero (and it is assumed that Player 1 can reach her target state from all states). The decrease in the weights is compensated by adding 1 unit to the initial energy level (see Fig. 3). Obviously, if Player 1 wins in the original game (with reachability objective), then she also wins in the game where the weights have been decreased (assuming ε is small enough). Conversely, if she wins in the latter game, then either she manages to reach the target, and can mimic the same behaviour in the original game, or she can maintain the energy level above L along infinite runs, which means ther she can make the energy level grow arbitrarily high in the original game, and then play her attractor strategy to reach the target. □

[1]The value of ε will generally be a rational, but it can be made integer by scaling up all constants.

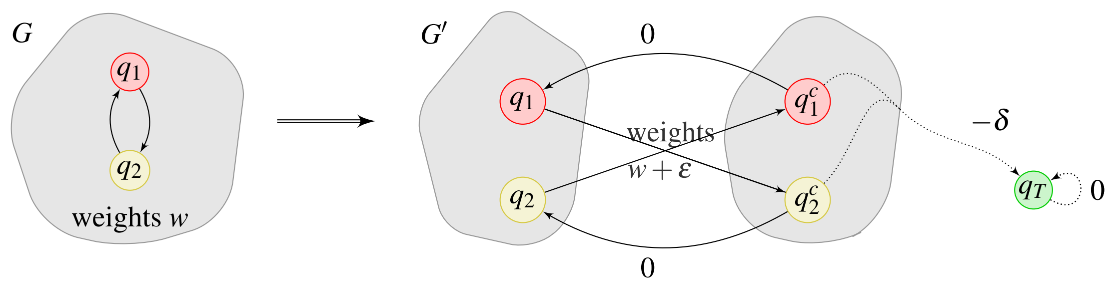

Figure 2: Schema of the reduction from L-energy to L-energy-reachability objectives

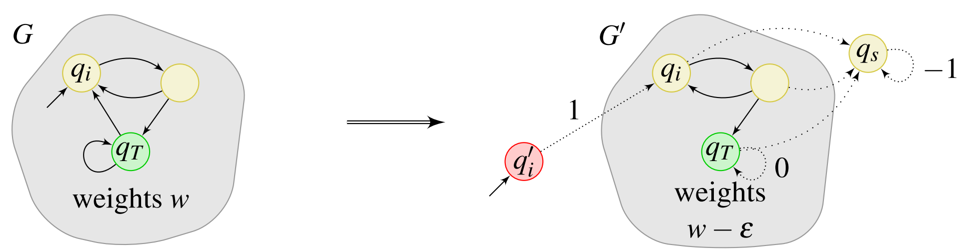

Figure 3: Schema of the reduction from L-energy-reachability to L-energy objectives (assuming Player 1 can reach the target state from all states in G)

Similarly, for LU-energy-reachability objectives, we prove the same complexities as with classical LU-energy objectives:

Theorem 6 *One-player* LU*-energy-reachability games are* PSPACE*-complete. Two-player* LU*-energy-reachability games are* EXPTIME*-complete.*

Proof. Membership in EXPTIME is proven by considering the *expanded game* G_{LU}: it can be used to check reachability for both the one- and the two-player cases. For the one-player case, this can be achieved by proceeding on-the-fly, without explicitly building the expanded game; the resulting algorithm runs in PSPACE. For the two-player case, we solve reachability in that exponential-size game, which results in an EXPTIME algorithm.

For both the one- and the two-player settings, the hardness proofs for LU-energy objectives are readily adapted to LU-energy-reachability objectives, since they are based on reachability-like problems (reachability in bounded one-counter automata [19] and countdown games [23], respectively). □

4 Energy reachability games with weak upper bound

Finding a strategy that satisfies an LW-energy constraint along an infinite run is conceptually easy: it suffices to find a cycle that can be iterated once with a positive effect. It follows that memoryless strategies are enough, and the LW-energy problem was shown to be in PTIME for one-player arenas, and in NP ∩ coNP for two-player arenas [3].

The situation is different when we have a reachability condition: players may have to keep track of the exact energy level in order to find their way to the target state. Obviously, considering the expanded arena G_{LW}, we easily get exponential-time algorithms for LW-energy reachability objectives. As we prove below, in the one-player case, a PTIME algorithm exists.

Example 7 *Consider the one-player arena of Fig. 4, where the lower bound is $L = 0$ and the weak-upper bound is $W = 5$, and the target state is q_T. Starting with initial credit 0, we first have to move to q_1, and then iterate the positive cycle $\beta_1 = (q_1, +2, q_2) \cdot (q_2, -2, q3) \cdot (q_3, +1, q_1)$ three times, ending up in q_1 with energy level 3. We then take the cycle $\beta_2 = (q_1, +2, q_2) \cdot (q_2, -5, q4) \cdot (q_4, +5, q_1)$, which raises the energy level to 5 when we come back to q_1, so that we can reach q_T. Notice that β_1 has to be repeated three times before taking cycle β_2, and that repeating β_1 more than three times maintains the energy level at 4, which is not sufficient to reach q_T. This shows that Player 1 needs memory and cannot rely on a single cycle to win LW-energy reachability games.*

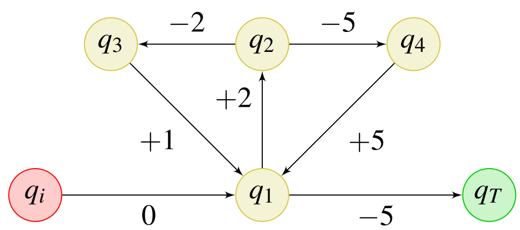

Figure 4: A one-player arena with LW-energy reachability objective

This example shows that winning strategies for Player 1 may have to monitor the exact energy level all along the computation, thereby requiring exponential memory (assuming that all constants are encoded in binary; with unary encoding, the expanded game G_{LW} would have polynomial size and directly give a polynomial-time algorithm).

Proposition 8 *In LW-energy-reachability games, exponential memory may be necessary for Player 1 (assuming binary-encoded constants).*

In order to get a polynomial-time algorithm also with binary-encoded constants, we analyze cycles in the graph, and prove that a path witnessing LW-energy reachability can have a special form, which can be represented compactly using polynomial size. We begin with a series of simple lemmas.

Lemma 9 *Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{\mathsf{LW}} (q',u')$, then for any $v \geq u$, $(q,v) \xrightarrow{\pi}_{\mathsf{LW}} (q',v')$ for some $v' \geq u'$.*

Notice that, even if we add condition $u' > u$ in the hypotheses of Lemma 9, it need not be the case that $v' > v$. In other terms, a sequence of transitions may have a positive effect on the energy level from some configuration, and a negative effect from some other configuration, due to the weak upper bound. Below, we prove a series of results related to this issue, and that will be useful for the rest of the proof.

Lemma 10 *Let π be a finite path in a one-player arena G, and consider two LW-runs $(q,u) \xrightarrow{\pi}_{\mathsf{LW}} (q',u')$ and $(q,v) \xrightarrow{\pi}_{\mathsf{LW}} (q',v')$ with $u \leq v$. Then $u' - u \geq v' - v$, and if the inequality is strict, then the energy level along the run $(q,v) \xrightarrow{\pi}_{\mathsf{LW}} (q',v')$ must have hit W.*

Lemma 11 *Let π be a finite path in a one-player arena G, for which there is an LW-runs $(q,u) \xrightarrow{\pi}_{\mathsf{LW}} (q',u')$. If u' is the maximal energy level along that run, then $(q,W) \xrightarrow{\pi}_{\mathsf{LW}} (q',W)$; if u is the maximal energy level along the run above, then $(q,W) \xrightarrow{\pi}_{\mathsf{LW}} (q',W+u'-u)$.*

From Lemma 9, it follows that any run witnessing LW-energy reachability can be assumed to contain no cycle with nonpositive effect. Formally:

Lemma 12 *Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ and π can be decomposed as $\pi_1 \cdot \pi_2 \cdot \pi_3$ in such a way that $(q,u) \xrightarrow{\pi_1}_{LW} (s,v) \xrightarrow{\pi_2}_{LW} (s,v') \xrightarrow{\pi_3}_{LW} (q',u')$ with $v' \leq v$, then $(q,u) \xrightarrow{\pi_1 \cdot \pi_3}_{LW} (q',u'')$ with $u'' \geq u'$.*

The following lemmas show that several occurrences of a cycle having positive effect along a path can be gathered together. This will be useful to prove the existence of a short witnessing path witnessing LW-energy reachability.

