Entropy stable Godunov numerical schemes for the Euler equations.

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Abstract This work concerns the design of entropy stable numerical schemes for the Euler equations. The fully discrete entropy inequality is reached involving a local condition incorporated in the scheme design. The method yields explicit schemes. Numerical experiments are carried out to assess the performances of the proposed schemes.

1 Introduction

The present work is devoted to the numerical approximation of the weak solutions of the Euler equations in one space dimension given by

$$\begin{aligned}
&\partial_t \rho + \partial_x(\rho u) = 0,\\
&\partial_t \rho u + \partial_x(\rho u^2 + p) = 0,\\
&\partial_t \rho E + \partial_x\big((\rho E + p)u\big) = 0.
\end{aligned}$$

This nonlinear model governs the density $\rho > 0$, the velocity $u \in \mathbb{R}$ and the total energy $\rho E > 0$ of a compressible fluid. The unknown state vector $w = (\rho, \rho u, \rho E)^{\mathrm{T}}$ is assumed to be in the set $\Omega = \left\{(\rho, \rho u, \rho E) \in \mathbb{R}^3 \mid \rho > 0, \rho u \in \mathbb{R}, E - {}^{u^2}\!/_2 > 0\right\}$. The function $p : \Omega \to \mathbb{R}^+_*$ denotes the fluid pressure. In this work, we consider the perfect gas model in which

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$$p(w) = (\gamma - 1)\Big(\rho E - \frac{\rho u^2}{2}\Big), \quad \forall w \in \Omega, \tag{1}$$

where $\gamma \in (1,3]$ stands for the adiabatic constant.

We consider $w_0 : \mathbb{R} \to \Omega$ a given measurable function of $L^1_{\text{loc}}(\mathbb{R})$ as initial condition of w at time $t = 0$ and we study the following Cauchy problem

$$\begin{cases} \partial_t w + \partial_x f(w) = 0, \quad x \in \mathbb{R},\ t > 0, \\ w(x, t = 0) = w_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{2}$$

where $f(w) = \big(\rho u, \rho u^2 + p, (\rho E + p)u\big)^{\mathrm{T}}$.

The Euler system (2) is a nonlinear hyperbolic system of conservation laws [9] and it is well-known that its solutions may develop discontinuities in a finite time. In this case, the uniqueness of the solution is lost [13, 14, 18, 15] and the selection of a solution is done according to entropy inequalities [13, 14, 18] that write

$$\partial_t \eta(w) + \partial_x G(w) \leq 0, \tag{3}$$

where the entropy $\eta : \Omega \to \mathbb{R}$ denotes a convex function and the entropy flux $G : \Omega \to \mathbb{R}$ is such that $\nabla G(w)^{\mathrm{T}} = \nabla \eta(w)^{\mathrm{T}} \nabla f(w)$ for all $w \in \Omega$. According to [6, 2, 13], and for a function $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ such that

$$\mathcal{K}'(y) > 0, \quad \mathcal{K}'(y) - \gamma \mathcal{K}''(y) > 0, \quad \forall y \in \mathbb{R},$$

the couple convex entropy-entropy flux (η, G) for the Euler system (2) are given by

$$\eta(w) = -\rho \mathcal{K}\big(\ln(p/\rho^\gamma)\big), \quad G(w) = -(\rho u)\mathcal{K}\big(\ln(p/\rho^\gamma)\big), \quad \forall w \in \Omega.$$

In the sequel, a single entropy inequality (3) is considered and it is defined by $\mathcal{K}(y) = y$ in the above formulations. As a consequence, the considered couple (η, G) is given by

$$\eta(w) = -\rho \ln(p/\rho^\gamma), \quad G(w) = -(\rho u) \ln(p/\rho^\gamma), \quad \forall w \in \Omega. \tag{4}$$

From a numerical point of view, we consider uniform meshes in space $(x_{i+\frac{1}{2}})_{i\in\mathbb{Z}} \subset \mathbb{R}$ of constant size $\Delta x > 0$. Thus, we have $x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + \Delta x$ for all $i \in \mathbb{Z}$. We also consider uniform meshes in time $(t^n)_{n\in\mathbb{N}} \subset [0, +\infty)$ of constant size $\Delta t > 0$, and we have $t^{n+1} = t^n + \Delta t$ for all n in $\mathbb{N}$. At time t^n, the quantity $w(\cdot, t^n)$ is approximated by a piecewise constant function defined by a sequence $(w_i^n)_{i\in\mathbb{Z}}$ such that

$$w_i^n \approx \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w(x, t^n)\, \mathrm{d}x, \quad \forall i \in \mathbb{Z}.$$

Since the sequence $(w_i^0)_{i\in\mathbb{Z}}$ is given by $w_i^0 = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} w_0(x)\, \mathrm{d}x$, a numerical approximation of $w(\cdot, t^{n+1})$ is entirely defined by a numerical scheme that gives the updated sequence $(w_i^{n+1})_{i\in\mathbb{Z}}$ from the sequence $(w_i^n)_{i\in\mathbb{Z}}$. Such schemes write

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}(\mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}}). \tag{5}$$

However, a suitable updated sequence $(w_i^{n+1})_{i\in\mathbb{Z}}$ has to satisfy a discrete version of the inequality (3). Denoting $\mathcal{G}_{i+1/2}$ a consistent approximation of G defined in (4), these discrete inequalities read

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{\mathcal{G}_{i+\frac{1}{2}} - \mathcal{G}_{i-\frac{1}{2}}}{\Delta x} \leq 0, \quad \forall i \in \mathbb{Z}, \tag{6}$$

and they are of main importance. On the hand, they avoid the scheme convergence toward a non-entropic solution [4, 12, 16, 21] and on the other hand, they drastically reduce spurious oscillations [7].

Several schemes verifying the discrete entropy inequality (6) or a semi-discrete version, have already been introduced [10, 11, 8, 20]. But, Riemann solvers made of two intermediate states and satisfying a fully discrete inequality are rare. They are composed of the HLLC scheme [22], relaxation schemes such as Suliciu relaxation approaches [1, 3, 5] and a scheme introduced in [10] (Section b.ii). However, the proof of the discrete entropy inequality (6) requires technical computations [3] or depends on implicit parameters [10, 22], which can make their implementation complex.

In this work, we propose to design numerical schemes from sufficient conditions that ensure the weak consistency and the discrete entropy inequality (6). The paper is organized as follows. In Section 2, we define consistent, entropic approximate Riemann solvers for the Riemann problem associated to the Euler system (2). In Section 3, we propose to define an approximate Riemann solver with quadratic equations to enforce the sufficient conditions derived in Section 2. These quadratic equations are always well-posed and they give explicit approximate Riemann solvers that are used in Section 4 to define consistent, entropic numerical schemes (5). Finally, in Section 5, numerical tests are carried out to illustrate the performance of the numerical schemes.

2 Two constant states approximate Riemann solver

In this section, we introduce two constant states approximate Riemann solver (ARS). Let consider two constant states $(w_L, w_R) \in \Omega^2$ such that the initial condition w_0 reads

$$w_0(x) = \begin{cases} w_L & \text{if } x < 0, \\ w_R & \text{otherwise,} \end{cases} \tag{7}$$

In this case, the Cauchy problem (2), (7) recasts into Riemann problem. An approximate Riemann solver is a given function $\tilde{w} : \mathbb{R} \times \Omega^2 \to \Omega$ that defines an (L^2)

approximation of the exact solution of the Riemann problem (2), (7). The choice of an approximate Riemann solver has to satisfy some criteria that are now defined.

Definition 1 (Approximate Riemann Solver (ARS) [10])

Assume $\Delta t > 0$ such that the following CFL condition holds:

$$\frac{\Delta t}{\Delta x} \max_{w \in \{w_L, w_R\}} \mu_{\nabla f(w)}(w) \leq \frac{1}{2}, \tag{8}$$

where $\mu_{\nabla f(w)}(w)$ denotes the spectral radius of the Jacobian matrix $\nabla f(w)$. A given function $\tilde{w} : \mathbb{R} \times \Omega^2 \to \Omega$ is an approximate Riemann solver (ARS) if the following statements hold.

i) $\xi \mapsto \tilde{w}(\xi, w_L, w_R) \in L^1_{\text{Loc}}(\mathbb{R})$ for all $(w_L, w_R) \in \Omega^2$.
ii) The function $\tilde{w}$ is consistent that writes $\tilde{w}(\xi, w, w) = w$ for all $(\xi, w) \in \mathbb{R} \times \Omega$.
iii) Under the CFL condition (8), the function $\tilde{w}$ satisfies a consistency integral relation that writes

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}\big(x/\Delta t, w_L, w_R\big)\, \mathrm{d}x = \frac{w_L + w_R}{2} - \frac{\Delta t}{\Delta x}\big(f(w_R) - f(w_L)\big). \tag{9}$$

iv) Under the CFL condition (8), the function $\tilde{w}$ satisfies the following entropy condition

$$\frac{1}{\Delta x} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \eta\big(\tilde{w}(x/\Delta t, w_L, w_R)\big)\, \mathrm{d}x \leq \frac{\eta(w_L) + \eta(w_R)}{2} - \frac{\Delta t}{\Delta x}\big(G(w_R) - G(w_L)\big). \tag{10}$$

The consistency given in *ii)* means that an approximate Riemann solver preserves the constants of Ω that are obvious solutions of the Euler system (2).

