Asymptotic preserving schemes for highly oscillating kinetic equations

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Abstract

Derivation of a micro-macro model which enables to design an AP scheme in the two-scale limit.

1 Introduction

The description of charged particles systems like plasmas or beams can be performed through different ways from a kinetic description to a fluid like interpretation. One choice could be the kinetic description which involves a distribution function $f = f(t, x, v)$ where $t \geq 0$ is time and $x, v \in \mathbb{R}^d$ (where $d$ is the dimension) denotes the spatial and velocity variables. Due to the high number of dimensions involved in kinetic models $(2d$ dimensions plus the time $t$), the numerical simulation of kinetic models is challenging. Moreover, since no relaxation term is considered, the unknown makes appear a lot of fine structures like vortices or filaments which is very difficult to capture at the numerical level.

We are interested here in the numerical study of collisionless kinetic equations which involve stiff transport terms. Considering the time evolution of the distribution function $f = f(t, x, v)$ with $x, v \in \mathbb{R}^d$, the kinetic equation we are interested in is a partial differential equation which can be written as follows

$$\partial_t f + v \cdot \nabla_x f + (E + E_{\text{app}}) \cdot \nabla_v f = 0,$$ \hspace{1cm} (1.1)

where $E = E(t, x)$ is the self-consistent electric field satisfying a Poisson equation

$$\nabla_x \cdot E = \int f dv,$$ \hspace{1cm} (1.2)

and $E_{\text{app}} = E_{\text{app}}(t, x)$ is an applied electric field. Under paraxial approximations (see [6, 7, 3, 9, 11]), the Vlasov-Poisson model (1.6)-(1.7) reduces to the following one dimensional axisymmetric Vlasov-Poisson system satisfied by $f(t, r, v_r)$ with $r \in \mathbb{R}_+$ and $v_r \in \mathbb{R}$

$$\partial_t f + \frac{v_r}{\varepsilon} \partial_r f + (E + E_{\text{app}}) \partial_{v_r} f = 0,$$ \hspace{1cm} (1.3)

where $E = E(t, r)$ is the self-consistent electric field satisfying a Poisson equation

$$\frac{1}{r} \partial_r (rE) = \int_{\mathbb{R}} f dv_r,$$ \hspace{1cm} (1.4)

and we consider an applied electric field $E_{\text{app}}$ which has the following form

$$E_{\text{app}}(t, r) = -\frac{r}{\varepsilon} + a \left( \frac{t}{\varepsilon} \right) r,$$ \hspace{1cm} (1.5)

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where $a$ is a given function (a so-called tension function).

In these standard notations (see \[6, 7, 3, 9, 11\]), $t \geq 0$ corresponds to the longitudinal position coordinate (the direction of propagation of the beam) and plays the role of time, $r \geq 0$ is the radial component of the position in the transverse plane to the propagation direction, $v_r$ is the corresponding velocity, $\varepsilon$ denotes the ratio between the characteristic length in the perpendicular and longitudinal directions of the beam. This system is defined for $r \geq 0$ but can be extended to $r \in \mathbb{R}$ by using the conventions $f(t, -r, -v_r) = f(t, r, v_r)$ and $E(t, -r) = -E(t, r)$. In the following we then consider the following Vlasov-Poisson system satisfied by $f(t, x, v)$ where $x, v \in \mathbb{R}$

$$
\partial_t f + \frac{v}{\varepsilon} \partial_x f + \left( E_{\text{tot}} - \frac{x}{\varepsilon} \right) \partial_v f = 0,
$$

where $E_{\text{tot}} = E_{\text{tot}}(t, x) = E_s(t, x) + E_a(t, x)$ is the total electric field composed of the self-consistent electric field $E_s$ satisfying a Poisson equation

$$
\frac{1}{x} \partial_x (xE_s) = \rho, \text{ for } x > 0 \text{ and } E_s(t, -x) = -E_s(t, x),
$$

with $\rho = \int_{\mathbb{R}} f dv$ and the applied electric field $E_a$ has the form

$$
E_a(t, x) = a \left( \frac{t}{\varepsilon} \right) x.
$$

The main purpose of this work is the study of stiff transport equations of type (1.6) in the limit $\varepsilon \to 0$. Contrary to relaxation type equations which involve diffusion or relaxation operator, the asymptotic behaviour $\varepsilon \to 0$ of stiff transport equations like (1.6) makes appear oscillations. In this context, the notion of two-scale convergence seems to be adapted ([1, 9, 12]. However, these asymptotic type models are valid only when $\varepsilon$ is uniformly small.

To overcome this domain of validity, an answer could be the design of asymptotic preserving schemes which, for a given set of numerical parameters, are valid for all $\varepsilon > 0$, and degenerates into the required asymptotic numerical model when $\varepsilon \to 0$. This is the main purpose of this work.

To do this, the strategy of micro-macro decomposition is used to derive an equivalent model which is a good candidate to design an asymptotic preserving scheme, according to previous works [10, 2, 4]. However this strategy has to be adapted to the case of highly oscillating case. First, according to the homogenization theory (see [1, 8, 12]), a change of variables is performed to transform the stiff part (which generates the highly oscillating behaviour) into a fast oscillating variable $\xi$. Then, a second time scale $\tau$ is introduced through a new variable of the unknown. Then, a new "augmented" kinetic equation on $F(t, \tau, \xi)$ is derived depending on the new phase space variables
(according to the change of variables), on the time \( t \) and on \( \tau \). This function \( F \) is related to the original equation (1.6) by the relation \( f(t, x, v) = F(t, t/\varepsilon, e^{t/\varepsilon M} \xi) \) where \((x, v)^T = e^{t/\varepsilon M} \xi\) denotes the change of variables. This equation satisfied by \( F \) makes appear a stiff term \((1/\varepsilon) \partial_\tau F\). Hence the micro-macro decomposition strategy can be applied directly: the function \( F \) is decomposed between a independent function of \( \tau \) (called \( G \)) and a function (called \( h \)) that contains all the other oscillations. Hence a coupled micro-macro model can be derived satisfied by \( G \) and \( h \) which is equivalent to the model satisfied by \( F \). Some links can be made with the strategy adopted in [5].

A numerical method has to be used for the approximation of the so-obtained micro-macro model. To that purpose, a semi-implicit numerical scheme is used. It can be easily proved that this numerical scheme is consistent and stable independently of \( \varepsilon > 0 \); moreover, this numerical scheme degenerates when \( \varepsilon \to 0 \) to a consistent discretization of the asymptotic two-scale model. As in previous works on the derivation of AP schemes using the micro-macro strategy, the numerical scheme is able to capture the corrective terms of order \( \varepsilon \).

The procedure is applied to different asymptotic which occur in magnetized plasma or plasma beam.