Lemma 13 *Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ with $u' > u$ and $(q,w) \xrightarrow{\pi}_{LW} (q',w')$ with $w' > w$, then for any $u \leq v \leq w$, it holds $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ with $v' > v$.*

Lemma 14 *Let π be a cycle on q such that $(q,u) \xrightarrow{\pi}_{LW} (q,v)$ for some $u \leq v$. Then $(q,u) \xrightarrow{\pi^{W-L}}_{LW} (q,v')$ for some v', and $(q,v') \xrightarrow{\pi}_{LW} (q,v')$.*

Fix a path π in G, and assume that some cycle φ appears (at least) twice along π: the first time from some configuration (q,u) to some configuration (q,u'), and the second time from (q,v) to (q,v'). First, we may assume that φ has length at most $|Q|$, since otherwise we can take an inner subcycle. We may also assume that $v > u'$, as otherwise we can apply Lemma 12 to get rid of the resulting nonpositive cycle between (q,u') and (q,v). For the same reason we may assume $u' > u$ and $v' > v$. As a consequence, by Lemma 13, by repeatedly iterating φ from (q,u), we eventually reach some configuration (q,w) with $w \geq v'$, from which we can follow the suffix of π after the second occurrence of φ. It follows that all occurrences of φ along π can be grouped together, and we can restrict our attention to runs of the form $\alpha_1 \cdot \varphi_1^{n_1} \cdot \alpha_2 \cdot \varphi_2^{n_2} \cdots \varphi_k^{n_k} \cdot \alpha_{k+1}$ where the cycles φ_j are distinct and have size at most $|Q|$, and the finite runs α_j are acyclic. Notice that by Lemma 14, we may assume $n_j = W - L$ for all j.

While this allows us to only consider paths of a special form, this does not provide *short* witnesses, since there may be exponentially many cycles of length less than or equal to $|Q|$, and the witnessing run may need to iterate several cycles looping on the same state (see Example 7). In order to circumvent this problem, we have to show that all cycles need not be considered, and that one can compute the "useful" cycles efficiently. For this, we introduce *universal* cycles, which are cycles that can be iterated from any initial energy level (above L).

Definition 15 *A* universal cycle *on q is a cycle φ with $\mathsf{first}(\varphi) = \mathsf{last}(\varphi) = q$ such that $(q,L) \xrightarrow{\varphi}_{LW} (q, v_{\varphi,L})$ for some $v_{\varphi,L}$. A universal cycle is* positive *if $v_{\varphi,L} > L$*

When a cycle φ is iterated $W-L$ times in a row, then some universal cycle σ is also iterated $W-L-1$ times (by considering the state with minimal energy level along φ). As a consequence, iterating only universal cycles is enough: we may now only look for runs of the form $\beta_1 \cdot \sigma_1^{n_1} \cdot \beta_2 \cdot \sigma_2^{n_2} \cdots \sigma_k^{n_k} \cdot \beta_{k+1}$ where σ_j's are *universal* cycles of length at most $|Q|$. Now, assume that some state q admits two universal cycles σ and σ', and that both cycles appear along a given run π. Write e (resp. e') for the energy levels reached after iterating σ (reap. σ') $W-L$ times. We define an order on universal cycles of q by letting $\sigma \triangleright \sigma'$ when $e > e'$. Then if $\sigma \triangleright \sigma'$, each occurrence of σ' along π can be replaced with σ, yielding a run π' that still satisfies the LW-energy condition (and has the same first and last states). Generalizing this argument, each state that admits universal cycles has an optimal universal cycle of length at most $|Q|$, and it is enough to iterate only this universal cycle to find a path witnessing reachability. This provides us with a *small witness*, of the form $\gamma_1 \cdot \tau_1^{W-L} \cdot \gamma_2 \cdot \tau_2^{W-L} \cdots \tau_k^{W-L} \cdot \gamma_{k+1}$ where τ_j are optimal universal cycles of length at most $|Q|$ and γ_j are acyclic paths. Since it suffices to consider at most one universal cycle per state, we have $k \leq |Q|$. From this, we immediately derive an NP algorithm for solving LW-energy reachability for one-player arenas: it suffices to non-deterministically select each portion of the path, and compute that each portion is LW-feasible (notice that there is no need for checking universality nor optimality of cycles; those properties were only used to prove that small witnesses exist). Checking LW-feasibility requires computing the final energy level reached after iterating a cycle $W-L$ times; this can be performed by detecting the highest energy level along that cycle, and computing how much the energy level decreases from that point on until the end of the cycle.

We now prove that optimal universal cycles of length at most $|Q|$ can be computed for a given state q_0. For this we unwind the graph from q as a DAG of depth $|Q|$, so that it includes all cycles of length at most $|Q|$. We name the states of this DAG $[q',d]$, where q' is the name of a state of the arena and d is the depth of this state in the DAG (using square brackets to avoid confusion with configurations (q,l) where l is the energy level); hence there are transitions $([q',d],w,[q'',d+1])$ in the DAG as soon as there is a transition (q',w,q'') in the arena.

We then explore this DAG from its initial state $[q_0,0]$, looking for (paths corresponding to) universal cycles. Our aim is to keep track of all runs from $[q_0,0]$ to $[q',d]$ that are prefixes of universal cycles starting from q_0. Actually, we do not need to keep track of those runs explicitly, and it suffices for each such run to remember the following two values:

- the maximal energy level M that has been observed along the run so far (starting from energy level L, with weak upper bound W);
- the difference m between the maximal energy level M and the final energy level in $[q',d]$. Notice that $m \geq 0$, and that the final energy level in $[q',d]$ is $M-m$.

Example 16 *Figure 5 shows two universal cycles from q_0 in an LW-energy game with $L=0$ and $W=5$. The first cycle, going via q_2, ends with $M_1=5$ (reached in q_2) and $m_1=4$, thus with a final energy level of* 1 *(when starting from energy level* 0*); actually, iterating this cycle will not improve this final energy level. The second cycle, via q_4 and q_5, has a maximal energy level $M_2=4$ (reached in q_3) and ends with $m_2=1$. Hence, after one iteration of this cycle, one can end in state q_0 with energy level $W-m_2=4$.*

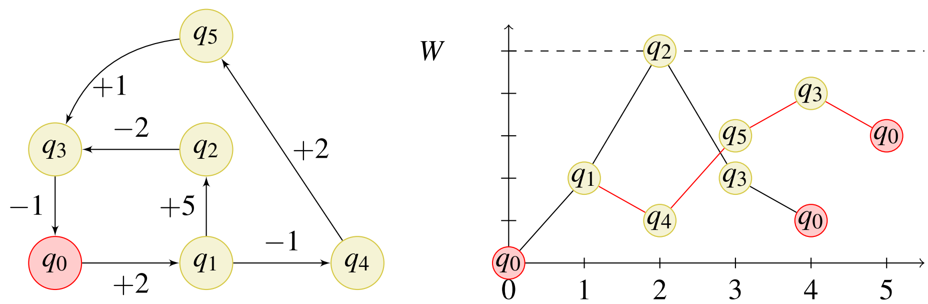

Figure 5: Two cycles with upper bound $W=5$

If we know the values (M,m) of some path from $[q_0,0])$ to $[q',d]$, we can decide if a given transition with weight w from $[q',d]$ to $[q'',d+1]$ can be taken (the resulting path can still be a prefix of a universal cycle if $M-m+w \geq L$), and how the values of M and m have to be updated: if $w>m$, the run will reach a new maximal energy level, and the new pair of values is $(\min(W;M-m+w),0)$; if $m+L-M \leq w \leq m$, then the transition can be taken: the new energy level $M-m+w$ will remain between L and M, and we update the pair of values to $(M,m-w)$; finally, if $w<m+L-M$, the energy level would go below L, and the resulting run would not be a prefix of a universal cycle.

Following these ideas, we inductively attach labels to the states of the DAG: initially, $[q_0,0]$ is labelled with $(M=L, m=0)$; then if a state $[q',d]$ is labelled with (M,m), and if there is a transition from $[q',d]$ to $[q'',d+1]$ with weight w:

- if $w>m$, then we label $[q'',d+1]$ with the pair $(\max(W; M-m+w), 0)$;
- if $m+L-M \leq w \leq m$, we label $[q'',d+1]$ with $(M, m-w)$.