Now, considering three given reals λ_L, v^*, λ_R such that $\lambda_L < v^* < \lambda_R$, an approximate Riemann solver $\tilde{w}(\cdot, w_L, w_R) : \mathbb{R} \to \Omega$ made of two intermediate constant states is displayed in Figure 1. Denoting $w_L^* := w_L^*(w_L, w_R)$ and $w_R^* := w_R^*(w_L, w_R)$ the two intermediate states, such an approximate Riemann solver writes

$$\tilde{w}(x/\Delta t, w_L, w_R) = \begin{cases} w_L \text{ if } \dfrac{x}{\Delta t} \leq \lambda_L, \\[2mm] w_L^* \text{ if } \lambda_L < \dfrac{x}{\Delta t} \leq v^*, \\[2mm] w_R^* \text{ if } v^* < \dfrac{x}{\Delta t} \leq \lambda_R, \\[2mm] w_R \text{ if } \lambda_R < \dfrac{x}{\Delta t}. \end{cases} \tag{11}$$

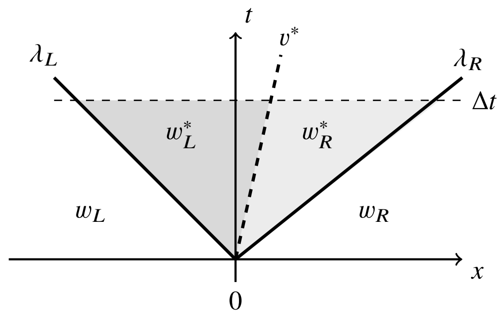


Fig. 1: Approximate Riemann solver made of two intermediate constant states w_L^* and w_R^* in the plane (x, t).

According to [10], the fan of the exact solution must be included within the (x, t)-plane cone defined by λ_L and λ_R (see Figure 1). This restriction, once again called CFL condition, is given by

$$\begin{aligned} &\lambda_L < \min_{\alpha\in\{L,R\}} \left(u_\alpha \pm \sqrt{\gamma p(w_\alpha)/\rho_\alpha}, u_\alpha\right) \leq \max_{\alpha\in\{L,R\}} \left(u_\alpha \pm \sqrt{\gamma p(w_\alpha)/\rho_\alpha}, u_\alpha\right) < \lambda_R, \\ &0 < \max_{\alpha\in\{L,R\}} |\lambda_\alpha| \frac{\Delta t}{\Delta x} \leq \frac{1}{2}. \end{aligned} \tag{12}$$

In addition, a two constant states approximate Riemann solver (11) obviously satisfies Definition 1-*i*).The consistency condition *ii*) will be immediate according to the definition which will be given to w_L^* and w_R^*.

As a consequence, we now have to determine the two intermediate states w_L^* and w_R^* according to the consistency integral relation (9) and to the entropy condition (10). The following lemma presents the satisfied constraints by entropic, consistent intermediate states w_L^* and w_R^*.

Lemma 1 *Consider an approximate Riemann solver* $\tilde{w}(\cdot, w_L, w_R)$ (11) *that approximates the solution of the Riemann problem for the Euler equations given by* (2), (7). *Assume the CFL condition* (12) *holds and let us set*

$$\delta_L = \frac{v^* - \lambda_L}{\lambda_R - \lambda_L} > 0, \quad \delta_R = \frac{\lambda_R - v^*}{\lambda_R - \lambda_L} > 0, \tag{13a}$$

$$w^{\mathrm{HLL}} = (\rho^{\mathrm{HLL}}, (\rho u)^{\mathrm{HLL}}, (\rho E)^{\mathrm{HLL}})^{\mathrm{T}} = \frac{\lambda_R w_R - \lambda_L w_L}{\lambda_R - \lambda_L} - \frac{f(w_R) - f(w_L)}{\lambda_R - \lambda_L} \in \mathbb{R}^3, \tag{13b}$$

$$\eta^{\mathrm{HLL}} = \frac{\lambda_R \eta(w_R) - \lambda_L \eta(w_L)}{\lambda_R - \lambda_L} - \frac{G(w_R) - G(w_L)}{\lambda_R - \lambda_L} \in \mathbb{R}. \tag{13c}$$

There is an equivalence between the relations (9), (10) *and the relations*

$$\delta_L w_L^* + \delta_R w_R^* = w^{\mathrm{HLL}}, \tag{14a}$$

$$\delta_L \eta(w_L^*) + \delta_R \eta(w_R^*) \leq \eta^{\mathrm{HLL}}. \tag{14b}$$

The relations (14) give four constraints on the intermediate states w_L^* and w_R^*. The three equations (14a) ensure the weak consistency while the inequality (14b) enforces the entropy condition. However, the states w_L^* and w_R^* are made of six unknowns that write

$$w_\alpha^* = (\rho_\alpha^*, \rho_\alpha^* u_\alpha^*, \rho_\alpha^* E_\alpha^*)^{\mathrm{T}}, \quad \forall \alpha \in \{L, R\}\,. \tag{15}$$

As a consequence, the relations (14) under-determine w_L^* and w_R^* and several additional relations will be given in the next section. The proof of Lemma 1 is given below.

Proof (Proof of Lemma 1) For an approximate Riemann solver (11), if the CFL condition (12) holds then the left hand side of the equality (9) writes

$$\begin{aligned}
&\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}\,(x/\Delta t, w_L, w_R)\,\mathrm{d}x \\
&= \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\lambda_L \Delta t} w_L\,\mathrm{d}x + \frac{1}{\Delta x}\int_{\lambda_L \Delta t}^{v^*\Delta t} w_L^*\,\mathrm{d}x + \frac{1}{\Delta x}\int_{v^*\Delta t}^{\lambda_R \Delta t} w_R^*\,\mathrm{d}x + \frac{1}{\Delta x}\int_{\lambda_R \Delta t}^{\frac{\Delta x}{2}} w_R\,\mathrm{d}x, \\
&= \Big(\frac{\lambda_L \Delta t}{\Delta x} + \frac{1}{2}\Big) w_L + \frac{\Delta t}{\Delta x}(v^* - \lambda_L) w_L^* + \frac{\Delta t}{\Delta x}(\lambda_R - v^*) w_R^* + \Big(\frac{1}{2} - \frac{\lambda_R \Delta t}{\Delta x}\Big) w_R.
\end{aligned}$$

Therefore, the equality (9) is equivalent to

$$\begin{aligned}
\Big(\frac{\lambda_L \Delta t}{\Delta x} + \frac{1}{2}\Big) w_L + \frac{\Delta t}{\Delta x}(v^* - \lambda_L) w_L^* + \frac{\Delta t}{\Delta x}(\lambda_R - v^*) w_R^* + \Big(\frac{1}{2} - \frac{\lambda_R \Delta t}{\Delta x}\Big) w_R = \\
\frac{w_L + w_R}{2} - \frac{\Delta t}{\Delta x}\left(f(w_R) - f(w_L)\right).
\end{aligned}$$

Using the quantities δ_L, δ_R and w^{HLL} defined in (13) in the above equality, we deduce the equivalent formulation (14a).