The rest of the paper is organized as follows. Plan

2 Filtering

Starting from (1.6)-(1.7)-(1.8), the first step consists in filtering the main oscillation of the system. To do that, we introduce the rotation matrix \( M \)

\[
M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

so that the model (1.6)-(1.7)-(1.8) rewrites

\[
\partial_t f + \frac{1}{\varepsilon} M \begin{pmatrix} x \\ v \end{pmatrix} \cdot \nabla_{x,v} f + E_{\text{tot}} \partial_v f = 0,
\]

where \( \nabla_{x,v} = (\partial_x \ \partial_v)^T \). Hence by applying the following change of variable

\[
\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = e^{-t/\varepsilon M} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} x \cos(t/\varepsilon) - v \sin(t/\varepsilon) \\ x \sin(t/\varepsilon) + v \cos(t/\varepsilon) \end{pmatrix},
\]

the function \( \tilde{f}(t, \xi) = f(t, x, v) \) satisfies

\[
\partial_t \tilde{f}(t, \xi) = \partial_t f(t, x, v) + \frac{1}{\varepsilon} M \begin{pmatrix} x \\ v \end{pmatrix} \cdot \nabla_{x,v} f(t, x, v) = -E_{\text{tot}} \partial_v f.
\]
But since $\nabla_{x,v} f(t,x,v) = \nabla_{x,v} \left[ \tilde{f}(t,e^{-t/\varepsilon M} \begin{pmatrix} x \\ v \end{pmatrix}) \right] = (e^{-t/\varepsilon M})^T \nabla_\xi \tilde{f}(t,\xi)$, the equation satisfied by $\tilde{f}(t,\xi)$ finally writes

$$\partial_t \tilde{f}(t,\xi) - A(t,t/\varepsilon,e^{t/\varepsilon M} \xi) \cdot \nabla_\xi \tilde{f}(t,\xi) = 0,$$

with

$$A(t,t/\varepsilon,e^{t/\varepsilon M} \xi) = -e^{-t/\varepsilon M} \begin{pmatrix} 0 \\ E_{tot}(t,\xi_1 \cos(t/\varepsilon) + \xi_2 \sin(t/\varepsilon)) \end{pmatrix}$$

and $E_{tot}(t,x) = E_s(t,x) + E_a(t,x)$ given by (1.7) and (1.8).

3 General formulation

We start from the following formulation satisfied by $u(t)$

$$\begin{cases} \dot{u} = A(t,t/\varepsilon,u), \\ u(t=0) = u_0, \end{cases}$$

where $A(t,t/\varepsilon,\cdot)$ is an operator acting on a $C$-Banach, periodic (of period $T$) with respect to the second variable, and $u_0$ is a given initial condition.

We introduce an extended function $U(t,\tau)$ which satisfies $U(t,t/\varepsilon) = u(t)$. The function $U$ then solves

$$\partial_t U + \frac{1}{\varepsilon} \partial_\tau U = A(t,\tau,U).$$

Let us remark that the initial condition $U(0,0) = u(0) = u_0$ is not sufficient to define $U(t,\tau)$ for all $t \geq 0$ and for all $\tau \in [0,T]$. However, every solution $U$ of (3.2) provides the same solution $u(t)$ of (3.1) using the relation $U(t,t/\varepsilon) = u(t)$.

3.1 Micro-macro decomposition

The main goal is to design a numerical scheme which is stable with respect to the small parameter $\varepsilon$ and which provides a consistent discretization of the asymptotic model. To do that, we adapt the micro-macro decomposition to (3.2). First, we decompose the solution $U$ as

$$U(t,\tau) = G(t) + h(t,\tau),$$

where $G$ is the projection of $U$ on the kernel of the operator $\partial_\tau$, i.e.

$$G(t) := (\Pi U)(t) = \frac{1}{T} \int_0^T U(t,\tau)d\tau.$$
By construction, we then have $\Pi h = 0$. Injecting this decomposition into (3.2) leads to
\[
\partial_t G + \partial_t h - A(t, \tau, G + h) = -\frac{1}{\varepsilon} \partial_\tau h.
\]
Averaging this last equation with respect to $\tau$ (or applying $\Pi$) and considering the difference between this last equation and its averaging provides the following micro-macro model satisfied by $(G, h)$
\[
\begin{aligned}
\partial_t G - (\Pi A)(t, G + h) &= 0, \\
\partial_t h - (I - \Pi) A(t, \tau, G + h) &= -\frac{1}{\varepsilon} \partial_\tau h.
\end{aligned}
\]
(3.3)

An asymptotic expansion in $\varepsilon$ can be performed to study the asymptotic behavior of the solution $U = G + h$. From the second equation of (3.3), we immediately have $h = \mathcal{O}(\varepsilon)$ since $\partial_\tau h = \varepsilon (I - \Pi) A(t, \tau, G + h) - \varepsilon \partial_\tau h$. Hence, the zero-th order limit model writes
\[
\partial_t G - (\Pi A)(t, G) = 0.
\]

Let us now consider the correction of order $\varepsilon$. As $\varepsilon$ goes to zero, we get
\[
h(t, \tau) = \varepsilon (I - \Pi) \int_0^\tau (I - \Pi) A(t, s, G(t)) ds + \mathcal{O}(\varepsilon^2),
\]
so that the first order limit model writes
\[
\partial_t G - (\Pi A)(t, G) - \varepsilon \Pi A \left( t, \tau, (I - \Pi) \int_0^\tau (I - \Pi) A(t, s, G) ds \right) = \mathcal{O}(\varepsilon^2).
\]
Up to the terms of $\mathcal{O}(\varepsilon^2)$, the original solution $U(t, \tau)$ is approximated by
\[
U(t, \tau) = G(t) + \varepsilon (I - \Pi) \int_0^\tau (I - \Pi) A(t, s, G(t)) ds + \mathcal{O}(\varepsilon^2),
\]
(3.4)
for all $t \geq 0$. Then a natural initial condition for (3.2) can be (3.4) evaluated at $t = 0$
\[
U(0, \tau) = G(0) + \varepsilon (I - \Pi) \int_0^\tau (I - \Pi) A(0, s, G(0)) ds
\]
\[
= u_0 + \varepsilon (I - \Pi) \int_0^\tau (I - \Pi) A(0, s, u_0) ds.
\]
3.2 Generalization to arbitrary order in $\varepsilon$

Here an improved micro-macro decomposition is proposed in order to take into account an arbitrary number of corrections. This decomposition writes

$$U(t, \tau) = G(t) + \sum_{i=1}^{k} \varepsilon^i G_i(t, \tau) + h(t, \tau),$$

where we have always $\Pi U = G$, but now $(I - \Pi)U = \sum_{i=1}^{k} \varepsilon^i G_i + h$. Injecting this decomposition in (3.2) and applying $\Pi$ leads to

$$\partial_t G - (\Pi A)(t, G) - \Pi A(t, \tau, \sum_{i=1}^{k} \varepsilon^i G_i + h) = 0.$$  (3.5)