The following lemma makes a link between runs in the DAG and labels computed by our algorithm:

Lemma 17 *Let $[q,d]$ be a state of the DAG, and M and m be two integers such that $0 \leq m \leq M$. Upon termination of this algorithm, state $[q,d]$ of the DAG is labelled with (M,m) if, and only if, there is an LW-run of length d from (q_0,L) to $(q,M-m)$ along which the energy level always remains in the interval $[L,M]$.*

Lemma 18 *Let $[q_0,d]$ be a state of the DAG, with $d>0$. Let m be a nonnegative integer such that $L+m<W$. Upon termination of this algorithm, state $[q_0,d]$ is labelled with (M,m) for some $M>L+m$ if, and only if, there is a universal cycle φ on q_0 of length d such that $(q_0,L) \xrightarrow{\varphi^{W-L}}_{\mathsf{LW}} (q_0,W-m)$.*

Proof. First assume that $[q_0,d]$ is labelled with (M,m) for some M such that $M-m>L$. From Lemma 17, there is a cycle φ on q_0 of length d generating a run $(q_0,L) \xrightarrow{\varphi}_{\mathsf{LW}} (q_0,M-m)$ along which the energy level is within $[L,M]$. Then $M-m \geq L$, so that Lemma 14 applies: we then get $(q_0,L) \xrightarrow{\varphi^{W-L}}_{\mathsf{LW}} (q_0,E)$ with $(q_0,E) \xrightarrow{\varphi}_{\mathsf{LW}} (q_0,E)$. Write $(p_i)_{0\leq i<|\varphi|}$ for the sequence of weights along φ. Also write ρ for the run $(q_0,L) \xrightarrow{\varphi}_{\mathsf{LW}} (q_0,M-m)$, and σ for the run $(q_0,E) \xrightarrow{\varphi}_{\mathsf{LW}} (q_0,E)$.

As $L<M-m$, then by Lemma 10, it must be the case that energy level W is reached along σ. Write i_0 and j_0 for the first and last positions along ρ for which the energy level along ρ is M. That is, the subpath from index 0 to index i_0 has growing energy level, and the subpath from j_0 to $|\rho|$ has a decreasing energy level. Assume $\widetilde{\sigma}_{i_0} \neq W$: by Lemma 9, we must have $M=\widetilde{\rho}_{i_0} \leq \widetilde{\sigma}_{i_0} < W$. Then for all $k \geq i_0$, $\sum_{l=i_0}^{k} p_l \leq 0$, and $\sum_{l=i_0}^{j_0} p_l = 0$. Since $\widetilde{\sigma}_{i_0} < W$, then also $\widetilde{\sigma}_k < W$ for all $k \geq i_0$. According to Lemma 10, energy level W is reached in σ, so there exists some $k_0 < i_0$ such that $\widetilde{\sigma}_{k_0} = W$. However, as i_0 is the index of the first maximal value in ρ, we have $\widetilde{\rho}_{k_0} < M$, and the energy level increases in run ρ between k_0 and i_0. So according to Lemma 11, we should have $\widetilde{\sigma}_{i_0} < W$, which raises a contradiction. Hence we proved $\widetilde{\sigma}_{i_0} = W$; applying the second result of Lemma 11, we get $E=W-m$.

Conversely, if there is a universal cycle φ satisfying the conditions of the lemma, then it must have positive effect when run from energy level L. Let F be such that $(q_0,L) \xrightarrow{\varphi}_{\mathsf{LW}} (q_0,F)$, and M be the the maximal energy level encountered along the run $(q_0,L) \xrightarrow{\varphi}_{\mathsf{LW}} (q_0,F)$. By Lemma 17, state $[q_0,d]$ is labelled with (M,m') for some $m' \geq 0$ such that $F=M-m'$. By Lemma 14, we must have $(q_0,L) \xrightarrow{\varphi^{W-L}}_{\mathsf{LW}} (q_0,W-m')$. $\square$

The algorithm above computes optimal universal cycles, but it still runs in exponential time (in the worst case) since it may generate exponentially many different labels in each state $[q,d]$ (one per path from $[q_0,0]$ to $[q,d]$). We now explain how to only generate polynomially-many pairs (M,m). This is based on the following partial order on labels: we let $(M,m) \preceq (M',m')$ whenever $M-m \leq M'-m'$ and $m' \leq m$. Notice in particular that

- if $M=M'$, then $(M,m) \preceq (M',m')$ if, and only if, $m' \leq m$;
- if $m=m'$, then $(M,m) \preceq (M',m')$ if, and only if, $M \leq M'$.

The following lemma entails that it suffices to store maximal labels:

Lemma 19 *Consider two paths π and π' such that $\mathsf{first}(\pi) = \mathsf{first}(\pi')$ and $\mathsf{last}(\pi) = \mathsf{last}(\pi')$, and with respective values (M,m) and (M',m') such that $(M,m) \preceq (M',m')$. If π is a prefix of a universal cycle φ, then π' is a prefix of a universal cycle φ' with $\varphi' \triangleright \varphi$.*

It remains to prove that by keeping only maximal labels, we only store a polynomial number of labels:

Lemma 20 *If the labelling of the DAG only keeps maximal labels (for $\preceq$), then it runs in polynomial time.*

Proof. We prove that, when attaching to each node $[q,d]$ of the DAG only the maximal labels (w.r.t $\preceq$) reached for a path of length d ending in state q, the number of values for the first component of the different labels that appear at depth $d > 0$ in the DAG is at most $d \cdot |Q|$. Since it only stores optimal labels, our algorithm will never associate to a state $[q,d]$ two labels having the same value on their first component. So, any state at depth d will have at most $d \cdot |Q|$ labels.

So we prove, by induction on d, that the number of different values for the first component among the labels appearing at depth $d > 0$ is at most $d \cdot |Q|$. This is true for $d = 1$ since the initial state $(q,0)$ only contains $(M = 0, m = 0)$, and each transition with nonnegative weight w will create one new label $(w,0)$ (transitions with negative weight are not prefixes of universal cycles). Now, since all those labels have value 0 as their second component, each state $[q,1]$ in the DAG will be attached at most one label. Hence, the total number of labels (and the total number of different values for their first component) is at most $|Q|$ at depth 1 in the DAG.

Now, assume that the labels appearing at depth $d > 1$ are all drawn from a set of labels $L = \{M_i, m_i) \mid 1 \le i \le n\}$ in which the number of different values of M_i is at most $d \cdot |Q|$. Consider a state $[q',d]$, labelled with $\{(M_i,m_i) \mid 1 \le i \le n_{q',d}\}$ (even if it means reindexing the labels). Pick a transition from $[q',d]$ to $[q'',d+1]$, with weight w. For each pair (M_i,m_i) associated with $[q,d]$, it creates a new label in $(q'',d+1)$; this label is

- either $(M_i - m_i + w, 0)$ if $m_i < w$;
- or $(M_i, m_i + w)$ if $m_i = M_i \le w \le m_i$.

Now, for a state $(q'',d+1)$, the set of labels created by all incoming transitions can be grouped as follows:

- labels having zero as their second component; among those, our algorithm only stores the one with maximal first component, as $(M_i,0) \preceq (M_j,0)$ as soon as $M_i \le M_j$;
- for each M_i appearing at depth d, labels having M_i as their first component; again, we only keep the one with minimal second component, as $(M,m_i) \preceq (M,m_j)$ when $m_j \le m_i$.

In the end, for this state $[q'',d+1]$, we keep at most one label for each distinct value among the first components M_i of labels appearing at depth d, and possibly one extra label with second value 0. In other terms, at depth $d+1$ the values that appear as first component of labels are obtained from values at depth d, plus possibly one per state; Hence, at depth $d+1$, there exists at most $(d+1) \cdot |Q|$ labels, which completes the proof of the induction step. ☐

Using the algorithm above, we can compute, for each state q of the original arena, the smallest value m_q for which there exists a universal cycle on q that, when iterated sufficiently many times, leads to configuration $(q, W - m_q)$. Since universal cycles can be iterated from any energy level, if q is reachable, then it is reachable with energy level $W - m_q$. We make this explicit by adding to our arena a special self-loop on q, labelled with $\mathsf{set}(W - m_q)$, which sets the energy level to $W - m_q$ (in the same way as *recharge transitions* of [17]).

In the resulting arena, we know that we can restrict to paths of the form $\gamma_1 \cdot \nu_1 \cdot \gamma_2 \cdot \nu_2 \cdots \nu_k \cdot \gamma_{k+1}$, where ν_i are newly added transitions labelled with $\mathsf{set}(W-m)$, and γ_i are acyclic paths. Such paths have length at most $(|Q|+1)^2$. We can then inductively compute the maximal energy level that can be reached (under our LW-energy constraint) in any state after paths of length less than or equal to $(|Q|+1)^2$. This can be performed by unwinding (as a DAG) the modified arena from the source state q_{init} up to depth $(|Q|+1)^2$, and labelling the states of this DAG by the maximal energy level with which that state can be reached from (q_{init}, L); this is achieved in a way similar to our algorithm for computing the effect of universal cycles, but this time only keeping the maximal energy level that can be reached (under LW-energy constraint). As there are at most $|Q|$ states per level in this DAG of depth at most $(|Q|+1)^2$, we can establish the following theorem:

Theorem 21 *The existence of a winning path in one-player* LW*-energy-reachability games can be decided in* PTIME.