Concerning the inequality (14b), according to the approximate Riemann solver structure (11), and since the CFL condition (12) holds, the left hand side of the inequality (10) writes

$$\frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \eta\left(\tilde{w}\left(x/\Delta t, w_L, w_R\right)\right) \mathrm{d}x = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\lambda_L \Delta t} \eta(w_L)\,\mathrm{d}x + \frac{1}{\Delta x}\int_{\lambda_L\Delta t}^{v^*\Delta t} \eta(w_L^*)\,\mathrm{d}x$$
$$+ \frac{1}{\Delta x}\int_{v^*\Delta t}^{\lambda_R\Delta t} \eta(w_R^*)\,\mathrm{d}x + \frac{1}{\Delta x}\int_{\lambda_R\Delta t}^{\frac{\Delta x}{2}} \eta(w_R)\,\mathrm{d}x,$$
$$= \left(\frac{\lambda_L\Delta t}{\Delta x} + \frac{1}{2}\right)\eta(w_L) + \frac{\Delta t}{\Delta x}(v^* - \lambda_L)\eta(w_L^*)$$
$$+ \frac{\Delta t}{\Delta x}(\lambda_R - v^*)\eta(w_R^*) + \left(\frac{1}{2} - \frac{\lambda_R\Delta t}{\Delta x}\right)\eta(w_R).$$

As a consequence, an equivalent form of the inequality (10) is given by

$$\left(\frac{\lambda_L\Delta t}{\Delta x}+\frac{1}{2}\right)\eta(w_L)+\frac{\Delta t}{\Delta x}(v^*-\lambda_L)\eta(w_L^*)+\frac{\Delta t}{\Delta x}(\lambda_R-v^*)\eta(w_R^*)+\left(\frac{1}{2}-\frac{\lambda_R\Delta t}{\Delta x}\right)\eta(w_R) \leq$$
$$\frac{\eta(w_L)+\eta(w_R)}{2} - \frac{\Delta t}{\Delta x}\left(G(w_R) - G(w_L)\right).$$

Using the quantities δ_L, δ_R and η^{HLL} given by (13) in the above inequality, we deduce the equivalent formulation (14b) that concludes the proof.

The state vector w^{HLL} defined in (13) refers to the standard HLL approximate Riemann solver detailed in [10]. According to the robustness properties of the HLL approximate Riemann solver, the following inequalities hold (see Appendix 7 for the proof):

$$\rho^{\mathrm{HLL}} > 0, \quad (\rho E)^{\mathrm{HLL}} > 0, \quad \eta(w^{\mathrm{HLL}}) \leq \eta^{\mathrm{HLL}}. \tag{16}$$

As a consequence, we can define

$$u^{\mathrm{HLL}} = \frac{(\rho u)^{\mathrm{HLL}}}{\rho^{\mathrm{HLL}}}, \quad E^{\mathrm{HLL}} = \frac{(\rho E)^{\mathrm{HLL}}}{\rho^{\mathrm{HLL}}} > 0, \tag{17}$$

and w^{HLL} given by (13) also reads

$$w^{\mathrm{HLL}} = (\rho^{\mathrm{HLL}}, \rho^{\mathrm{HLL}} u^{\mathrm{HLL}}, \rho^{\mathrm{HLL}} E^{\mathrm{HLL}})^{\mathrm{T}}.$$

The next section derives additional equations to entirely define the intermediate states w_L^*, w_R^* and to satisfy the relations (14).

3 Consistent, entropy stable approximate Riemann solver

In this section, we consider the conditions (14) to design an approximate Riemann solver (ARS) (11) for the Euler equations (2). As the conditions (14) are under-determined, below the first lemma gives two closure equations that impose the velocity u and the pressure p continuity in the intermediate states of the solver. Such continuities have already been introduced for the HLLC scheme [22] for instance.

Lemma 2 *Consider an approximate Riemann solver* $\tilde{w}(\cdot, w_L, w_R)$ (11) *that approximates the solution of the Riemann problem for the Euler equations* (2), (7) *endowed with the pressure function* $w \mapsto p(w)$ *given by* (1). *Assume the CFL condition* (12) *holds and let also consider the quantities* $\delta_L > 0$, $\delta_R > 0$, w^{HLL} *and* u^{HLL} *respectively defined in* (13),(17). *If the intermediate states denoted* w_L^* *and* w_R^* *are defined by the equations*

$$w_L^* \delta_L + w_R^* \delta_R = w^{\mathrm{HLL}}, \tag{18a}$$

$$u_L^* = u_R^* \tag{18b}$$

$$p(w_L^*) = p(w_R^*), \tag{18c}$$

then,

$$u_L^* = u_R^* = u^{\mathrm{HLL}}, \tag{19a}$$

$$p(w_L^*) = p(w_R^*) = p(w^{\mathrm{HLL}}). \tag{19b}$$

Proof Writing component by component the consistency integral relation (18a) verified by the states w_L^*, w_R^*, we have

$$\delta_L \rho_L^* + \delta_R \rho_R^* = \rho^{\mathrm{HLL}}, \tag{20a}$$

$$\delta_L \rho_L^* u_L^* + \delta_R \rho_R^* u_R^* = \rho^{\mathrm{HLL}} u^{\mathrm{HLL}}, \tag{20b}$$

$$\delta_L \rho_L^* E_L^* + \delta_R \rho_R^* E_R^* = \rho^{\mathrm{HLL}} E^{\mathrm{HLL}}, \tag{20c}$$

where u^{HLL} and E^{HLL} are given in (17). As $u_L^* = u_R^*$, let denote $u^* = u_L^* = u_R^*$. The equation (20b) associated to (20a) gives

$$\begin{aligned} \rho^{\mathrm{HLL}} u^{\mathrm{HLL}} &= \delta_L \rho_L^* u_L^* + \delta_R \rho_R^* u_R^*, \\ &= \left(\delta_L \rho_L^* + \delta_R \rho_R^*\right) u^*, \\ &= \rho^{\mathrm{HLL}} u^*. \end{aligned} \tag{21}$$

Since $(w_L, w_R) \in \Omega^2$, the robustness of the standard HLL solver [10] enforces the strict inequality $\rho^{\mathrm{HLL}} > 0$ (see Lemma 5-*ii*) for the proof). Therefore, multiplying the equation (21) by $1/\rho^{\mathrm{HLL}}$, we deduce the result related to u_L^*, u_R^*.

Using the equalities (19a), we also deduce that $p(w_L^*)$ and $p(w_R^*)$ write

$$p(w_\alpha^*) = (\gamma - 1)\left(\rho_\alpha^* E_\alpha^* - \frac{\rho_\alpha^* (u^{\mathrm{HLL}})^2}{2}\right), \quad \forall \alpha \in \{L, R\}\,. \tag{22}$$

With the above result, we can compute the quantity $\delta_L p(w_L^*) + \delta_R p(w_R^*)$ and using the equations (20a)-(20c), we obtain

$$\begin{aligned}
\delta_L p(w_L^*) + \delta_R p(w_R^*) &= (\gamma-1)\Big(\delta_L \rho_L^* E_L^* + \delta_R \rho_R^* E_R^* - \frac{\delta_L \rho_L^* + \delta_R \rho_R^*}{2}(u^{\mathrm{HLL}})^2\Big),\\
&= (\gamma-1)\Big(\rho^{\mathrm{HLL}} E^{\mathrm{HLL}} - \frac{\rho^{\mathrm{HLL}}}{2}(u^{\mathrm{HLL}})^2\Big),\\
&= p(w^{\mathrm{HLL}}).
\end{aligned}$$

As $\delta_L + \delta_R = 1$ and since the equality $p(w_L^*) = p(w_R^*)$ holds, we deduce the result.□

The closure equations of the above lemma give the formulation of the quantities u_L^*, u_R^*, $p(w_L^*)$ and $p(w_R^*)$. In addition, if the quantities ρ_L^* and ρ_R^* are known, then the computations of E_L^*, E_R^* is direct with the equations (22). As a consequence, we only have to determine ρ_L^*, ρ_R^* according to the constrained equation

$$\rho_L^* \delta_L + \rho_R^* \delta_R = \rho^{\mathrm{HLL}}, \tag{23a}$$

$$\delta_L \eta(w_L^*) + \delta_R \eta(w_R^*) \le \eta^{\mathrm{HLL}}. \tag{23b}$$

Since the entropy non-linearity complexifies the solving of the constrained equation (23), the constrain (23b) will be substituted by a simpler quadratic equation. This new equation is defined from an upper-bound of $\delta_L \eta(w_L^*) + \delta_R \eta(w_R^*)$ that is now detailed.