Applying now $(I - \Pi)$ to (3.2) leads to

$$\partial_t h + \sum_{i=1}^{k} \varepsilon^i \partial_t G_i - (I - \Pi)A(t, \tau, \sum_{i=1}^{k} \varepsilon^i G_i + h) =$$

$$-\frac{1}{\varepsilon} \partial_\tau h - \sum_{i=0}^{k-1} \varepsilon^i \partial_\tau G_{i+1}. \quad (3.6)$$

A Chapman-Enskog expansion enables to express $G_i$ for $i = 1, \ldots, k$ as functions of $G$. Since $h = O(\varepsilon)$, we get from (3.6)

$$\begin{cases}
\partial_\tau G_1 = (I - \Pi)A(t, \tau, G), \\
\partial_t G_{i+1} = -\partial_t G_i + (I - \Pi)A(t, \tau, G_i), & \text{for } i = 1, \ldots, k - 1.
\end{cases} \quad (3.7)$$

The micro-macro model (3.5)-(3.6) can be simplified to

$$\begin{cases}
\partial_t G - (\Pi A)(t, G) - \Pi A(t, \tau, \sum_{i=1}^{k} \varepsilon^i G_i + h) = 0, \\
\partial_t h + \varepsilon^k \partial_t G_k - (I - \Pi)A(t, \tau, \varepsilon^k G_k + h) = -\frac{1}{\varepsilon} \partial_\tau h.
\end{cases} \quad (3.8)$$

In particular, as $\varepsilon$ goes to zero, the micro equation gives $\partial_\tau h = -\varepsilon \partial_t h + \varepsilon(I - \Pi)A(t, \tau, h) + O(\varepsilon^{k+1})$, which means that $h = O(\varepsilon^{k+1})$. In the macro equation, we then have the following $k$-th order limit model

$$\partial_t G - (\Pi A)(t, G) - \Pi A \left( t, \tau, \sum_{i=1}^{k} \varepsilon^i G_i \right) = O(\varepsilon^{k+1}).$$
4 Numerical scheme

4.1 Derivation of the micro-macro model

Starting from (2.1), and following the reasoning of the previous section, we introduce $F(t, \tau, \xi)$ which is periodic in $\tau$ such that $F(t, t/\varepsilon, \xi) = \tilde{f}(t, \xi)$ where $\tilde{f}$ solves (2.1)-(2.2). The equation satisfied by $F$ is

$$\partial_t F - A(t, \tau, e^{\tau M} \xi) \cdot \nabla_\xi F = -\frac{1}{\varepsilon} \partial_\tau F,$$

(4.1)

where $A(t, \tau, e^{\tau M} \xi)$ is given by

$$A(t, \tau, e^{\tau M} \xi) = -e^{\tau M} \left( \begin{array}{c} 0 \\ E_{\text{tot}}(t, \xi_1 \cos(\tau) + \xi_2 \sin(\tau)) \end{array} \right)$$

(4.2)

with $E_{\text{tot}} = E_s + E_a$.

The micro-macro decomposition of $F$ writes $F(t, \tau, \xi) = G(t, \xi) + h(t, \tau, \xi)$ and leads to the following micro-macro model

$$\begin{cases} 
\partial_t G - (\Pi A) \cdot \nabla_\xi G - \Pi (A \cdot \nabla_\xi h) = 0, \\
\partial_t h - (I - \Pi)(A \cdot \nabla_\xi G) - (I - \Pi)(A \cdot \nabla_\xi h) = -\frac{1}{\varepsilon} \partial_\tau h. 
\end{cases}$$

(4.3)

Taking into account higher order terms in $\varepsilon$, we get $F(t, \tau, \xi) = G(t, \xi) + \sum_{i=1}^k \varepsilon^i G_i(t, \tau, \xi) + h(t, \tau, \xi)$ and the corresponding model is

$$\begin{cases} 
\partial_t G - (\Pi A) \cdot \nabla_\xi G - \Pi \left( A \cdot \nabla_\xi \left( \sum_{i=1}^k \varepsilon^i G_i \right) \right) = 0, \\
\partial_t h + \varepsilon^k \partial_\tau G_k - \varepsilon^k (I - \Pi)(A \cdot \nabla_\xi G_k) - (I - \Pi)(A \cdot \nabla_\xi h) = -\frac{1}{\varepsilon} \partial_\tau h. 
\end{cases}$$

(4.4)

with the following relations between $G_i$ $i = 1, \ldots, k$ and $G$

$$\begin{cases} 
\partial_\tau G_1 = (I - \Pi)(A \cdot \nabla_\xi G), \\
\partial_\tau G_{i+1} = -\partial_\tau G_i + (I - \Pi)(A \cdot \nabla_\xi G_i), \text{ for } i = 1, \ldots k - 1. 
\end{cases}$$

(4.5)

The coupling with $E_s$ through the Poisson equation has to be considered. First, the density $\rho(t, \tau, x)$ is computed for each $\tau \in [0, T]$ as an integral of $F$

$$\rho(t, \tau, x) = \int_{\mathbb{R}} F(t, \tau, x \cos(\tau) + v \sin(\tau), x \sin(\tau) + v \cos(\tau)) dv,$$
which enables to compute the self-consistent electric field \( E_s(t, \tau, x) \) with (1.7). Then, \( E_s \) is evaluated in terms of the variables \( \xi \) using the relation \( x = \xi_1 \cos(\tau) + \xi_2 \sin(\tau) \).

### 4.2 Time discretization

#### 4.2.1 Unsplitting approach for (4.3)

As in [10, 2], the time discretization for (4.3) writes

\[
\left( I + \frac{\Delta t}{\varepsilon} \partial_\tau \right) h^{n+1} = h^n + \Delta t (I - \Pi)(A) \cdot \nabla_\xi G^n \Delta t(I - \Pi)(A) \cdot \nabla_\xi h^n.
\]

As \( \varepsilon \) goes to zero, we have

\[
h^{n+1} = \varepsilon (I - \Pi) \int_0^T (I - \Pi)(A) \cdot \nabla_\xi G^n + \mathcal{O}(\varepsilon^2),
\]

which injected in the macro equation discretized as

\[
G^{n+1} = G^n + \Delta t \Pi(A) \cdot \nabla_\xi G^n + \Delta t \Pi(A) \cdot \nabla_\xi h^{n+1},
\]

leads to the following

\[
G^{n+1} = G^n + \Delta t \Pi(A) \cdot \nabla_\xi G^n + \varepsilon \Delta t \Pi[A \cdot \nabla_\xi (I - \Pi) \int_0^T (I - \Pi)(A) \cdot \nabla_\xi G^n],
\]

which is a consistent discretization of the first order (in \( \varepsilon \)) limit model.