Example 22 *Consider the one-player arena of Fig. 6. We assume $L = 0$, and fix an even weak upper bound W. The state s has $W/2$ disjoint cycles: for each odd integer i in $[0;W-1]$, the cycle c_i is made of three consecutive edges with weights $-i$, $+W$ and $-W+i+1$. Similarly, the state s' has $W/2$ disjoint cycles: for even integers i in $[0;W-1]$, the cycle c'_i has weights $-i$, $+W$ and $-W+i+1$. Finally, there are: two sequences of k edges of weight 0 from s to s' and from s' to s; an edge from the initial state to s of weight 1, and from s' to target state t of weight $-W$. The total number of states then is $2W+2k+2$.*

In order to go from the initial state, with energy level 0 to the final state, we have to first take the cycle c_1 (with weights -1, $+W$, $-W+2$) on s (no other cycles c_i can be taken). We then reach configuration $(s,2)$. Iterating c_1 has no effect, and the only next interesting cycle is c_2, for which we have to go to s'. After running c_2, we end up in $(s',3)$. Again, iterating c_2 has no effect, and we go back to s, take c_3, and so on. We have to take each cycle c_i (at least) once, and take the sequences of k edges between s and s' $W/2$ times each. In the end, we have a run of length $3W+Wk+2$.

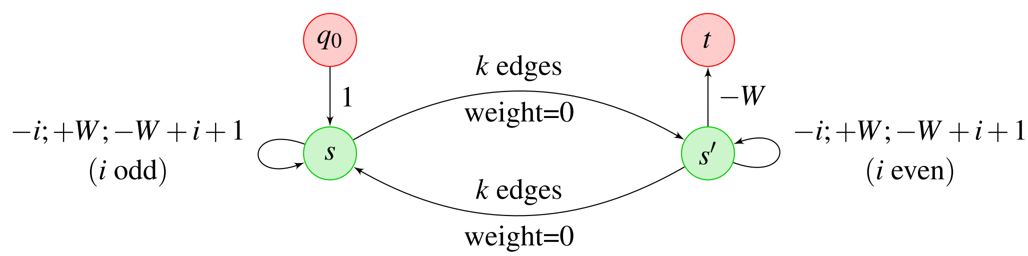

Figure 6: An example showing that more than one cycle per state can be needed.

Let us look at the universal cycles that we have in this arena: besides the cycles made of the $2k$ edges with weight zero between s and s', the only possible universal cycles can only depart from the first state of each cycle c_i (as there are the only states having a positive outgoing edge). As we proved, such cycles can be iterated arbitrarily many times, and set the energy level to some value in $[L;w]$. Since the only edge available at the end of a universal cycle has weight $+W$, the exact value of the universal cycles is unimportant: the energy level will be W anyway when reaching the second state of each cycle c_i. As a consequence, using set*-edges in this example does not shorten the witnessing run, which then cannot be shorter than $3W+Wk+2$ (which is more than $2|Q|$ but less than $|Q|^2$). This demonstrates that we cannot avoid looking for quadratic-size runs in the modified arena at the end of our algorithm.*

We now move to the two-player setting. We begin with proving a result similar to Lemma 9:

Lemma 23 *Let G be a two-player arena, equipped with an LW-energy-reachability objective. Let q be a state of G, and $u \leq u'$ in $[L;W]$. If Player 1 wins the game from (q,u), then she also wins from (q,u').*

By Martin's theorem [26], our games are determined. It follows that if Player 2 wins from some configuration (q,v), she also wins from (q,v') for all $L \leq v' \leq v$ (assuming the contrary, i.e. (q,v') winning for Player 1, would lead to the contradictory statement that (q,v) is both winning for Player 1 and Player 2). Using classical techniques [9], we prove that Player 2 can be restricted to play memoryless strategies:

Proposition 24 *For two-player LW-energy-reachability games, memoryless strategies are sufficient for Player 2.*

A direct consequence of this result and of Theorem 21 is the following:

Corollary 25 *Two-player LW-energy reachability games are in* coNP.

5 Energy reachability games with soft upper bound

We now consider games with limited violations, i.e. (reachability) games with $\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\overline{\#}}$-energy and LSU^{Σ}-energy objectives. We address the problems of deciding the winner in the one-player and two-player settings, and consider the existence and minimizations questions.

Theorem 26 *$\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\overline{\#}}$-energy and LSU^{Σ}-energy (reachability) games are* PSPACE*-complete for one-player arenas, and* EXPTIME*-complete for two-player arenas.*

Proof. Membership in PSPACE and EXPTIME can be obtained by building a variant G_{LSU} of the G_{LU} arena: besides storing the energy level in each state, we can also store the amount of violations (for any of the three measures we consider). More precisely, given an arena G, lower and upper bounds L and U on the energy level, a soft bound S, and a bound V on the measure of violations, we define a new arena[2] G_{LSU} with set of states $(Q \times ([L;U] \cup \{\perp\}) \times ([0;V] \cup \{\perp\})^3)$, and each transition (q,w,q') of the original arena generates a transition from state $(q,l,(n,c,s))$ to state $(q',l',(n',c',s'))$ whenever

- l' correctly encodes the evolution of the energy level: if l and $l+w$ are in $[L;U]$, then $l' = l+w$; If the energy level leaves interval $[L;U]$ ($l+w < L$ or $l+w > U$) or has formerly leaved interval $[L;U]$ (in this case $l = \perp$), then $l' = \perp$.
- n' correctly stores the number of violations: $n' = \perp$ if $l' \in (S;U]$ and $n+1 > V$ (the number of violations allowed is exceeded); Once the number of violations is exceeded ($n = \perp$) or the maximal energy level is exceeded ($l' = \perp$), we have $n' = \perp$; Last, $n' = n$ if $l' \in [L;S]$ (the current state does not violate bound S), and $n' = n+1$ if $l' \in (S;U]$ and $n+1 \leq V$ (the current state is a additional violation of bound S);

We can similarly update c' to count the current number of consecutive violations, and s' for the sum of all violations. We refer interested readers to Appendix for the complete construction. In this arena, values n, c and s keep tack of the number of violations, number of consecutive violations and sum of violations; their values range in $[0,V]$ and are set to $\perp$ as soon as they exceed bound V, or if the energy level has exceeded its bounds. The arena G_{LSU} has size exponential, and $\mathsf{LSU}^{\star}$-energy-reachability problems can then be reduced to solving reachability of corresponding sets of states in G_{LSU}. The announced complexity

[2] In order to factor our proof, we store all three measures of violations in one single arena, event if only one measure per type of LSU-energy game is needed.

results follow. Hardness results are obtained by a straightforward encoding of LU-energy reachability problems, taking $S = U$ and $V = 0$.

Solving $\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\overline{\#}}$-energy, LSU^{Σ}-energy games (without reachability objective) can be performed with arena G_{LSU} built above. Now, the objective in $\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\overline{\#}}$-energy, LSU^{Σ}-energy games is to enforce infinite runs, that avoid states with $l = \perp$ and with $n = \perp$, $c = \perp$ or $s = \perp$, depending on the chosen measure of violations. Our result follows. □

When the strong upper bound U is not given, the existence problem consists in deciding if such a bound exist under which Player 1 wins the $\mathsf{LSU}^{\star}$-energy game. We have:

Theorem 27 *The existence problems for $\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\overline{\#}}$-energy, and LSU^{Σ}-energy (reachability) games are* PSPACE *complete for the one-player case and* EXPTIME*-complete for the two-player case.*

Proof. Along any outcome of a winning strategy, the energy level remains below $S + V \cdot w_{\max}$, where $w_{\max}$ is the maximal weight appearing on transitions of G. This gives a strong upper bound U, with which we can apply the construction above and check the existence of a winning strategy for Player 1. □

Theorem 28 *Let G be an arena, L and S be integer bounds, and $V_{\max}$ be an integer. There exist algorithms that compute the value of U (if any) that minimizes the value of V (below $V_{\max}$) for which Player 1 has a winning strategy in a $\mathsf{LSU}^{\star}$-energy (reachability) game. These algorithm runs in* PSPACE *for one-player games and in* EXPTIME *for two-player games. These bounds are sharp.*

Proof. We perform a binary search for an optimal value for V when the strict upper energy bound U varies between S and $S + V_{\max}.w_{\max}$. For each value U, we discard from G_{LSU} transitions for which the energy level exceeds U. One can remark that when U grows, the minimal amount of violation may decrease, both in reachability and infinite-run games. We can hence discover optimal values with a polynomial number of PSPACE checks (for the one-player games), and EXPTIME checks in the two-player case. □

6 Conclusion

This paper has considered several variants of energy games. The first variant defines games with upper and lower bound constraints, combined with reachability or infinite runs objectives. The second variant proposed defines games with strong lower lower an upper bound that can be temporarily exceeded, reachability or infinite run objectives, and constraints on violations of upper bound. In the one player case, complexities ranges from PTIME to PSPACE-complete. and in the two-player case from $\mathsf{NP} \cap co\mathsf{NP}$ to EXPTIME-complete. In general, the complexity is the same for a reachability and for an infinite run objective. Interestingly, for LW-energy games, the complexity of the single player case is PTIME, but reachability objectives require exponential memory (in the size of the weak upper bound) while strategies are memoryless for infinite run objectives.