Lemma 3 (Quadratic upper-bound of $\delta_L \eta(w_L^*) + \delta_R \eta(w_R^*)$)

Consider an approximate Riemann solver $\tilde{w}(\cdot, w_L, w_R)$ (11) *defined by the system* (18)*. Let also consider the entropy* $w \mapsto \eta(w)$ *given by* (4) *and the quantities* $\delta_L > 0$*,* $\delta_R > 0$*,* w^{HLL} *defined in* (13)*. If* $\rho_L^* > 0$ *and* $\rho_R^* > 0$ *then*

$$\eta(w_L^*)\delta_L + \eta(w_R^*)\delta_R \le \frac{\gamma \delta_L}{\rho^{\mathrm{HLL}}}\big(\rho_L^* - \rho^{\mathrm{HLL}}\big)^2 + \frac{\gamma \delta_R}{\rho^{\mathrm{HLL}}}\big(\rho_R^* - \rho^{\mathrm{HLL}}\big)^2 + \eta(w^{\mathrm{HLL}}). \tag{24}$$

Proof At first, if $\rho_L^* = \rho_R^*$ then the equation (20a) imposes $\rho_L^* = \rho_R^* = \rho^{\mathrm{HLL}}$. In this case, the inequality (24) is obviously verified in the sens of the equality. Let now consider the case $\rho_L^* \neq \rho_R^*$. Using the definition of the entropy $\eta(w)$ given by (4) and the equations (19b)-(20a), we have

$$\delta_L \eta(w_L^*) + \delta_R \eta(w_R^*) = -\delta_L \rho_L^* \ln\Big(\frac{p(w^{\mathrm{HLL}})}{(\rho_L^*)^\gamma}\Big) - \delta_R \rho_R^* \ln\Big(\frac{p(w^{\mathrm{HLL}})}{(\rho_R^*)^\gamma}\Big). \tag{25}$$

Considering the equation (20a) in the quantity $\eta(w^{\mathrm{HLL}})$, we also have

$$\eta(w^{\mathrm{HLL}}) = -\delta_L \rho_L^* \ln\Big(\frac{p(w^{\mathrm{HLL}})}{(\rho^{\mathrm{HLL}})^\gamma}\Big) - \delta_R \rho_R^* \ln\Big(\frac{p(w^{\mathrm{HLL}})}{(\rho^{\mathrm{HLL}})^\gamma}\Big).$$

Therefore, substracting the above equation to the equation (25), we obtain

$$\frac{\delta_L \eta(w_L^*) + \delta_R \eta(w_R^*) - \eta(w^{\mathrm{HLL}})}{\gamma \rho^{\mathrm{HLL}}} = \delta_L \frac{\rho_L^*}{\rho^{\mathrm{HLL}}} \ln\Big(\frac{\rho_L^*}{\rho^{\mathrm{HLL}}}\Big) + \delta_R \frac{\rho_R^*}{\rho^{\mathrm{HLL}}} \ln\Big(\frac{\rho_R^*}{\rho^{\mathrm{HLL}}}\Big). \tag{26}$$

Up to a permutation, we can assume $\rho_L^* < \rho_R^*$. In this case, a direct proof by contradiction associated to the equation (20a) gives $\rho_L^* < \rho^{\text{HLL}}$. Since the inequalities $\rho_L^* > 0$ and $\rho_R^* > 0$ are assumed satisfied, we deduce the existence of reals r_L^*, r_R^* such that

$$\frac{\rho_L^*}{\rho^{\text{HLL}}} = 1 - r_L^*, \quad \frac{\rho_R^*}{\rho^{\text{HLL}}} = 1 + r_R^*, \quad \text{with,} \quad r_R^* \geq 0,\, r_L^* \in [0,1). \tag{27}$$

Reformulating the equation (26) with r_L^* and r_R^*, we have

$$\frac{\delta_L \eta(w_L^*) + \delta_R \eta(w_R^*) - \eta(w^{\text{HLL}})}{\gamma \rho^{\text{HLL}}} = \delta_L \left(1 - r_L^*\right) \ln\left(1 - r_L^*\right) + \delta_R \left(1 + r_R^*\right) \ln\left(1 + r_R^*\right).$$

But, using a Taylor expansion with an integral form of the reminder, we obtain

$$\ln(1 - r_L^*) = -r_L^* - \int_0^{r_L^*} \frac{(r_L^* - s)}{(1-s)^2}\, \mathrm{d}s,$$
$$\left(1 + r_R^*\right) \ln\left(1 + r_R^*\right) = r_R^* + \frac{(r_R^*)^2}{2} - \frac{1}{2}\int_0^{r_R^*} \left(\frac{r_R^* - s}{1+s}\right)^2 \mathrm{d}s.$$

Since $0 \leq r_L^* < 1$ and $0 \leq r_R^*$, both residues of the above Taylor expansions are signed and consequently, we deduce

$$\begin{aligned}
&\delta_L \left(1 - r_L^*\right) \ln\left(1 - r_L^*\right) + \delta_R \left(1 + r_R^*\right) \ln\left(1 + r_R^*\right) \\
&\qquad \leq -\delta_L r_L^* \left(1 - r_L^*\right) + \delta_R \left(r_R^* + \frac{(r_R^*)^2}{2}\right), \\
&\qquad \leq \delta_R r_R^* - \delta_L r_L^* + \delta_L (r_L^*)^2 + \delta_R (r_R^*)^2.
\end{aligned}$$

Rewriting the equation (20a) with the quantities r_L^*, r_R^* defined in (27), we obtain $\delta_L r_L^* = \delta_R r_R^*$. Using this result in the above inequality, we have

$$\begin{aligned}
\frac{\delta_L \eta(w_L^*) + \delta_R \eta(w_R^*) - \eta(w^{\text{HLL}})}{\gamma \rho^{\text{HLL}}} &= \delta_L \left(1 - r_L^*\right) \ln\left(1 - r_L^*\right) + \delta_R \left(1 + r_R^*\right) \ln\left(1 + r_R^*\right), \\
&\leq \delta_L (r_L^*)^2 + \delta_R (r_R^*)^2.
\end{aligned}$$

Using once again the definition (27) to rewrite the above inequality with ρ_L^*, ρ_R^* we eventually deduce the inequality (24) that concludes the proof. □

The upper-bound of the inequality (24) is not the sharpest but it is proposed in this work for the sake of simplicity. Imposing this upper-bound equals to η^{HLL}, we obtain a last equation that preserves the inequality $\delta_L \eta(w_L^*) + \delta_R \eta(w_R^*) \leq \eta^{\text{HLL}}$. Therefore, the approximate Riemann solver (11) proposed for the Euler system (2) is defined by the equations

$$w_L^* \delta_L + w_R^* \delta_R = w^{\mathrm{HLL}}, \tag{28a}$$

$$u_L^* = u_R^*, \tag{28b}$$

$$p(w_L^*) = p(w_R^*), \tag{28c}$$

$$\frac{\gamma \delta_L}{\rho^{\mathrm{HLL}}} \left(\rho_L^* - \rho^{\mathrm{HLL}}\right)^2 + \frac{\gamma \delta_R}{\rho^{\mathrm{HLL}}} \left(\rho_R^* - \rho^{\mathrm{HLL}}\right)^2 + \eta(w^{\mathrm{HLL}}) = \eta^{\mathrm{HLL}}. \tag{28d}$$

Considering the above system and the results of Lemma 2, we now achieve the approximate Riemann solver design.

Theorem 1 (Entropic preserving two intermediate states approximate Riemann solver for the Euler equations)

Consider an approximate Riemann solver $\tilde{w}(\cdot, w_L, w_R)$ (11) *that approximates the solution of the Riemann problem for the Euler equations* (2), (7) *endowed with the pressure function* $w \mapsto p(w)$ *given by* (1) *and supplemented with the convex entropy-entropy flux* (η, G) *given by* (4). *Assume the CFL condition* (12) *holds and let also consider the quantities* $\delta_L > 0$, $\delta_R > 0$, w^{HLL}, η^{HLL}, u^{HLL} *respectively defined in* (13),(17). *If the intermediate states denoted* w_L^* *and* w_R^* *write*

$$u_L^* = u_R^* = u^{\mathrm{HLL}}, \tag{29a}$$

$$p(w_L^*) = p(w_R^*) = p(w^{\mathrm{HLL}}), \tag{29b}$$

$$\rho_L^* = \rho^{\mathrm{HLL}} \mp \sqrt{\frac{\rho^{\mathrm{HLL}}}{\gamma} \frac{\delta_R}{\delta_L} \left(\eta^{\mathrm{HLL}} - \eta(w^{\mathrm{HLL}})\right)}, \tag{29c}$$

$$\rho_R^* = \rho^{\mathrm{HLL}} \pm \sqrt{\frac{\rho^{\mathrm{HLL}}}{\gamma} \frac{\delta_L}{\delta_R} \left(\eta^{\mathrm{HLL}} - \eta(w^{\mathrm{HLL}})\right)}, \tag{29d}$$

then the following statements are verified.

i) The approximate Riemann solver (11) *is consistent with the Euler equations* (2).
ii) If $\rho_L^* > 0$ *and* $\rho_R^* > 0$ *then approximate Riemann solver* (11) *verifies the entropy condition* (10).

In the definitions (29), both symbols $\pm$ and $\mp$ mean $+$ or $-$ but both are self-dependent. If $\pm$ is positive (*resp.* negative) then $\mp$ is negative (*resp.* positive). As a consequence, the above theorem yields two distinct approximate Riemann solver and the selection of one of them is done with an arbitrary choice for $\pm$. The proof of Theorem 1 is given below.