#### 4.2.2 Splitting approach for (4.3)

In this part, we focus on a splitting approach for (4.3). We split the operator \( A \cdot \nabla_\xi \) between \( A_1 \partial_{\xi_1} \) and \( A_2 \partial_{\xi_2} \). The time discretization then writes

- First step of the splitting
  - Advance \( h \): \( \partial_t h - (I - \Pi)(A_1 \partial_{\xi_1}(G + h)) = -\frac{1}{2\varepsilon} \partial_\tau h \)

\[
\left( I + \frac{\Delta t}{2\varepsilon} \partial_\tau \right) h^* = h^n + \Delta t (I - \Pi)(A_1 \partial_{\xi_1}(G^n + h^n))
\]

- Advance \( G \): \( \partial_t G - \Pi(A_1 \partial_{\xi_1}(G + h)) = 0 \)

\[
G^* = G^n - \Delta t \Pi(A_1 \partial_{\xi_1}(G^n + h^*))
\]

- Second step of the splitting
– Advance $h$: $\partial_t h - (I - \Pi)(A_2 \partial_{\xi_2}(G + h)) = -\frac{1}{2\varepsilon} \partial_t h$

$$\left( I + \frac{\Delta t}{2\varepsilon} \partial_t \right) h^{n+1} = h^* + \Delta t(I - \Pi)(A_2 \partial_{\xi_2}(G^* + h^*))$$

– Advance $G$: $\partial_t G - (I - \Pi)(A_2 \partial_{\xi_2}(G + h)) = 0$:

$$G^{n+1} = G^* + \Delta t\Pi(A_2 \partial_{\xi_2}(G^* + h^{n+1}))$$

The first step gives, as $\varepsilon$ goes to zero $h^* = 2\varepsilon(I - \Pi) \int_0^\tau (I - \Pi)(A_1 \partial_{\xi_1} G^n) ds$ and $G^* = G^n - \Delta t\Pi(A_1 \partial_{\xi_1}(G^n + 2\varepsilon(I - \Pi) \int_0^\tau (I - \Pi)(A_1 \partial_{\xi_1} G^n) ds))$. With the second step, we get $h^{n+1} = 2\varepsilon(I - \Pi) \int_0^\tau (I - \Pi)(A_2 \partial_{\xi_2} G^n) ds$ and

$$G^{n+1} = G^n + \Delta t\Pi(A_2 \partial_{\xi_2}(G^* + 2\varepsilon(I - \Pi) \int_0^\tau (I - \Pi)(A_2 \partial_{\xi_2} G^n) ds))$$

$$= G^n + \Delta t\Pi \left[ A_1 \partial_{\xi_1}(G^n + 2\varepsilon(I - \Pi) \int_0^\tau (I - \Pi)(A_1 \partial_{\xi_1} G^n) ds) \right]$$

$$+ \Delta t\Pi \left[ A_2 \partial_{\xi_2}(G^n + 2\varepsilon(I - \Pi) \int_0^\tau (I - \Pi)(A_2 \partial_{\xi_2} G^n) ds) \right] + \mathcal{O}(\Delta t^2)$$

$$= G^n + \Delta t\Pi A \cdot \nabla G^n + 2\varepsilon \Delta t\Pi \left[ A_1 \partial_{\xi_1}((I - \Pi) \int_0^\tau (I - \Pi)(A_1 \partial_{\xi_1} G^n) ds) \right]$$

$$+ 2\varepsilon \Delta t\Pi \left[ A_2 \partial_{\xi_2}((I - \Pi) \int_0^\tau (I - \Pi)(A_2 \partial_{\xi_2} G^n) ds) \right].$$

We can observe that the time discretization of the first order term in $\varepsilon$ is not consistent with the limit model. This motivates the use of the model (4.4) with $k = 1$ to get a AP scheme up to the first term in $\varepsilon$.

4.2.3 Splitting approach for (4.4)

First order splitting

In this part, we focus on a splitting approach for (4.4). We split the operator $A \cdot \nabla \xi$ between $A_1 \partial_{\xi_1}$ and $A_2 \partial_{\xi_2}$. The time discretization then writes

• First step of the splitting

  – Advance $G$: $\partial_t G - \Pi(A_1 \partial_{\xi_1}(G + \varepsilon G_1 + h)) = 0$:

    $$G^* = G^n - \Delta t\Pi(A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n + h^n))$$
– Advance \( G_1: \) \(-\partial_r G_1 = (I - \Pi)(A \cdot \nabla \xi G)\)

\[
G_1^* = \varepsilon(I - \Pi) \int_0^T (I - \Pi)(A \cdot \nabla \xi)G^*.
\]

– Advance \( h: \) \(\partial_t h + \varepsilon \partial_t G_1 - (I - \Pi)(A_1 \partial_{\xi_1}(\varepsilon G_1 + h)) = -\frac{1}{2\varepsilon} \partial_r h
\)

\[
\left(I + \frac{\Delta t}{2\varepsilon} \partial_r\right) h^* = h^n + \Delta t(I - \Pi)(A_1 \partial_{\xi_1}(\varepsilon G^n_1 + h^n)) + \varepsilon(G^*_1 - G^n_1).
\]

• Second step of the splitting

– Advance \( G: \) \(\partial_t G - \Pi(A_2 \partial_{\xi_2}(G + \varepsilon G_1 + h)) = 0:\)

\[
G^{n+1} = G^* - \Delta t \Pi(A_2 \partial_{\xi_2}(G^* + \varepsilon G^n_1 + h^*))
\]

– Advance \( G_1 \) (optionnel): \(-\partial_r G_1 = (I - \Pi)(A \cdot \nabla \xi G)\)

\[
G_1^{n+1} = \varepsilon(I - \Pi) \int_0^T (I - \Pi)(A \cdot \nabla \xi G^{n+1}).
\]

– Advance \( h: \) \(\partial_t h + \varepsilon \partial_t G_1 - (I - \Pi)(A_2 \partial_{\xi_2}(\varepsilon G_1 + h)) = -\frac{1}{2\varepsilon} \partial_r h
\)

\[
\left(I + \frac{\Delta t}{2\varepsilon} \partial_r\right) h^{n+1} = h^* + \Delta t(I - \Pi)(A_2 \partial_{\xi_2}(\varepsilon G^n_1 + h^n)) + \varepsilon(G^{n+1}_1 - G^*_1).
\]

The first step gives, as \(\varepsilon\) goes to zero \(h^* = \mathcal{O}(\varepsilon^2)\) and \(G^* = G^n - \Delta t \Pi(A_1 \partial_{\xi_1}(G^n + \varepsilon G^n_1))\).