A possible extension of this work is to consider energy games with mean-payoff functions and discounted total payoff, both for the energy level and for the violation constraints, and the associated minimization and existence problems.

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A Proofs of Section 3

Theorem 4 *Two-player* L*-energy-reachability games are decidable in* $\mathsf{NP} \cap \mathsf{coNP}$*. The one-player version is in* PTIME*.*

Proof. We prove that L-energy-reachability and L-energy games are interreducible. The theorem then follows from the results of [3].

First consider a two-player arena $G = (Q_1, Q_2, E)$, an initial state q_{init}, and an L-energy objective. We define a new arena $G' = (Q_1 \cup Q_c \cup \{q_T\}, Q_2, E')$ (assuming $q_T \notin Q$) where $Q_c = \{q_c \mid q \in Q\}$ is a copy of all the vertices of G. Note that, q_c is always a Player 1 vertex; intuitively, states in Q_c are used to allow Player 1 to stop the game and reach the target state q_T, if enough energy has been stored. The set of transitions E' is obtained from E as follows (where the (positive) rational[3] value of ε will be fixed later):

- for each $(q, w, q') \in E$, there is a transition $(q, w+\varepsilon, q'_c)$ and $(q'_c, 0, q')$ in E';
- for each $q_c \in Q_c$, there is a transition $(q_c, -\delta, q_T)$ in E', where $\delta = 1 + \sum_{(q,w,q') \in E} |w|$;
- finally, E' contains an edge $(q_T, 0, q_T)$.

We claim that Player 1 has a winning strategy from q_0 for the L-energy-reachability objective in G' if, and only if, she has a winning strategy from q_0 for the L-energy objective in G.

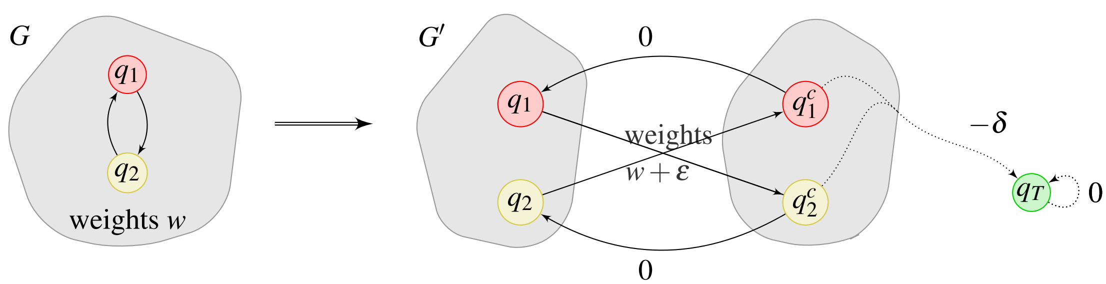

Figure 7: Schema of the reduction from L-energy to L-energy-reachability objectives

First assume that Player 1 has a winning strategy σ in G for the L-energy objective; then we can assume that this strategy is memoryless [3]; we define the strategy σ' as follows: for any state q of G, letting $q' = \sigma(q)$, we define $\sigma'(\pi \cdot q) = q'_c$, and

$$\sigma'(\pi \cdot q \cdot q'_c) = \begin{cases} q' & \text{if } |\pi| \leq \frac{\delta}{2\varepsilon} - 1 \\ q_T & \text{otherwise.} \end{cases}$$

Obviously, any outcome μ' of σ' from q_0 reaches q_T. First note that, by construction of σ', the prefix ν' of μ' just before reaching q_T has odd length, say of length $2n-1$. Also note that it corresponds to an outcome ν of σ in G of length n. Since σ is assumed winning, ν must be L-feasible; moreover, we have

$$\widetilde{\nu}'_{2i} = \widetilde{\nu}'_{2i-1} = \widetilde{\nu}_i + i \cdot \varepsilon.$$

for all $0 \leq i < n$. Now, $\widetilde{\nu}_i \geq L$ for all i, since ν is an outcome of σ, so that also $\widetilde{\nu}'_i \geq L$ for all i. Moreover, $|\nu|' = \delta/2\varepsilon - 1$ implies that, $|\nu| = \delta/\varepsilon$, so that $\widetilde{\nu}'_{2n-1} \geq L + \delta$, and $\widetilde{\mu}'_{2n} \geq L$. It follows that σ' is winning in G' for the L-energy-reachability objective.

[3]Our definition of arenas do not allow for rational weights, but by scaling up all constants (including the energy bounds), we get an equivalent instance of our problem with only integer bounds.

Conversely, assume that Player 1 wins the L-energy-reachability game G', and write σ' for a winning strategy in G' from q_0. We may assume that no negative cycle occurs along any outcome of σ': indeed, consider the (finite) execution tree of σ', and assume that it involves a negative cycle starting and ending at some state q; then there must exists a subtree rooted at q which contains no other occurrences of q; by redefining σ' so as to play as in this subtree after any occurrence of q, we remove all occurrences of our negative cycle, while preserving reachability of q_T and still satisfying the energy constraint (since removing negative cycles may only increase the energy level).

Now, take any outcome ρ' of σ' from q_0, it must eventually reach q_T. First note that, any prefix of ρ' looks like $q_0 {q_1}^c q_1 \ldots q_T$. Hence, if we take any prefix π' of ρ' before reaching q_T and drop the alternate vertices, we get a corresponding path in G. Now, as ρ' eventually reaches q_T and since the edge leading to q_T has weight $-\delta$, a positive cycle must have been visited along ρ' in G'. From σ', we can then build a strategy σ that, intuitively, repeats the first positive cycle it visits(after dropping the alternate vertices). Formally, $\sigma(\pi.q) = q'$ if $\sigma'(\pi', q) = q'_c$ where π is obtained dropping alternate vertices from π' and π' contains no positive cycle. When π is a run of the form $\pi = \rho_1.\beta_1 \ldots \beta_{k-1}.\rho_k$, where each β_i is a positive cycle, we take $\sigma(\rho_1.\beta_1) = \sigma(\rho_1)$. The resulting strategy σ then never takes the edge to q_T, since it only plays moves returned by σ' along outcomes that do not contain positive cycles. Moreover, all simple cycles generated by σ in G' are positive cycles; by taking $\varepsilon < \frac{1}{|Q|+1}$, these cycles still are positive cycles in G, so that σ is winning in G for the L-energy objective.

We now prove the converse reduction, which relies on similar ideas: we consider a two-player arena $G = (Q_1, Q_2, E)$, an initial state q_{init}, and an L-energy-reachability objective; we assume without loss of generality that there is a unique target state q_T, and write $Attr_1(q_T)$ for the Player 1-attractor of q_T in G. We build (in polynomial time) a two-player arena $G' = (Q'_1, Q'_2, E')$ from G as follows:

- $Q'_1 = (Q_1 \cap Attr_1(q_T)) \cup \{q'_{init}, q_s\}$ and $Q'_2 = Q_2$. State q_{init} will serve as the new initial state, and q_s is a sink state that guarantees existence of an infinite run;
- letting $E_0 = \{(q, w - \varepsilon, q') \mid (q, w, q') \in E \text{ and } q \in Q'_1 \cup Q'_2 \setminus \{q_T\}\} \cup \{(q_T, 0, q_T), (q_s, -1, q_s)\} \cup \{(q'_{init}, q) \mid q \in \{q_{init}\} \cap Attr_1(q_T)\}$, we define $E' = E_0 \cup \{(q, 0, q_s) \mid qE_0 = \emptyset\}$. This way, all states have an outgoing edge, possibly to the sink state if no other transitions exist.