Proof (Proof of Theorem 1) At first, we show that the formulations (29) are the solutions of the system (28). The results (29a)-(29b) that respectively concern u_L^*, u_R^* and $p(w_L^*)$, $p(w_R^*)$ have already proved in Lemma 2. Now, we have to prove the formulations of ρ_L^*, ρ_R^* from the equations (28a), (28d) that read

$$\delta_L \rho_L^* + \delta_R \rho_R^* = \rho^{\mathrm{HLL}}, \tag{30a}$$

$$\frac{\gamma\delta_L}{\rho^{\mathrm{HLL}}}\left(\rho_L^* - \rho^{\mathrm{HLL}}\right)^2 + \frac{\gamma\delta_R}{\rho^{\mathrm{HLL}}}\left(\rho_R^* - \rho^{\mathrm{HLL}}\right)^2 + \eta(w^{\mathrm{HLL}}) = \eta^{\mathrm{HLL}}. \tag{30b}$$

Writing the equation (30a) in the equation (30b), we obtain

$$\frac{\gamma\delta_R\delta_L}{\rho^{\mathrm{HLL}}}(\rho_R^* - \rho_L^*)^2 = \eta^{\mathrm{HLL}} - \eta\left(w^{\mathrm{HLL}}\right). \tag{31}$$

As the CFL condition (12) holds, the inequality $\eta^{\mathrm{HLL}} - \eta\left(w^{\mathrm{HLL}}\right) \geq 0$ is ensured thanks to the robustness the standard HLL approximate Riemann solver [10] (see Lemma 5-*iii*) for the proof). In addition, since $\delta_L > 0$ and $\delta_R > 0$, we can multiply the equation (31) by $\rho^{\mathrm{HLL}}/\gamma\delta_L\delta_R$ then using the square root on both sides of the result, we deduce

$$\rho_R^* - \rho_L^* = \pm\sqrt{\rho^{\mathrm{HLL}}\left(\eta^{\mathrm{HLL}} - \eta(w^{\mathrm{HLL}})\right)/\gamma\delta_L\delta_R}.$$

Since $\delta_L + \delta_R = 1$, the above equation associated to the equation (30a) leads to the formulations of ρ_L^*, ρ_R^*. This last remark achieves to show that (29) are the solutions of the system (28).

For the consistency *i*), according to the intermediate states definition given by (29), the equality $\tilde{w}(\cdot, w, w) = w$ is satisfied. In addition, the approximate Riemann solver are designed from the equation (14a) which is equivalent to the consistency integral relation (9) (see Lemma 1). As a consequence, the approximate Riemann solver are consistent with the Euler system (2).

Concerning statement *ii*), according to Lemma 3, the equation (30b) ensures the inequality $\delta_L\eta(w_L^*) + \delta_R\eta(w_R^*) \leq \eta^{\mathrm{HLL}}$. Hence, using Lemma 1, the approximate Riemann solver satisfy the entropy condition (10) that achieves the proof □

Both approximate Riemann solver of the above theorem are entropic but their definitions require the strict inequalities $\rho_L^* > 0$ and $\rho_R^* > 0$ that may not be satisfied. However, these strict inequalities can be enforced with a local correction that uses the standard HLL approximate Riemann solver [10]. Indeed, denoting $\widetilde{w_R^*}$ and $\widetilde{w_L^*}$ two states defined by the formulations (29), a robust version of Theorem 1 approximate Riemann solver is given by

$$(w_L^*, w_R^*) = \begin{cases} (\widetilde{w_L^*}, \widetilde{w_R^*}), & \text{if } (\widetilde{w_L^*}, \widetilde{w_R^*}) \in \Omega^2, \\ (w^{\mathrm{HLL}}, w^{\mathrm{HLL}}), & \text{otherwise.} \end{cases} \tag{32}$$

According to Lemma 1, the above procedure ensures $\rho_L^*, \rho_R^* > 0$, preserves the consistency and the entropy condition and it also guarantees the preservation of the set Ω.

Theorem 1 achieves to define the approximate Riemann solver that is now used to define a numerical scheme (5).

4 Consistent, entropy stable Godunov type scheme

At time t^n, the solution $w(\cdot, t^n)$ of the Euler system (2) is approximated by a piecewise constant function defined by a sequence $(w_i^n)_{i\in\mathbb{Z}}$ in Ω. As a consequence, the sequence $(w_i^n)_{i\in\mathbb{Z}}$ can be reinterpreted as a juxtaposition of Riemann problems each of them defined on one interface of the mesh.

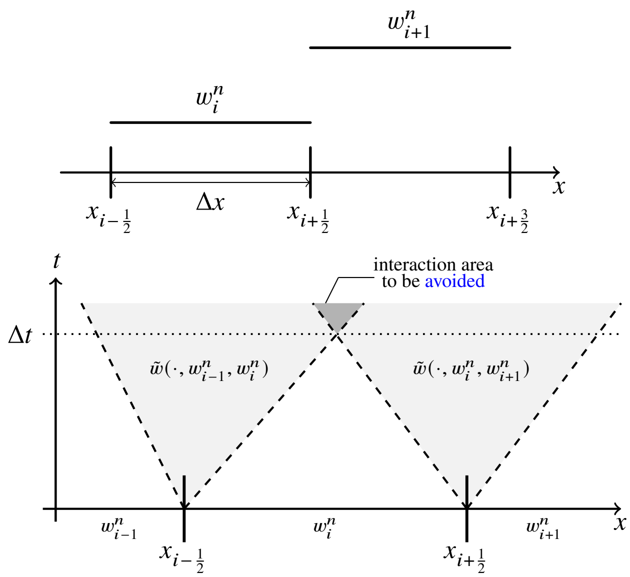


Fig. 2: On the top, piecewise constant approximation of $w(\cdot, t^n)$ on the mesh defined by interfaces $(x_{i+\frac{1}{2}})_{i\in\mathbb{Z}}$. On the bot, Riemann problems juxtaposition in the plane (x, t).

As displayed in Figure 2, a restriction of the time step $\Delta t > 0$ imposes a non interaction between each Riemann problems. This restriction refers to the CFL condition enunciated in Definition 1-*i*). As a consequence, and by definition of the approximate Riemann solver detailed in Sections 2 and 3, an approximation of $w(\cdot, t^{n+1})$ is given by the L^2-projection of the sequence $\big(\tilde{w}(\cdot, w_i^n, w_{i+1}^n)\big)_{i\in\mathbb{Z}}$ on piecewise constant functions that reads

$$w_i^{n+1} = \frac{1}{\Delta x}\int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n)\,\mathrm{d}x + \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^0 \tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\,\mathrm{d}x. \tag{33}$$

The above numerical scheme is a Godunov type scheme [10] which is entirely defined by an approximate Riemann solver $\tilde{w}$. The next Lemma establishes a more usual formulation of the Godunov type scheme (33) and also shows the discrete entropy inequality from the entropy condition (10).

Lemma 4 (Godunov type schemes properties)

Consider a Godunov type scheme (33) *defined by an approximate Riemann solver* $\tilde{w}$ *satisfying Definition 1 for the Euler equations* (2) *endowed with the convex entropy-entropy flux* (η, G) *given by* (4)*. Let also consider the functions* $\mathcal{F} : (\mathbb{R}^3)^2 \to \mathbb{R}^3$ *and* $\mathcal{G} : (\mathbb{R}^3)^2 \to \mathbb{R}$ *such that*

$$\begin{aligned}\mathcal{F}(w_i^n, w_{i+1}^n) = \ & \frac{f(w_{i+1}^n) + f(w_i^n)}{2} - \frac{\Delta x}{4\Delta t}\left(w_{i+1}^n - w_i^n\right) \\ & + \frac{1}{2\Delta t}\int_0^{\frac{\Delta x}{2}} \tilde{w}\big(x/\Delta t, w_i^n, w_{i+1}^n\big)\, \mathrm{d}x \\ & - \frac{1}{2\Delta t}\int_{-\frac{\Delta x}{2}}^{0} \tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\, \mathrm{d}x,\end{aligned} \tag{34}$$

$$\begin{aligned}\mathcal{G}(w_i^n, w_{i+1}^n) = \ & \frac{G(w_{i+1}^n) + G(w_i^n)}{2} - \frac{\Delta x}{4\Delta t}\big(\eta(w_{i+1}^n) - \eta(w_i^n)\big) \\ & + \frac{1}{2\Delta t}\int_0^{\frac{\Delta x}{2}} \eta\big(\tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\big)\, \mathrm{d}x \\ & - \frac{1}{2\Delta t}\int_{-\frac{\Delta x}{2}}^{0} \eta\big(\tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\big)\, \mathrm{d}x.\end{aligned} \tag{35}$$