With the second step, we also get (neglecting the second order term in \(\varepsilon\)) \(h^{n+1} = \mathcal{O}(\varepsilon^2)\) and then

\[
G^{n+1} = \begin{align*}
G^* + \Delta t \Pi(A_2 \partial_{\xi_2}(G^* + \varepsilon G^n_1)) \\
G^n + \Delta t \Pi [A_1 \partial_{\xi_1}(G^n + \varepsilon G^n_1)] + \Delta t \Pi [A_2 \partial_{\xi_2}(G^n + \varepsilon G^n_1)] + \mathcal{O}(\Delta t^2)
\end{align*}
\]

which is consistent with a time discretization of the first order limit model.

**Second order splitting**

1. First step of the splitting
• Advance $G$: $\partial_t G - \Pi(A_1 \partial_{\xi_1}(G + \varepsilon G_1 + h)) = 0$:

$$G^* = G^n - \frac{\Delta t}{2} \Pi(A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n + h^n))$$

• Advance $G_1$: $-\partial_r G_1 = (I - \Pi)(A \cdot \nabla \xi G)$

$$G_1^* = \varepsilon (I - \Pi) \int_0^T (I - \Pi)(A \cdot \nabla \xi) G^*.$$ 

• Advance $h$: $\partial_t h + \varepsilon \partial_t G_1 - (I - \Pi)(A_2 \partial_{\xi_2}(\varepsilon G_1 + h)) = -\frac{1}{2\varepsilon} \partial_r h$

$$\left(I + \Delta t \frac{\partial}{4\varepsilon} \partial_r \right) h^* = h^n + \frac{\Delta t}{2}(I - \Pi)(A_1 \partial_{\xi_1}(\varepsilon G_1^n + h^n)) + \varepsilon(G_1^* - G_1^n).$$

2. Second step of the splitting

• Advance $G$: $\partial_t G - \Pi(A_2 \partial_{\xi_2}(G + \varepsilon G_1 + h)) = 0$:

$$G^{n+1} = G^* - \Delta t \Pi(A_2 \partial_{\xi_2}(G^* + \varepsilon G_1^* + h^*))$$

• Advance $G_1$ (optionnel): $-\partial_r G_1 = (I - \Pi)(A \cdot \nabla \xi G)$

$$G_1^{n+1} = \varepsilon (I - \Pi) \int_0^T (I - \Pi)(A \cdot \nabla \xi G^{n+1}).$$

• Advance $h$: $\partial_t h + \varepsilon \partial_t G_1 - (I - \Pi)(A_2 \partial_{\xi_2}(\varepsilon G_1 + h)) = -\frac{1}{2\varepsilon} \partial_r h$

$$\left(I + \Delta t \frac{\partial}{2\varepsilon} \partial_r \right) h^{n+1} = h^n + \Delta t(I - \Pi)(A_2 \partial_{\xi_2}(\varepsilon G_1^* + h^*)) + \varepsilon(G_1^{n+1} - G_1^*).$$

3. Third step of the splitting

• Advance $G$: $\partial_t G - \Pi(A_1 \partial_{\xi_1}(G + \varepsilon G_1 + h)) = 0$:

$$G^{n+1} = G^{**} - \frac{\Delta t}{2} \Pi(A_1 \partial_{\xi_1}(G^{**} + \varepsilon G_1^{**} + h^{**}))$$

• Advance $G_1$: $-\partial_r G_1 = (I - \Pi)(A \cdot \nabla \xi G)$

$$G_1^{n+1} = \varepsilon (I - \Pi) \int_0^T (I - \Pi)(A \cdot \nabla \xi) G^{n+1}.$$
• Advance $h$: $\partial_t h + \varepsilon \partial_t G_1 - (I - \Pi)(A_1 \partial_{\xi_1}(\varepsilon G_1 + h)) = \frac{-1}{2\varepsilon} \partial_r h$

$$
\left( I + \frac{\Delta t}{4\varepsilon} \partial_r \right) h^{n+1} = h^{**} + \frac{\Delta t}{2}(I - \Pi)(A_1 \partial_{\xi_1}(\varepsilon G_1^{**} + h^{**})) + \varepsilon((G_1^{n+1} - G_1^{**})).
$$

The three step gives, as $\varepsilon$ goes to zero $h^* = h^{**} = h^{n+1} = \mathcal{O}(\varepsilon^2)$ and $G^* = G^n - \frac{\Delta t}{2}\Pi(A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n))$. With the second step, we also get

$$
G^{n+1} = G^{**} + \frac{\Delta t}{2}\Pi(A_1 \partial_{\xi_1}(G^{**} + \varepsilon G_1^{**}))
$$

$$
= G^* + \Delta t\Pi[A_2 \partial_{\xi_2}(G^* + \varepsilon G_1^*)] + \frac{\Delta t}{2}\Pi[A_1 \partial_{\xi_1}(G^{**} + \varepsilon G_1^{**})]
$$

$$
= G^n + \frac{\Delta t}{2}\Pi[A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n)] + \Delta t\Pi\left[A_2 \partial_{\xi_2}(G^n + \varepsilon G_1^n) + \frac{\Delta t}{2}\Pi[A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n)] + \varepsilon G_1^*ight]
$$

$$
+ \frac{\Delta t}{2}\Pi A_1 \partial_{\xi_1}[G^n + \frac{\Delta t}{2}\Pi[A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n)] + \Delta t\Pi(A_2 \partial_{\xi_2}(G^n + \varepsilon G_1^n)) + \varepsilon G_1^*]
$$

$$
= G^n + \frac{\Delta t}{2}\Pi[A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n)] + \Delta t\Pi[A_2 \partial_{\xi_2}(G^n + \varepsilon G_1^n)] + \frac{\Delta t^2}{2}\Pi A_2 \partial_{\xi_2} \Pi[A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n)]
$$

$$
+ \frac{\Delta t}{2}\Pi A_1 \partial_{\xi_1}[G^n + \frac{\Delta t}{2}\Pi[A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n)] + \Delta t\Pi(A_2 \partial_{\xi_2}(G^n + \varepsilon G_1^n)) + \varepsilon G_1^*]
$$

$$
= G^n + \frac{\Delta t}{2}\Pi[A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n)] + \Delta t\Pi[A_2 \partial_{\xi_2}(G^n + \varepsilon G_1^n)] + \frac{\Delta t^2}{2}\Pi A_2 \partial_{\xi_2} \Pi[A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n)]
$$

$$
+ \frac{\Delta t}{2}\Pi A_1 \partial_{\xi_1}[G^n + \frac{\Delta t}{2}\Pi[A_1 \partial_{\xi_1}(G^n + \varepsilon G_1^n)] + \Delta t\Pi(A_2 \partial_{\xi_2}(G^n + \varepsilon G_1^n)) + \varepsilon G_1^*]
$$

which is consistent with Strang splitting time discretization of the first order limit model (...)