Again, the exact value of ε will be fixed below. We prove that Player 1 wins the L-energy-reachability game in G from q_{init} if, and only if, she wins the L-energy game in G' from q'_{init}.

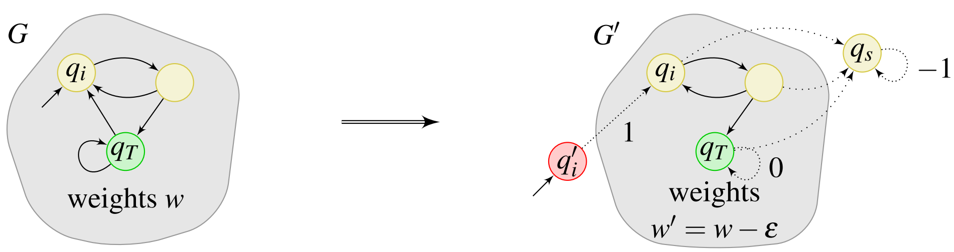

Figure 8: Schema of the reduction from L-energy-reachability to L-energy objectives

For the first direction, if Player 1 has a winning strategy to reach q_T from q_0 in G while maintaining the energy level above L, then she has such a strategy σ along whose outcomes the energy level is bounded above by $L + 2\delta$ (where $\delta = 1 + \sum_{(q,w,q') \in E} |w|$): indeed, if energy level $L + \delta$ is reached along some outcome, then Player 1 can achieve the reachability objective by playing her memoryless *attractor strategy*. Choosing the attractor strategy ensures reaching q_T, and will decrease the energy level by at most δ along any outcome. Similarly, following the attractor strategy can increase the energy level by no more than δ. Similarly, strategy σ can be assumed to yield no negative cycles, so that we can bound the length of the outcomes by $(\delta+1)\cdot|Q|$. Now, by taking $\varepsilon < \frac{1}{(\delta+1)\cdot|Q|}$, we can mimic strategy σ in G': all outcomes only visits states in the attractor of q_T, and reach q_T in at most $(\delta+1)\cdot|Q|+1$ steps (the extra step is the transition from q'_{init} to q_{init}). The ε difference in the weights is compensated by the initial credit 1 harvested when moving from q'_{init} to q_{init}, so that all outcomes satisfy the L-energy constraint.

Conversely, if Player 1 has a winning strategy σ' from q'_{init} in G', then we can assume that this strategy is memoryless [3]. Some of the outcomes may reach q_T, some may not. Since σ' is memoryless, it cannot take any negative cycle, as this would yield an outcome whose energy level tends to $-\infty$. Hence it may only take nonnegative cycles in G', which correspond to positive cycles in G (since $\varepsilon > 0$). As a consequence, when mimicking σ' in G, for those outcomes that do not reach q_T, the energy level will grow arbitrarily high; when it exceeds δ, Player 1 can play her attractor-strategy to reach q_T. This concludes our proof for two-player games. □

B Proofs of Section 4

Lemma 9 *Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$, then for any $v \geq u$, $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ for some $v' \geq u'$.*

Proof. Write $\pi = (e_i)_{0\leq i<n}$, with $e_i = (q_i, p_i, q'_i)$ for each i. The sequence defined as

$$u_0 = u \qquad\qquad u_{i+1} = \min W, u_i + p_i$$

is the sequence of energy levels along the run $(q,u) \xrightarrow{\pi}_{\mathsf{LW}} (q',u')$. For $v \geq u$, letting

$$v_0 = v \qquad\qquad v_{i+1} = \min W, v_i + p_i,$$

we easily prove by induction that for all i, $u_i \leq v_i \leq W$, which entails that $(q,v) \xrightarrow{\pi}_{\mathsf{LW}} (q',v')$ with $v' = v_n \geq u_n = u'$. □

Lemma 10 *Let π be a finite path in a one-player arena G, and consider two LW-runs $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ and $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ with $u \leq v$. Then $u'-u \geq v'-v$, and if the inequality is strict, then the energy level along the run $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ must have hit W.*

Proof. The first statement is proven by induction: we again write $\pi = (e_i)_{0\leq i<n}$, with $e_i = (q_i, p_i, q'_i)$ for each i, and

$$u_0 = u \qquad\qquad u_{i+1} = \min(W, u_i + p_i)$$
$$v_0 = v \qquad\qquad v_{i+1} = \min(W, v_i + p_i).$$

Then $u_{i+1} - u_i = \min(W - u_i, p_i)$ and $v_{i+1} - v_i = \min(W - v_i, p_i)$. Since $u_i \leq v_i$ for all i, we also have $W - u_i \geq W - v_i$, and $u_{i+1} - u_i \geq v_{i+1} - v_i$. By summing up these inequalities, we get $u_{i+1} - u_0 \geq v_{i+1} - v_0$. Now, as long as $W - v_i \geq p_i$ (then also $W - u_i \geq p_i$), the inequalities above are equalities. It follows that if the inequality is strict, then the run $(q,v) \xrightarrow{\pi}_{\mathsf{LW}} (q',v')$ must have hit W. □

Lemma 11 *Let π be a finite path in a one-player arena G, for which there is an LW-runs $(q,u) \xrightarrow{\pi}_{LW} (q',u')$. If u' is the maximal energy level along that run, then $(q,W) \xrightarrow{\pi}_{LW} (q',W)$; if u is the maximal energy level along the run above, then $(q,W) \xrightarrow{\pi}_{LW} (q',W+u'-u)$.*

Proof. Write $\pi = (e_i)_{0\leq i<n}$, with $e_i = (q_i, p_i, q_i')$ for each i. If u' is the maximal energy level, then for all i, we have $\sum_{j=i}^{n-1} p_j \geq 0$. Now, define

$$v_0 = W \qquad\qquad v_{i+1} = \min(W, v_i + p_i).$$

If $v_n < W$, then by induction we also have $v_i < W$ for all i, contradicting the fact that $v_0 = W$. This proves our first result.

Similarly, if u is the maximal energy level, then for all i, we have $\sum_{j=0}^{i} p_j \leq 0$. Then for all i, $v_{i+1} = v_i + p_i \leq W$, so that $v_n - v_0 = u' - u$. Our second result follows. □

Lemma 12 *Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ and π can be decomposed as $\pi_1 \cdot \pi_2 \cdot \pi_3$ in such a way that $(q,u) \xrightarrow{\pi_1}_{LW} (s,v) \xrightarrow{\pi_2}_{LW} (s,v') \xrightarrow{\pi_3}_{LW} (q',u')$ with $v' \leq v$, then $(q,u) \xrightarrow{\pi_1\cdot\pi_3}_{LW} (q',u'')$ with $u'' \geq u'$.*

Proof. Since $(s,v') \xrightarrow{\pi_3}_{\mathsf{LW}} (q',u')$ and $v' \leq v$, by Lemma 9 we also have $(s,v) \xrightarrow{\pi_3}_{\mathsf{LW}} (q',u'')$ for some $u'' \geq u'$. The result follows. □

Lemma 13 *Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ with $u' > u$ and $(q,w) \xrightarrow{\pi}_{LW} (q',w')$ with $w' > w$, then for any $u \leq v \leq w$, it holds $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ with $v' > v$.*

Proof. Using Lemma 9, we immediately have $(q,v) \xrightarrow{\pi}_{\mathsf{LW}} (q',v')$. As in the previous proof, we define sequences

$$u_0 = u \qquad\qquad u_{i+1} = \min(W, u_i + p_i)$$
$$v_0 = v \qquad\qquad v_{i+1} = \min(W, v_i + p_i)$$
$$w_0 = w \qquad\qquad w_{i+1} = \min(W, w_i + p_i).$$

We still have $u_i \leq v_i \leq w_i$ for all i. Moreover, if $v_j < W$ for all $j \leq i$, then $v_i - u_i = v - u$. As a consequence, if $v' \leq v$, then it must be the case that $v_j = W$ for some j; but then $w_j = v_j$, since $v_j \leq w_j \leq W$. It follows that $w_k = v_k$ for all $k \geq j$, so at the end of π we have $w' = v'$. Assuming $v' \leq v$ raises a contradiction since we have $v' = w' > w \geq v$. Hence $v' > v$. □

Lemma 14 *Let π be a cycle on q such that $(q,u) \xrightarrow{\pi}_{LW} (q,v)$ for some $u \leq v$. Then $(q,u) \xrightarrow{\pi^{W-L}}_{LW} (q,v')$ for some v', and $(q,v') \xrightarrow{\pi}_{LW} (q,v')$.*

Proof. The case where $u = v$ is trivial. We assume $u < v$. Applying Lemma 9 inductively, we get that the cycle can be iterated arbitrarily many times; this also proves that the sequence of energy levels reached at the end of each iteration is non-decreasing.