We have the following statements:

i) *Both functions* $\mathcal{F}$ *and* $\mathcal{G}$ *are respectively consistent with* f *and* G *that writes*

$$\mathcal{F}(w, w) = f(w), \quad \mathcal{G}(w, w) = G(w), \quad \forall w \in \Omega. \tag{36}$$

ii) *The Godunov type scheme* (33) *is equivalent to*

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\big(\mathcal{F}(w_i^n, w_{i+1}^n) - \mathcal{F}(w_{i-1}^n, w_i^n)\big), \quad \forall i \in \mathbb{Z}. \tag{37}$$

ii) *The updated sequence* $(w_i^{n+1})_{i\in\mathbb{Z}}$ *given by the Godunov type scheme* (33) *satisfies the following discrete entropy inequality:*

$$\frac{\eta(w_i^{n+1}) - \eta(w_i^n)}{\Delta t} + \frac{\mathcal{G}(w_i^n, w_{i+1}^n) - \mathcal{G}(w_{i-1}^n, w_i^n)}{\Delta x} \leq 0, \quad \forall i \in \mathbb{Z}. \tag{38}$$

Proof For i), using the consistency of the approximate Riemann solver given by Definition 1-iii), we have

$$\begin{aligned}\mathcal{F}(w,w) &= \frac{f(w)+f(w)}{2} - \frac{\Delta x}{4\Delta t}(w-w)\\ &\quad + \frac{1}{2\Delta t}\int_0^{\frac{\Delta x}{2}} \tilde{w}(x/\Delta t, w, w)\,\mathrm{d}x - \frac{1}{2\Delta t}\int_{-\frac{\Delta x}{2}}^{0} \tilde{w}(x/\Delta t, w, w)\,\mathrm{d}x,\\ &= f(w) + \frac{1}{2\Delta t}\int_0^{\frac{\Delta x}{2}} w\,\mathrm{d}x - \frac{1}{2\Delta t}\int_{-\frac{\Delta x}{2}}^{0} w\,\mathrm{d}x,\\ &= f(w).\end{aligned}$$

Since a similar computation gives the consistency of $\mathcal{G}$ with G that shows statement i).

For the reformulation ii), since the CFL condition (12) holds, we can split the integrals in (33) to rewrite a Godunov type scheme (33) under the form

$$\begin{aligned}w_i^{n+1} &= \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \tilde{w}\big(x/\Delta t, w_i^n, w_{i+1}^n\big)\,\mathrm{d}x + \frac{1}{2\Delta x}\int_0^{\frac{\Delta x}{2}} \tilde{w}\big(x/\Delta t, w_{i-1}^n, w_i^n\big)\,\mathrm{d}x\\ &+ \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}\big(x/\Delta t, w_{i-1}^n, w_i^n\big)\,\mathrm{d}x + \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \tilde{w}\big(x/\Delta t, w_i^n, w_{i+1}^n\big)\,\mathrm{d}x\\ &- \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \tilde{w}\big(x/\Delta t, w_{i-1}^n, w_i^n\big)\,\mathrm{d}x - \frac{1}{2\Delta x}\int_0^{\frac{\Delta x}{2}} \tilde{w}\big(x/\Delta t, w_i^n, w_{i+1}^n\big)\,\mathrm{d}x.\end{aligned}$$

Using the consistency integral relation (9) in the above equality, we deduce

$$\begin{aligned}w_i^{n+1} &= \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \tilde{w}\big(x/\Delta t, w_i^n, w_{i+1}^n\big)\,\mathrm{d}x + \frac{1}{2\Delta x}\int_0^{\frac{\Delta x}{2}} \tilde{w}\big(x/\Delta t, w_{i-1}^n, w_i^n\big)\,\mathrm{d}x\\ &+ \frac{1}{2}\Big(\frac{w_{i-1}^n + w_i^n}{2} - \frac{\Delta t}{\Delta x}\big(f(w_i^n) - f(w_{i-1}^n)\big)\Big)\\ &+ \frac{1}{2}\Big(\frac{w_i^n + w_{i+1}^n}{2} - \frac{\Delta t}{\Delta x}\big(f(w_{i+1}^n) - f(w_i^n)\big)\Big)\\ &- \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \tilde{w}\big(x/\Delta t, w_{i-1}^n, w_i^n\big)\,\mathrm{d}x - \frac{1}{2\Delta x}\int_0^{\frac{\Delta x}{2}} \tilde{w}\big(x/\Delta t, w_i^n, w_{i+1}^n\big)\,\mathrm{d}x.\end{aligned}$$

Considering the definition of the function $\mathcal{F}$ given by (34) in the above formulation, we deduce the form (37). The converse is direct.

Concerning the discrete entropy inequality iii), as the CFL condition (12) holds, we can split the integrals in (33) and since the entropy $w \mapsto \eta(w)$ is a convex function, the Jensen inequality [17] gives

$$\begin{aligned}
\eta(w_i^{n+1}) \leq \\
&\frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \eta\big(\tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\big)\,\mathrm{d}x + \frac{1}{2\Delta x}\int_{0}^{\frac{\Delta x}{2}} \eta\big(\tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n)\big)\,\mathrm{d}x \\
&+ \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \eta\big(\tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n)\big)\,\mathrm{d}x + \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \eta\big(\tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\big)\,\mathrm{d}x \\
&- \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \eta\big(\tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n)\big)\,\mathrm{d}x - \frac{1}{2\Delta x}\int_{0}^{\frac{\Delta x}{2}} \eta\big(\tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\big)\,\mathrm{d}x.
\end{aligned}$$

Using the entropy condition (10) in the above inequality, we obtain

$$\begin{aligned}
\eta(w_i^{n+1}) \leq \\
&\frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \eta\big(\tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\big)\,\mathrm{d}x + \frac{1}{2\Delta x}\int_{0}^{\frac{\Delta x}{2}} \eta\big(\tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n)\big)\,\mathrm{d}x \\
&+ \frac{1}{2}\Big(\frac{\eta(w_{i-1}^n) + \eta(w_i^n)}{2} - \frac{\Delta t}{\Delta x}\big(G(w_i^n) - G(w_{i-1}^n)\big)\Big) \\
&+ \frac{1}{2}\Big(\frac{\eta(w_i^n) + \eta(w_{i+1}^n)}{2} - \frac{\Delta t}{\Delta x}\big(G(w_{i+1}^n) - G(w_i^n)\big)\Big) \\
&- \frac{1}{2\Delta x}\int_{-\frac{\Delta x}{2}}^{0} \eta\big(\tilde{w}(x/\Delta t, w_{i-1}^n, w_i^n)\big)\,\mathrm{d}x - \frac{1}{2\Delta x}\int_{0}^{\frac{\Delta x}{2}} \eta\big(\tilde{w}(x/\Delta t, w_i^n, w_{i+1}^n)\big)\,\mathrm{d}x.
\end{aligned}$$

Reorganized the above terms with the definition of the function $\mathcal{G}$ given by (35), we deduce the discrete entropy inequality (38) that achieves the proof. □

Associating the above Lemma and the results of Sections 2 and 3, we finally define a consistent entropic Godunov type scheme for the Euler equations (2).

Theorem 2 (Consistent, entropic Godunov type scheme for the Euler equations) *Consider a Godunov type scheme* (33) *that approximates the solution of the Euler equations* (2) *endowed with the pressure function* $w \mapsto p(w)$ *given by* (1) *and completed with the convex entropy-entropy flux* (η, G) *given by* (4)*. Assume this numerical scheme is defined by an approximate Riemann solver made of two intermediate states* (11)*. Let also assume the CFL condition* (12) *holds and the intermediate states of the approximate Riemann solver are given by* (32)*. The following statements are satisfied:*

i) The Godunov type scheme (33) *is consistent with the Euler equations* (2)*.*
ii) The updated sequence $(w_i^{n+1})_{i\in\mathbb{Z}}$ *given by the Godunov type scheme* (33) *satisfies a discrete entropy inequality given by* (38)*.*
iii) If the sequence $(w_i^n)_{i\in\mathbb{Z}}$ *defines a stationary contact state characterized by* $(u_i^n)_{i\in\mathbb{Z}} = 0$ *and* $\big(p(w_i^n)\big)_{i\in\mathbb{Z}} = \mathrm{cste}$ *then the Godunov type scheme* (33) *exactly preserves the variables u and p. This property writes*

$$\textit{if} \begin{cases} u_i^n = 0, \\ p(w_i^n) = \text{cste}, \end{cases} \quad \forall i \in \mathbb{Z}, \quad \textit{then} \begin{cases} u_i^{n+1} = 0, \\ p(w_i^{n+1}) = \text{cste}, \end{cases} \quad \forall i \in \mathbb{Z}.$$

Proof For statements *i*) and *ii*), according to Theorem 1, the approximate Riemann solver design includes the weak consistency and the entropy condition. As a consequence the proof of *i*) and *ii*) is direct with Lemma 4.