### 4.3 Phase-space discretization

**Discretization $\xi_1, \xi_2$**

In this section, we detail the numerical we use for the phase space discretization of (4.3). As precised above, the use of a splitting procedure enables to reduce the discretization to a problem of the form

$$
\partial_t G - \Pi A_1 \partial_{\xi_1}(G + h) = 0,
$$
and
\[ \partial_t h - (I - \Pi) A_1 \partial_{\xi_1} (G + h) = \frac{1}{2\varepsilon} \partial_{\tau} h. \]

A Lax-Wendroff method is employed to that purpose. Let us recall this method in our context. Introducing a mesh for the $\xi_1$ direction: $(\xi_1)_i, i = 0, \ldots, N$, with a uniform step $\Delta \xi_1$ together with a time discretization $t^n = n \Delta t$. The quantity $G^n_i$ denotes an approximation of $G(t^n, (\xi_1)_i, \xi_2)$ (the same notation occurs for $h$ and the vector field $A$). The notation $G^{n+1/2}_{i+1/2}$ can be viewed as an approximation of $G$ at time $t^{n+1/2} = (n + 1/2) \Delta t$ at the position $(\xi_1)_{i+1/2} = 1/2[(\xi_1)_i + (\xi_1)_{i+1}]$.

The first step consists in a Lax-Friedrichs numerical scheme
\[
\begin{cases}
G^{n+1/2}_{i+1/2} = \frac{G^n_i + G^n_{i+1}}{2} + \frac{\Delta t}{2\Delta \xi_1} \Pi \left[ A_{1,i+1}(G^n_{i+1} + h^n_{i+1}) - A_{1,i}(G^n_i + h^n_i) \right], \\
h^{n+1/2}_{i+1/2} = \frac{h^n_i + h^n_{i+1}}{2} + \frac{\Delta t}{\varepsilon \Delta \xi_1} (I - \Pi) \left[ A_{1,i+1}(G^n_{i+1} + h^n_{i+1}) - A_{1,i}(G^n_i + h^n_i) \right] - \frac{\Delta t}{2\varepsilon} \partial_{\tau} (h^{n+1/2}_i + h^n_i),
\end{cases}
\]

whereas the second step writes
\[
\begin{cases}
G^{n+1} = G^n_i + \frac{\Delta t}{2\Delta \xi_1} \Pi \left[ A_{1,i+1/2}(G^{n+1/2}_{i+1/2} + h^{n+1/2}_{i+1/2}) - A_{1,i-1/2}(G^{n+1/2}_{i-1/2} + h^{n+1/2}_{i-1/2}) \right], \\
h^{n+1} = h^n_i + \frac{\Delta t}{\varepsilon \Delta \xi_1} (I - \Pi) \left[ A_{1,i+1/2}(G^{n+1/2}_{i+1/2} + h^{n+1/2}_{i+1/2}) - A_{1,i-1/2}(G^{n+1/2}_{i-1/2} + h^{n+1/2}_{i-1/2}) \right] - \frac{\Delta t}{2\varepsilon} \partial_{\tau} (h^{n+1}_i + h^n_i),
\end{cases}
\]

The same is performed in the $\xi_2$ direction which, combined with a Strang splitting, enables to achieve second order in time and phase space.

**Discretization $\tau$**

Since the $\tau$ direction is periodic, Fourier decomposition is used. Hence, denoting by $\mathcal{F}(h)(k)$ the Fourier transform of $h(\tau)$ and using the relation $\mathcal{F}(\partial_{\tau} h) = ik \mathcal{F}(h)$, the second step of (4.6) then writes
\[
\left(1 + \frac{ik\Delta t}{4\varepsilon}\right) \mathcal{F}(h^{n+1/2}_{i+1/2})(k) = \\
\mathcal{F} \left( \frac{h^n_i + h^n_{i+1}}{2} + \frac{\Delta t}{2\Delta \xi_1} (I - \Pi) \left[ A_{1,i+1}(G^n_{i+1} + h^n_{i+1}) - A_{1,i}(G^n_i + h^n_i) \right] \right)(k).
\]
The inversion of the operator \(-\frac{\partial}{\partial \tau}\) is then performed easily using fast Fourier transform. In the same spirit, the second step of (4.7) is solved as

\[
(1 + \frac{i k \Delta t}{2 \epsilon}) \mathcal{F}(h_i^{n+1})(k) = -\frac{i k \Delta t}{2 \epsilon} \mathcal{F}(h_i^n)(k) + \mathcal{F}\left(h_i^n + \frac{\Delta t}{\Delta \xi_1} (I - \Pi) \left[A_{1,i+1/2}(G_{i+1/2}^{n+1/2} + h_{i+1/2}^{n+1/2}) - A_{1,i-1/2}(G_{i-1/2}^{n+1/2} + h_{i-1/2}^{n+1/2})\right]\right)(k)
\]

4.4 Computation of high order terms for three examples

As in [9], we consider two examples for \(E_a(t, \tau, Y)\) for which the limit model is explicit:

- \(E_a(t, \tau, \xi) = \cos(\tau)\xi_1,\)
- \(E_a(t, \tau, \xi) = \cos^2(n\tau)\xi_1, \quad n \geq 2,\)
- \(E_a(t, \tau, \xi) = \xi_1.\)

**First case:** \(E_a(t, \tau, \xi) = \cos(\tau)\xi_1.\) In this case, the limit model is \(\partial_t G = 0.\) Indeed, following (4.2) with \(E_s = 0\) and \(E_a(t, \tau, \xi) = \cos(\tau)\xi_1,\) we have

\[
A(t, \tau, e^{\tau M} \xi) = \begin{pmatrix} -\sin(\tau) \cos(\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau) \\ \cos(\tau) \cos(\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau) \end{pmatrix} = \begin{pmatrix} A_1(t, \tau, e^{\tau M} \xi) \\ A_2(t, \tau, e^{\tau M} \xi) \end{pmatrix},
\]

and the limit model writes

\(\partial_t G - \Pi(A) \cdot \nabla_\xi G = 0.\)

As performed in [9], we have \(\Pi(A_1) = \Pi(A_2) = 0.\) Let us now compute the corrective term \(G_1.\) It is given by