Now, assume that $(q,v') \xrightarrow{\pi}_{\mathsf{LW}} (q,v'')$ for some $v'' \neq v'$. Then $v'' > v'$. Lemma 13 then entails that the sequence of energy levels reached at the end of each iteration is increasing. Since the loop has been iterated $W - L$ times, the energy level in v'' would exceed W, which is impossible. This proves our result.

□

Lemma 17 *Let $[q,d]$ be a state of the DAG, and M and m be two integers such that $0 \leq m \leq M$. Upon termination of this algorithm, state $[q,d]$ of the DAG is labelled with (M,m) if, and only if, there is an LW-run of length d from (q_0,L) to $(q,M-m)$ along which the energy level always remains in the interval $[L,M]$.*

Proof. The proof is by induction on d. The result is trivial for $d = 0$. Now, assume it holds for some depth $d-1$, and pick a state $[q,d]$. For the first direction, if $[q,d]$ is labelled with (M,m), then this label was added using some transition $([q',d-1],w,[q,d])$ and some label (M',m') of $[q',d-1]$. By induction, there is an LW-run ρ of length $d-1$ from (q_0,L) to $(q',M'-m')$ in G along which the energy level remains in the interval $[L,M']$. We consider two cases, corresponding to the two ways of updating the pair of values:

- if $w > m'$, then we have $M = \min(W, M'-m'+w)$ and $m = 0$. Now, the transition $([q',d-1],w,[q,d])$ in the DAG originates from a transition (q',w,q) in G; taking this transition after ρ provides us with the run of length d from (q_0,L) to $(q,M-m)$ along which the energy level remains in $[L,M]$, as required;
- if $m'+L-M' \leq w \leq m'$, then $M = M'$ and $m = m'-w$. Again, taking transition (q',w,q) after ρ provides us with the LW-run we are looking for.

Conversely, if there is an LW-run ρ of length d from (q_0,L) to $(q,M-m)$ along which the energy level always remains in the interval $[L,M]$, then we write $\rho = \rho' \cdot ((q',l'),w,(q,M-m))$, distinguishing its last transition. By induction, $[q',d-1]$ must have been labelled with a pair (M',m') such that $l' = M'-m'$ and the energy level along ρ' remained within $[L,M']$. Now, from the existence of a transition $((q',l'),w,(q,M-m))$, we know that there is a transition $([q',d-1],w,[q,d])$ in the DAG, which will generate the required label of $[q,d]$. □

Lemma 19 *Consider two paths π and π' such that $\mathsf{first}(\pi) = \mathsf{first}(\pi')$ and $\mathsf{last}(\pi) = \mathsf{last}(\pi')$, and with respective values (M,m) and (M',m') such that $(M,m) \preceq (M',m')$. If π is a prefix of a universal cycle φ, then π' is a prefix of a universal cycle φ' with $\varphi' \triangleright \varphi$.*

Proof. Let $q = \mathsf{first}(\pi)$ and $q' = \mathsf{last}(\pi)$. We write ψ for the path such that $\varphi = \pi \cdot \psi$; ψ is a path from q' to q. Then $(q,L) \xrightarrow{\pi}_{\mathsf{LW}} (q'',M-m) \xrightarrow{\psi}_{\mathsf{LW}} (q',F)$. Also, $(q,L) \xrightarrow{\pi}_{\mathsf{LW}} (q'',M'-m')$. Since $M-m \leq M'-m'$, we have $(q'',M'-m') \xrightarrow{\psi}_{\mathsf{LW}} (q',F')$. We can thus let $\varphi' = \pi' \cdot \psi$: by Lemma 18, the final energy level reached after iterating φ' is higher than the energy level reached after iterating φ, since $m' \leq m$. Hence $\varphi' \triangleright \varphi$. □

Lemma 23 *Let G be a two-player arena, equipped with an LW-energy-reachability objective. Let q be a state of G, and $u \leq u'$ in $[L;W]$. If Player 1 wins the game from (q,u), then she also wins from (q,u').*

Proof. Let σ be a winning strategy for Player 1 from (q,u). If she plays the same strategy from (q,u'), then for any strategy of Player 2, the resulting outcome from (q,u') follows the same transitions as the outcome of the same strategies from u, with higher energy level. Since σ is winning from (q,u), it is also winning from (q,u'). □

Proposition 24 *For two-player LW-energy-reachability games, memoryless strategies are sufficient for Player 2.*

Proof. According to Lemma 23, for each state q, there is an integer $v_q \in [L;W+1]$ such that Player 1 wins the game from any configuration (q,v) satisfying $v_q \leq v \leq W$, while Player 2 wins the game from any configuration (q,v) with $L \leq v < v_q$.

Assume that Player 2 wins the game from some state (q,v), with $L \leq v \leq v_q$. Denote with $(q,p_i,q_i)_{1\leq i\leq m}$ for the set of outgoing transitions from q. By definition of v_{q_i}, Player 1 wins the game from any configuration of the form (q_i,v) with $v \geq v_{q_i}$. Since Player 2 wins from (q,v), there must exist an index $1 \leq i \leq m$ such that $v + p_i \leq v_{q_i}$. This defines a winning move for Player 2 from (q,v). The same argument applies in all states, and yields a memoryless winning strategy for Player 2. □

C Proofs of Section 5

Theorem 26 *$\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\bar{\#}}$-energy and LSU^{Σ}-energy (reachability) games are* PSPACE*-complete for one-player arenas, and* EXPTIME*-complete for two-player arenas.*

Proof. Membership in PSPACE and EXPTIME can be obtained by building a variant G_{LSU} of the G_{LU} arena: besides storing the energy level in each state, we can also store the amount of violations (for any of the three measures we consider). More precisely, given an arena G, lower and upper bounds L and U on the energy level, a soft bound S, and a bound V on the measure of violations, for any of our three measures of violations, the maximal energy level that can be reached along a path with violations smaller than or equal to V is $S+V\cdot w_{\max}$, where $w_{\max}$ is the maximal weight in our arena. We then define a new arena[4] G_{LSU} with set of states $(Q\times([L;U]\cup\{\bot\})\times([0;V]\cup\{\bot\})^3)$, and each transition (q,w,q') of the original arena generates a transition from state $(q,l,(n,c,s))$ to state $(q',l',(n',c',s'))$ whenever

- l' correctly encodes the evolution of the energy level:
  - $l'=l+w$ if l and $l+w$ are in $[L;U]$;
  - $l'=\bot$ if either $l=\bot$ or $l+w<L$ or $l+w>U$;
- n' correctly stores the number of violations:
  - $n'=\bot$ if $l'=\bot$ or $n=\bot$;
  - $n'=n$ if $l'\in[L;S]$;
  - $n'=n+1$ if $l'\in(S;U]$ and $n+1\le V$;
  - $n'=\bot$ if $l'\in(S;U]$ and $n+1>V$.
- c' is updated to count the current number of consecutive violations:
  - $c'=\bot$ if $l'=\bot$ or $c=\bot$;
  - $c'=0$ if $l'\in[L;S]$;
  - $c'=c+1$ if $l'\in(S;U]$ and $c+1\le V$;
  - $c'=\bot$ if $l'\in(S;U]$ and $c+1>V$.
- s' encodes the sum of all violations:
  - $s'=\bot$ if $l'=\bot$ or $s=\bot$;
  - $s'=s$ if $l'\in[L;S]$;
  - $s'=s+(l'-U)$ if $l'\in(S;U]$ and $s+(l'-S)\le V$;
  - $s'=\bot$ if $l'\in(S;U]$ and $s+(l'-S)>V$.

In this arena, n, c and s keep tack of the number of violations, number of consecutive violations and sum of violations; their values are set fo $\bot$ as soon as they exceed the bound, or if the energy level has exceeded its bounds $[L'U]$. The arena G_{LSU} has size exponential, and our $\mathsf{LSU}^{\star}$-energy-reachability problems can be reduced to solving reachability of the relevant set of states in that arena (e.g., Player 1 wins the $\mathsf{LSU}^{\#}$-energy reachability game if, and only if, she wins in the modified game G_{LSU} for the objective of reaching the target set without visiting states where $n=\bot$).

Hardness results are obtained by setting the number/amount of allowed violations to zero, thereby recovering the classical LU-energy-reachability games, which we proved are PSPACE-complete and EXPTIME-complete for one-player and two-player arenas, respectively.