Concerning statement *iii*), for sake of clarity, the local notations $w_i^n = w_L$ and $w_{i+1}^n = w_R$ are adopted. It is now sufficient to show that if

$$u_L = u_R = 0, \tag{39a}$$

$$p(w_L) = p(w_R) = p_0, \tag{39b}$$

then $u_L^* = u_R^* = 0$ and $\rho_\alpha E_\alpha = \rho_\alpha^* E_\alpha^*$ for all α in $\{L, R\}$. At first, the equalities (29a) reads $u_L^* = u_R^* = u^{\text{HLL}}$ but according to the definition of u^{HLL} given by (17), we have

$$\begin{aligned} u^{\text{HLL}} &= \frac{(\rho u)^{\text{HLL}}}{\rho^{\text{HLL}}}, \\ &= \frac{1}{\rho^{\text{HLL}}(\lambda_R - \lambda_L)}\Big(\lambda_R \rho_R u_R - \lambda_L \rho_L u_L - \rho_R u_R^2 + \rho_L u_L^2 - \big(p(w_R) - p(w_L)\big)\Big), \\ &= 0. \end{aligned} \tag{40}$$

As a consequence, using the above equation, we deduce that if the equalities (39) are satisfied then $u_L^* = u_R^* = 0$. In addition, and according to the pressure function $p(w)$ defined in (1), if the conditions (39) hold then a straightforward computation shows

$$\rho_\alpha E_\alpha = \frac{p_0}{\gamma - 1}, \quad \forall \alpha \in \{L, R\}\,. \tag{41}$$

Combining the above result with the equalities (40)-(39a) in the definition of w^{HLL} given by (13), we obtain

$$\begin{aligned} \frac{p(w^{\text{HLL}})}{\gamma - 1} &= (\rho E)^{\text{HLL}} - \frac{\rho^{\text{HLL}}(u^{\text{HLL}})^2}{2}, \\ &= \frac{\lambda_R \rho_R E_R - \lambda_L \rho_L E_L}{\lambda_R - \lambda_L} - \frac{\big(\rho_R E_R + p(w_R)\big)u_R - \big(\rho_L E_L + p(w_L)\big)u_L}{\lambda_R - \lambda_L}, \\ &= \frac{\lambda_R \rho_R E_R - \lambda_L \rho_L E_L}{\lambda_R - \lambda_L}, \\ &= \frac{p_0}{\gamma - 1}. \end{aligned}$$

Since, the equalities $u_L^* = u_R^* = 0$ have been proved, the association of the above equality to the results (29b)-(41) infers that if the conditions (39) hold then

$$\rho_\alpha^* E_\alpha^* = \frac{p(w_\alpha^*)}{\gamma - 1} = \frac{p(w^{\mathrm{HLL}})}{\gamma - 1} = \frac{p_0}{\gamma - 1} = \rho_\alpha E_\alpha, \quad \forall \alpha \in \{L, R\}.$$

These last equalities conclude the proof of statement *iii)* and the proof of Theorem 2. □

The next section concerns the numerical experiments.

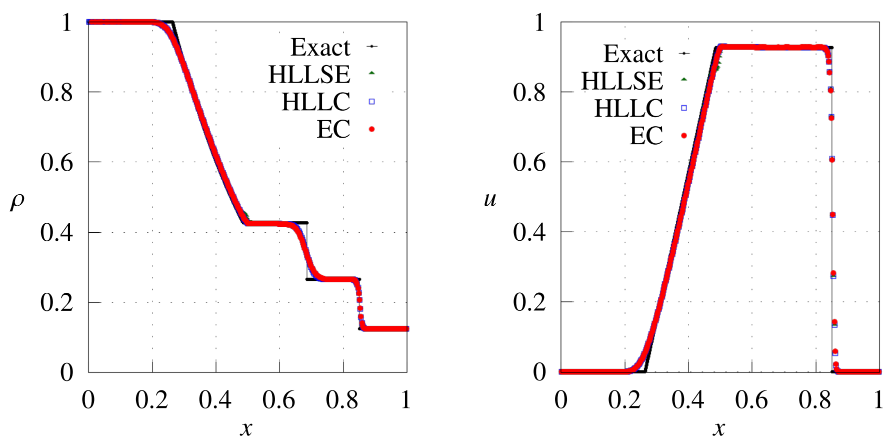
5 Numerical results

This section presents some numerical experiments to assess the performances of the numerical schemes derived in Theorem 2. The CFL condition is given by (12) and for all the interfaces of the mesh, we fix $v^* = u^{\mathrm{HLL}}$ where u^{HLL} is defined in (17). For each test case, one of the two approximate Riemann solvers given by (29) is randomly selected. The results are compared to the HLLC scheme [22] and to the scheme detailed in [10] (Section b.ii). This last scheme is symmetrized fixing $\lambda_R = -\lambda_L$ and its two implicit constants are both fixed to 10^{-7}.

The first test case is the Sod shock tube problem for a diatomic gas ($\gamma = 1.4$) [19]. The spatial domain $[0, 1]$ is discretized with 400 cells. The initial condition is

$$\rho_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.125 & \text{otherwise,} \end{cases} \qquad u_0(x) = 0, \qquad p_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.1 & \text{otherwise.} \end{cases} \tag{42}$$

Homogeneous Neumann conditions are imposed on both boundaries of the spatial domain. The final time is $t = 0.2$ and Figure 3 displays the results.



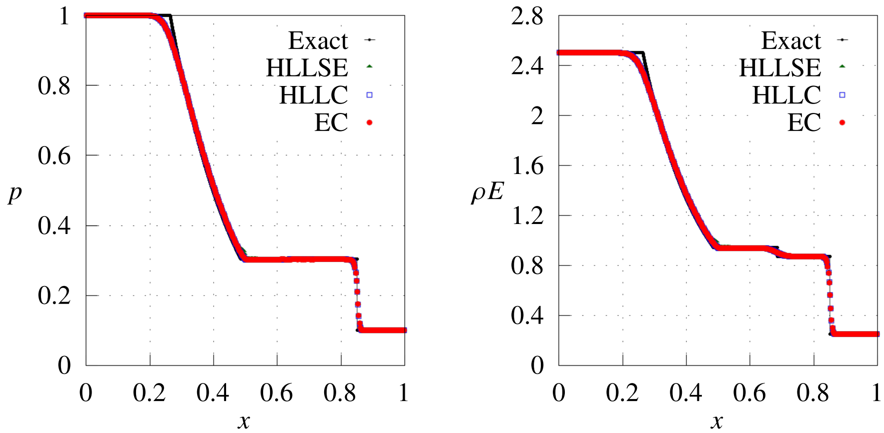


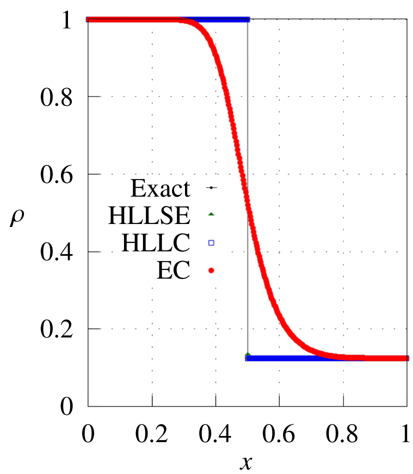
Fig. 3: Numerical results for the Sod test case (42), at time $t = 0.2$, with the legend EC : schemes derived in Theorem 2, HLLC and HLLSE : schemes respectively detailed in [22]-[10] (Section b.ii).

The results of the schemes defined in Theorem 2 are very satisfying and conform to the literature standards.

The second test case concerns the simulation of a stationary contact wave defined by the following initial condition:

$$\rho_0(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 0.125 & \text{otherwise,} \end{cases} \qquad u_0(x) = 0, \qquad p_0(x) = 1. \tag{43}$$

Homogeneous Neumann conditions are imposed on both boundaries of the spatial domain. The final time is $t = 1.0$ and Figure 4 displays the results and the numerical errors for several norms.



Errors on (ρ, u, p) for the preservation of the stationary contact waves.

	ρ		
	L^1	L^2	L^∞
EC	5.92E-02	1.49E-02	0.46E-00
HLLC	0.00E-00	0.00E-00	0.00E-00
HLLSE	4.34E-05	1.68E-07	8.18E-03

	u		
	L^1	L^2	L^∞
EC	4.16E-15	2.35E-29	8.07E-015
HLLC	0.00E-00	0.00E-00	0.00E-00
HLLSE	6.27E-05	4.39E-09	1.01E-04

	p		
	L^1	L^2	L^∞
EC	1.20E-015	1.97E-30	2.66E-015
HLLC	0.00E-00	0.00E-00	0.00E-00
HLLSE	6.74E-05	4.80E-09	8.17E-05

Fig. 4: On the left, numerical results at time $t = 1.0$ for the stationary contact problem (43) with the legend EC : schemes derived in Theorem 2, HLLC, HLLSE : schemes respectively detailed in [22]-[10] (Section b.ii). On the right, numerical errors for the variables ρ, u and p at time $t = 1.0$.