\(\partial_t G_1 = (I - \Pi)A \cdot \nabla_\xi G,\)

so that we have \(F(t, \tau, \xi) = G(t, \xi) + \epsilon G_1(t, \tau, \xi).\) In the macro model, we then have to compute the term

\[
\Pi(A \cdot G_1) = \Pi \left( A \cdot \nabla_\xi \left[ (I - \Pi) \int_0^\tau (I - \Pi) A \cdot \nabla_\xi G \right] \right).
\]
\[ \Pi(g \cdot \nabla F_1) = -\varepsilon \{ D_{1,2}, G \} = \varepsilon (\partial_{\xi_2} D_{1,2} \partial_{\xi_1} G - \partial_{\xi_1} D_{1,2} \partial_{\xi_2} G) \text{ with } D_{1,2} = \Pi(g \int_0^\tau g_1 d\sigma). \]

Let us compute \( D_{1,2} \)

\[
D_{1,2} = \frac{1}{2\pi} \int_0^{2\pi} (\xi_1 \cos^3 \tau + \xi_2 \cos^2 \tau \sin \tau) [\xi_1/3(\cos^3 \sigma - 1) - \xi_2/3(\sin^3 \sigma)]
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\xi_2^2}{3}(\cos^6 \tau - \cos^3 \tau) - \xi_1 \xi_2/3 \cos^3 \tau \sin^3 \tau \right.
\]
\[
+ \xi_1 \xi_2/3(\cos^5 \tau \sin \tau - \cos^2 \tau \sin^2 \tau) - \xi_2^2/3 \cos^2 \tau \sin^4 \tau) \right) d\tau
\]
\[
= \frac{\xi_2^2}{3}(5/16) + 0 + 0 - \frac{\xi_2^2}{3}(1/16)
\]
\[
= \frac{5 \xi_2^2}{48} + \frac{\xi_2^2}{48}.
\]

Then, the corrective term is \( \{ D_{1,2}, G \} = 5 \xi_1/24 \partial_{\xi_2} G - \xi_2/24 \partial_{\xi_1} G \) so that the first order model is

\[ \partial_t G + \varepsilon \frac{\xi_2}{24} \partial_{\xi_1} G - \varepsilon \frac{5 \xi_1}{24} \partial_{\xi_2} G = 0. \]

**Second case:**

\[ E_a(t, \tau, \xi) = \cos^2(2\tau)\xi_1. \]

In the second case, we have for \( n = 2 \)

\[ A(t, \tau, \xi) = -\left( \begin{array}{c}
- \sin(\tau) \cos^2(2\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau) \\
\cos(\tau) \cos^2(2\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau)
\end{array} \right), \]

and the limit model is \( \partial_t G - (\xi_2/4) \partial_{\xi_1} G + (\xi_1/4) \partial_{\xi_2} G = 0. \) Indeed, if we consider \( n = 2 \), we have

\[
\Pi(A_1)(t, \xi) = \frac{1}{2\pi} \int_0^{2\pi} \sin(\tau) \cos^2(2\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau)
\]
\[
= \frac{\xi_1}{2\pi} \int_0^{2\pi} \sin(2\tau) \cos^2(2\tau) d\tau + \frac{\xi_2}{2\pi} \int_0^{2\pi} \sin^2(\tau) \cos^2(2\tau) d\tau
\]
\[
= \frac{\xi_2}{4},
\]

and

\[
\Pi(A_2)(t, \xi) = -\frac{1}{2\pi} \int_0^{2\pi} \cos(\tau) \cos^2(2\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau)
\]
\[
= -\xi_1 \frac{1}{2\pi} \int_0^{2\pi} \cos^2(\tau) \cos^2(2\tau) d\tau - \frac{\xi_2}{2\pi} \frac{1}{2\pi} \int_0^{2\pi} \sin(2\tau) \cos^2(2\tau) d\tau
\]
\[
= -\xi_1/4.
\]
The corrective term needs $D_{1,2} = \Pi((I - \Pi)(g_2) \int_0^\tau (I - \Pi)(g_1) d\sigma)$

\[
D_{1,2} = \frac{1}{2\pi} \int_0^{2\pi} (g_2 - \xi_1/4) \int_0^\tau [g_1(\sigma) + \xi_2/4] d\sigma \\
= \frac{1}{2\pi} \int_0^{2\pi} (\xi_1 \cos^2 \tau \cos^2(2\tau) + \xi_2/2 \sin(2\tau) \cos^2(2\tau) - \xi_1/4) \\
\int_0^\tau (-\xi_1/2 \sin(2\sigma) \cos^2(2\sigma) - \xi_2 \sin^2 \sigma \cos^2(2\sigma) + \xi_2/4) d\sigma \\
= \frac{1}{2\pi} \int_0^{2\pi} (\xi_1 \cos^2 \tau \cos^2(2\tau) + \xi_2/2 \sin(2\tau) \cos^2(2\tau) - \xi_1/4) \\
(-\xi_1/12(1 - \cos^3(2\tau)) - \xi_2(1/16 \sin(4\tau) - 3/16 \sin(2\tau) - 1/48 \sin(6\tau) + \tau/4) + \xi_2/4\tau) \\
= \frac{1}{2\pi} \int_0^{2\pi} (\xi_1 \cos^2 \tau \cos^2(2\tau) + \xi_2/2 \sin(2\tau) \cos^2(2\tau) - \xi_1/4) \\
(-\xi_1/12 + \xi_1/12 \cos^3(2\tau) - \xi_2/16 \sin(4\tau) + 3\xi_2/16 \sin(2\tau) + \xi_2/48 \sin(6\tau)) \\
= \frac{1}{2\pi} \int_0^{2\pi} \xi_1^2/12 (-\cos^2 \tau \cos^2(2\tau) + \cos^2 \tau \cos^2(2\tau) + 1/4 - 1/4 \cos^3(2\tau)) \\
+\xi_2^2(-1/32 \sin(2\tau) \sin(4\tau) \cos^2(2\tau) + 3/32 \sin^2(2\tau) \cos^2(2\tau) + 1/92 \sin(2\tau) \sin(6\tau) \cos^2(2\tau)) \\
+\xi_1\xi_2(-1/16 \cos^2 \tau \cos^2(2\tau) \sin(4\tau) + 3/16 \cos^2 \tau \cos^2(2\tau) \sin(2\tau) \\
+1/48 \cos^2 \tau \cos^2(2\tau) \sin(6\tau) - 1/24 \sin(2\tau) \cos^2(2\tau) + 1/24 \sin(2\tau) \cos^2(2\tau) \cos^2(2\tau)) \\
= \xi_1^2/12(-1/4 + 5/32 + 1/4) + \xi_2^2(3/(32 \times 8) + 1/(92 \times 8)) \\
= \frac{5\xi_1^2}{384} + \frac{77\xi_2^2}{5888}.
\]