[4] In order to factor our proof, we store all three measures of violations in one single arena; we actually have four variables because two of them are used to compute and store the maximal number of consecutive violations (which for solving the decision problem is not needed).

Solving $\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\overline{\#}}$-energy, LSU^{Σ}-energy games (without reachability objective) can be performed with arena G_{LSU} built above. Now, the objective in $\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\overline{\#}}$-energy, LSU^{Σ}-energy games is to enforce infinite runs, that avoid states with $l = \perp$ and with $n = \perp, c = \perp, s = \perp$, depending on the chosen criterion on violation. Again, these strategies can be found in PSPACE for the one-player case, and in $\mathsf{EXPTIME}$ in the two-player case. For the hardness part, reduction from LU-energy still works. □

Theorem 27 *The existence problems for $\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\overline{\#}}$-energy, and LSU^{Σ}-energy (reachability) games are* PSPACE *complete for the one-player case and* $\mathsf{EXPTIME}$*-complete for the two-player case.*

Proof. Given an arena G, a lower bound L, a soft upper bound S, and a maximal number of violations V (resp. maximal number of consecutive violations, maximal sum of overloads), we know that the energy level in a winning run cannot exceed $U_{\max} = S + V * w_{max}$. So, we can reuse the expanded arena G_{LSU} of Theorem 26, with strong upper bound on energy level $U_{\max}$. If there is a strategy in a $\mathsf{LSU}^{\#}$-energy reachability game, it will visit only states of G with an energy level smaller than U_{max} and with a number of violations n that cannot exceed V. Hence, in G_{LSU}, this corresponds to a path visiting states of the form $(q, l, (n, c, s))$ in which $l \leq U_{\max}$, and $n \leq V$. Clearly, one can find an acyclic path from $(q_0, L, (0, 0, 0))$ to states of the form $(q_T, l, (n, c, s))$ in PSPACE. Similar reasoning holds for $\mathsf{LSU}^{\overline{\#}}$-energy,$\mathsf{LSU}^{\Sigma}$-energy reachability games, considering the c and s component of states in G_{LSU}.

For the two-player case, an attractor for T in G_{LSU} can be computed in polynomial time (but on an arena of exponential size w.r.t. U_{max} and V). If state $(q_0, L, (0, 0, 0))$ appears in the attractor, then there exists a strategy to reach $q_T \in T$ without exceeding U_{max}, and with a number of violations n is smaller than V. So existence for $\mathsf{LSU}^{\overline{\#}}$-energy in the two players setting is in $\mathsf{EXPTIME}$. Notice that the maximal energy level reached when using a strategy needs not be U_{max}. Similar reasoning holds for $\mathsf{LSU}^{\overline{\#}}$-energy,$\mathsf{LSU}^{\Sigma}$-energy reachability games, considering the c and s component of states in G_{LSU}.

Let us now address the existence question for $\mathsf{LSU}^{\#}$-energy (resp. $\mathsf{LSU}^{\overline{\#}}$-energy,$\mathsf{LSU}^{\Sigma}$-energy) games without reachability objective. Player 1 wins these games iff there exists an infinite path in which the number of violations (resp. the consecutive number of violations, the sum of violations) never exceeds V. As already mentioned, the energy level in these runs cannot exceed $U_{\max}$. So, solving the existence for $\mathsf{LSU}^{\#}$-energy (resp. $\mathsf{LSU}^{\overline{\#}}$-energy,$\mathsf{LSU}^{\Sigma}$-energy) games amounts to finding cycles in G_{LSU}. This can be done in PSPACE for the one player case and in $\mathsf{EXPTIME}$ for the two-player case.

For the hardness part, one can easily transform a LU-energy game into an existence question. For a LU-energy (reachability) game with lower bound L and strong upper bound S, we can build a $\mathsf{LSU}^{\#}$-energy (reachability) game with lower bound L, soft bound S, an arbitrary upper bound $U > S + 1$, and set $V = 0$. Then a winning run in G_{LSU} is a run that forbids any violation in G is also a winning run for the LU-energy game. The reduction works similarly for $\mathsf{LSU}^{\overline{\#}}$-energy and LSU^{Σ}-energy (reachability) games. □

Theorem 28 *Let G be an arena, L and S be integer bounds, and $V_{\max}$ be an integer. There exist algorithms that compute the value of U (if any) that minimizes the value of V (below $V_{\max}$) for which Player 1 has a winning strategy in a $\mathsf{LSU}^{\star}$-energy (reachability) game. These algorithm runs in* PSPACE *for one-player games and in* $\mathsf{EXPTIME}$ *for two-player games. These bounds are sharp.*

Proof. Let us first consider $\mathsf{LSU}^{\#}$-energy reachability games with strong lower bound L and soft bound S. Given U and V, one can check in PSPACE for the single player version whether a solution exists without exceeding energy level U nor bound V, and in $\mathsf{EXPTIME}$ for the two-payer version. This can be done by computing an arena G^{U}_{LSU} that has maximal energy level $\min(U_{\max}, U)$ (where $U_{\max} = S + V_{\max} * w_{\max}$).

To find the value for U that minimizes V, we can first notice that, for $U' > U$, the minimal possible number of violations is smaller in $G^{U'}_{\mathsf{LSU}}$ than in G^{U}_{LSU}, as increasing the maximal value of energy allowed can only add new runs in the arena. One can then perform a binary search for an optimal value for a bound U in $[S, U_{\max}]$. Given a tested value K, one can associate with each state (q, E, v) of Player 1 in the attractor of T the value v (the number of violations) if $E \leq K$, or $+\infty$ otherwise. Similarly, one can associate value v to each state (q, E, v) of Player 2 where $E \leq K$ and whose successors all have an energy level smaller than K, and $+\infty$ otherwise. Then it suffices to find the path from $(q_0, L, (0,0,0))$ to T in the attractor whose number of violations is minimal. This can be done with a slight adaptation of Dijkstra's algorithm [15], where the shortest distance only maintains the maximal number of violations encountered from $(q_0, L, (0,0,0))$ to the current state. This is done in polynomial time in the size of the attractor (which can be as large as arena G_{LSU}.

If the value found is $+\infty$ for a search with energy bound K in the attractor, then there is no way to win $\mathsf{LSU}^{\#}$-energy with maximal number of violations K. Otherwise, one gets the optimal number of violations with this bound K. One can perform a binary search in $[S, U_{\max}]$, to test successive values for K. The number of violations in arena G^{K}_{LSU} decreases as K increases. Consider a particular value K tested in an interval $[K_{\min}, K_{\max}]$. If there are no solutions for $\mathsf{LSU}^{\#}$-energy with bound K then the optimal value is higher than K, i.e. is in $(K, K_{\max}]$. Otherwise, other values for U and V may be found after a test of another bound in $[K_{min}, K)$. At every step, we hence search an optimal value in an interval $[K_{\min}, K_{\max}]$, whose size is divided by 2 at each iteration of the binary search. We can return value $U = K_{\min}$ as soon as the optimal number of violations is the same in $G^{K_{\min}}_{\mathsf{LSU}}$ and $G^{K_{\max}}_{\mathsf{LSU}}$. i.e. after a logarithmic (in $U_{\max} - S$) number of steps. One can hence find optimal values for V and U in an arena in PSPACE for the one-player version of the game and in $\mathsf{EXPTIME}$ for the two-player game.

The minimization problems for $\mathsf{LSU}^{\#}$-energy, $\mathsf{LSU}^{\overline{\#}}$-energy, and LSU^{Σ}-energy games without reachability objectives follow the same lines. Given V and U, one can check existence of a strategy for Player 1 in an $\mathsf{LSU}^{\star}$-energy game in PSPACE for the one player setting, and in $\mathsf{EXPTIME}$ for the two-payer setting. If Player 1 has no winning strategies with bound $U_{\max}$ then she has no winning strategies for larger values. As for reachability games, we can show that the number of violations decreases when U increases. One can hence perform up to $\log(U_{\max} - S)$ test to find the minimal value U such that Player 1 has a winning strategy for some bound U and none for bound $U - 1$. Hence, the minimization question is also in $\mathsf{EXPTIME}$.

The complexity of our algorithm is easily shown to be optimal, since LU-energy (reachability) games can be seen as LSU-energy (reachability) games in which the number of allowed violations is 0. Hence, if we impose $V_{\max} = 0$, there is a single test needed to find an optimal value for the number of violations. If this test is successful, the optimal value for U is then the soft bound S. So, there is a winning strategy in the LSU-energy (reachability) game if and only if the optimal number of violations in the associated LSU-energy (reachability) game is $V_{\max} = 0$, and the associated strong upper bound is $U = S$. $\square$