The HLLC scheme [22] exactly preserves the contact discontinuity defined by the initial condition (43). The scheme derived in [10] (Section b.ii) commits a slight error on the variable ρ that is clearly inferior to the first order error committed by the schemes defined in Theorem 2. For this variable, the schemes of Theorem 2 are more diffusive than the existing numerical schemes. Nevertheless, the schemes proposed in Theorem 2 exactly preserve the quantities u and p that it is not the case of the scheme detailed in [10] (Section b.ii).

6 Conclusion

We have presented entropic Godunov type scheme for the Euler equations. The schemes are defined with approximate Riemann solvers (ARS) made of two constant intermediate states and that include sufficient conditions for the consistency and for the discrete entropy stability. The approximate Riemann solver are defined by quadratic equations that always admit explicit reals solutions. As a consequence, the entropic schemes are simple. Their numerical results are conform to the literature standards devoted to the first-order schemes.

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7 Robustness of the Harten, Lax, van Leer approximate Riemann solver

This appendix concerns the standard HLL approximate Riemann solver [10]. Considering the formalism of Section 2, this approximate Riemann solver is denoted $\tilde{w}^{\mathrm{HLL}}(x/\Delta t, w_L, w_R)$ and it writes

$$\tilde{w}^{\mathrm{HLL}}(x/\Delta t, w_L, w_R) = \begin{cases} w_L \text{ if } \dfrac{x}{\Delta t} \le \lambda_L, \\ w^{\mathrm{HLL}} \text{ if } \lambda_L < \dfrac{x}{\Delta t} < \lambda_R, \\ w_R \text{ if } \lambda_R < \dfrac{x}{\Delta t}. \end{cases} \tag{44}$$

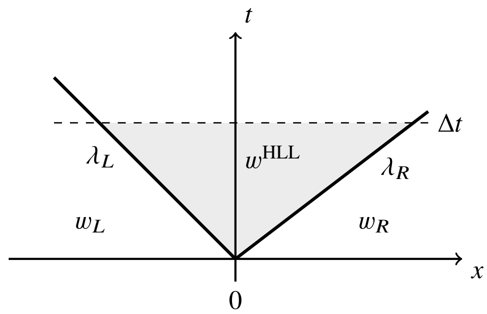

Fig. 5: HLL Approximate Riemann solver in the plane (x, t).

The CFL condition for the HLL approximate Riemann solver (44) is also given by (12). Using the consistency integral relation (9) in the definition of the HLL approximate Riemann solver given by (44), a straightforward computation shows

$$w^{\mathrm{HLL}} = (\rho^{\mathrm{HLL}}, (\rho u)^{\mathrm{HLL}}, (\rho E)^{\mathrm{HLL}})^{\mathrm{T}} = \frac{\lambda_R w_R - \lambda_L w_L}{\lambda_R - \lambda_L} - \frac{f(w_R) - f(w_L)}{\lambda_R - \lambda_L}. \quad (45)$$

As a consequence, the HLL approximate Riemann solver (44) satisfies the consistency condition $\tilde{w}^{\mathrm{HLL}}(\cdot, w, w) = w$ and the following Lemma gives additional robustness properties of the state vector w^{HLL}.

Lemma 5 (Robustness properties of the HLL approximate Riemann solver)

Consider the Riemann problem for the Euler equations given by (2), (7)*. Assume the CFL condition* (12) *holds and let also assume the solution of the Riemann problem* $(\rho, \rho u, \rho E)^{\mathrm{T}}(x, \Delta t)$ *is in the set* Ω *for all* $x \in [\lambda_L \Delta t, \lambda_R \Delta t]$*. The following statements are satisfied:*

i) *an equivalent form of* w^{HLL} *is given by*

$$w^{\mathrm{HLL}} = \frac{1}{(\lambda_R - \lambda_L)\Delta t} \int_{\lambda_L \Delta t}^{\lambda_R \Delta t} w(x, \Delta t)\, \mathrm{d}x.$$

ii) $\rho^{\mathrm{HLL}} > 0, \quad (\rho E)^{\mathrm{HLL}} > 0,$
iii)$\eta(w^{\mathrm{HLL}}) \leq \eta^{\mathrm{HLL}}$.

Proof Integrating the equation (2) over the domain $[0, \Delta t] \times [-\Delta x/2, \Delta x/2]$ and using the initial condition (7), we have

$$
\begin{aligned}
0 &= \int_0^{\Delta t} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \big(\partial_t w + \partial_x f(w)\big)\, \mathrm{d}x\mathrm{d}t, \\
&= \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \big(w(x,\Delta t) - w_0(x)\big)\, \mathrm{d}x + \int_0^{\Delta t} \Big(f\big(w(\Delta x/2, t)\big) - f\big(w(-\Delta x/2, t)\big)\Big)\, \mathrm{d}t, \\
&= \int_{-\frac{\Delta x}{2}}^{\lambda_L \Delta t} w_L\, \mathrm{d}x + \int_{\lambda_L \Delta t}^{\lambda_R \Delta t} w(x,\Delta t)\, \mathrm{d}x + \int_{\lambda_R \Delta t}^{\frac{\Delta x}{2}} w_R\, \mathrm{d}x - \Big(\int_{-\frac{\Delta x}{2}}^{0} w_L\, \mathrm{d}x + \int_0^{\frac{\Delta x}{2}} w_R\, \mathrm{d}x \Big) \\
&\quad + \Delta t \big(f(w_R) - f(w_L)\big), \\
&= \int_{\lambda_L \Delta t}^{\lambda_R \Delta t} w(x,\Delta t)\, \mathrm{d}x + \lambda_L \Delta t w_L - \lambda_R \Delta t w_R + \Delta t\big(f(w_R) - f(w_L)\big).
\end{aligned} \tag{46}
$$

Using the above result in the definition of w^{HLL} given by (45), we deduce the formulation given by i).

For the inequalities ii), since the solution of the Riemann problem (2), (7) is assumed to belong to Ω, writing the first and the last component of w^{HLL} given by i), we have

$$
\rho^{\mathrm{HLL}} = \frac{1}{(\lambda_R - \lambda_L)\Delta t} \int_{\lambda_L \Delta t}^{\lambda_R \Delta t} \rho(x,\Delta t)\, \mathrm{d}x > 0,
$$

$$
(\rho E)^{\mathrm{HLL}} = \frac{1}{(\lambda_R - \lambda_L)\Delta t} \int_{\lambda_L \Delta t}^{\lambda_R \Delta t} (\rho E)(x,\Delta t)\, \mathrm{d}x > 0.
$$

Concerning the inequality iii), as $\lambda_L < \lambda_R$, the convexity of the entropy $w \mapsto \eta(w)$ associated to the Jensen inequality [17] gives

$$
\eta(w^{\mathrm{HLL}}) \le \frac{1}{(\lambda_R - \lambda_L)\Delta t} \int_{\lambda_L \Delta t}^{\lambda_R \Delta t} \eta\big(w(x,\Delta t)\big)\, \mathrm{d}x. \tag{47}
$$

Since $w(x,\Delta t)$ is the exact solution of the Riemann problem (2), (7), it verifies the entropy inequality (3). As a consequence, integrating this inequality (3) over the domain $[0,\Delta t] \times [-\Delta x/2, \Delta x/2]$, an analogous computation to (46) yields the following inequality:

$$
\int_{\lambda_L \Delta t}^{\lambda_R \Delta t} \eta\big(w(x,\Delta t)\big)\, \mathrm{d}x + \lambda_L \Delta t \eta(w_L) - \lambda_R \Delta t \eta(w_R) + \Delta t\big(G(w_R) - G(w_L)\big) \le 0.
$$

Finally, associating the above inequality to (47), we deduce

$$\begin{aligned}
\eta(w^{\mathrm{HLL}}) &= \eta\Big(\frac{1}{(\lambda_R-\lambda_L)\Delta t}\int_{\lambda_L\Delta t}^{\lambda_R\Delta t} w(x,\Delta t)\,\mathrm{d}x\Big),\\
&\le \frac{1}{(\lambda_R-\lambda_L)\Delta t}\int_{\lambda_L\Delta t}^{\lambda_R\Delta t} \eta\big(w(x,\Delta t)\big)\,\mathrm{d}x,\\
&\le \frac{\lambda_R\eta(w_R)-\lambda_L\eta(w_L)}{\lambda_R-\lambda_L}-\frac{G(w_R)-G(w_L)}{\lambda_R-\lambda_L}.
\end{aligned}$$

As the right hand side of this last inequality is the definition of η^{HLL} given by (13) that achieves the proof. □