Then, the corrective term is $\{D_{1,2}, G\} = (5\xi_1/192)\partial_{\xi_2} G - (77\xi_2/2944)\partial_{\xi_1} G$. And the first order model writes

\[
\partial_t G + \left[ -\frac{\xi_2}{4} + \varepsilon \frac{77\xi_2}{2944} \right] \partial_{\xi_1} G + \left[ \frac{\xi_1}{4} - \varepsilon \frac{5\xi_1}{192} \right] \partial_{\xi_2} G = 0
\]

**Third case:** $E(t, \tau, \xi) = \xi_1$. We consider here a simple case for which we detail the computations relative to the high order terms. First, the vector field $A$ is

\[
A(t, \tau, \xi) = -\begin{pmatrix} -\sin(\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau) \\ \cos(\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau) \end{pmatrix},
\]

and the average is

\[
-\Pi(A_1) = -\frac{1}{2\pi} \int_0^{2\pi} \sin(\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau) = -\xi_2/2,
\]
and

\[-\Pi(A_2) = \frac{1}{2\pi} \int_0^{2\pi} \cos(\tau) (\xi_1 \cos \tau + \xi_2 \sin \tau) = \xi_1/2.\]

The limit model is

\[\partial_t G - \frac{\xi_2}{2} \partial_{\xi_1} G + \frac{\xi_1}{2} \partial_{\xi_2} G = 0.\]

The corrective term needs

\[D_{1,2} = \Pi((I - \Pi)(g_2) \int_0^\tau (I - \Pi)(g_1) d\sigma)\]

\[= \frac{1}{2\pi} \int_0^{2\pi} (g_2 - \xi_1/2) \int_0^\tau [g_1(\sigma) + \xi_2/2] d\sigma d\tau\]

\[= \frac{1}{2\pi} \int_0^{2\pi} (\xi_1 \cos^2 \tau + \xi_2/2 \sin(2\tau) - \xi_1/2)\]

\[\int_0^\tau (-\xi_1/2 \sin(2\sigma) - \xi_2 \sin^2 \sigma + \xi_2/2) d\sigma d\tau\]

\[= \frac{1}{2\pi} \int_0^{2\pi} (\xi_1 \cos^2 \tau + \xi_2/2 \sin(2\tau) - \xi_1/2)\]

\[(\xi_1/4 \cos(2\tau) + \xi_2/4 \sin(2\tau)) d\tau\]

\[= \frac{1}{2\pi} \int_0^{2\pi} (1/4) [\xi_1^2 \cos(2\tau)(\cos^2 \tau - 1/2) + \xi_2^2/2 \sin^2(2\tau)\]

\[+ \xi_1 \xi_2 \sin(2\tau)(\cos^2 \tau - 1/2) + 1/2 \sin(2\tau) \cos(2\tau)] d\tau\]

\[= \xi_1^2/16 + \xi_2^2/16.\]

The first order two-scale model is

\[\partial_t G + \left(\frac{-\xi_2 + \varepsilon \xi_2/4}{2}\right) \partial_{\xi_1} G + \frac{\xi_1 - \varepsilon \xi_2/4}{2} \partial_{\xi_2} G = 0.\]

### 5 Appendix

Et puis, calcul des termes d’ordre 1 en $\varepsilon$

- calcul générique et expliciter les termes
- lien avec Perko
- preuve structure hamiltonienne
5.1 Equivalence with the original model

**Proposition 5.1** (i) If $U = U(t, \tau)$ is a solution of (3.2) with the initial data $U(t = 0, \tau)$, then $(G, h)$ is a solution of (3.3) with the initial data

$$G(t = 0) = (\Pi U)(t = 0), \quad h(t = 0, \tau) = (I - \Pi)U(t = 0, \tau). \quad (5.1)$$

(ii) Conversely, if $(G, h)$ is a solution of (3.3) with the initial data (5.1), then $(\Pi h) = 0$ and $U(t, \tau) = G(t) + h(t, \tau)$ is a solution of (3.2).

5.2 Divergence free

It is interesting to remark that $\nabla_\xi \cdot A(t, \tau, \xi) = 0$, assuming the initial vector field $a$ is divergence free $\nabla_X \cdot a(t, X) = 0$ where $X = (x, v)^T$. We then rewrite the initial kinetic equation as $\partial_t f + \nabla_X \cdot (af) = 0$. Hence,

$$\nabla_\xi \cdot A(\xi) = \nabla_\xi \cdot [e^{-\tau M}a(e^{\tau M} \xi)]$$

$$= \sum_{i,j=1}^{N} \partial_{\xi_i}[(e^{-\tau M})_{i,j}(a(e^{\tau M} \xi))_{j}]$$

$$= \sum_{i,j=1}^{N} (e^{-\tau M})_{i,j} \partial_{X_i}[(a(e^{\tau M} \xi))_{j}]$$

$$= \sum_{i,j,k=1}^{N} (e^{-\tau M})_{i,j}(\partial_{X_k}a_j(e^{\tau M} \xi))(e^{\tau M})_{k,i}$$

$$= \text{tr}(e^{-\tau M}(\nabla_X a)e^{\tau M})$$

$$= \text{tr}(\nabla_X a)$$

$$= \nabla_X \cdot a = 0,$$

where the notation $(\nabla_X a)_{j,k} = \partial_{X_k}a_j$ has been used.

5.3 High order terms and Hamiltonian structure

**First order term**

The first corrective term $G_1$ writes

$$\partial_\tau G_1 = (I - \Pi)(A \cdot \nabla_\xi G),$$
and its contribution in the limit model is \( \Pi[A \cdot \nabla_\xi G_1] \). Using the notation \( L = \partial_\tau \), we have \( G_1 = L^{-1}(I - \Pi)(A \cdot \nabla_\xi G) \), so that the term \( \Pi[A \cdot \nabla_\xi G_1] \) becomes

\[
\Pi[A \cdot \nabla_\xi G_1](t, \xi) = \Pi[\nabla_\xi \cdot (AG_1)]
\]

\[
\frac{1}{2\pi} \int_0^{2\pi} \nabla_\xi \cdot (AG_1) d\tau = \frac{1}{2\pi} \nabla_\xi \cdot \int_0^{2\pi} A(t, \tau, \xi)G_1(t, \tau, \xi) d\tau
\]

\[
= \frac{1}{2\pi} \nabla_\xi \cdot \int_0^{2\pi} A(t, \tau, \xi)L^{-1}(I - \Pi)(A(t, \tau, \xi) \cdot \nabla_\xi G(t, \xi)) d\tau
\]

\[
= \frac{1}{2\pi} \nabla_\xi \cdot \int_0^{2\pi} [(I - \Pi)A(t, \tau, \xi)]L^{-1} [(I - \Pi)(A(t, \tau, \xi) \cdot \nabla_\xi G(t, \xi))] d\tau
\]

\[
= \frac{1}{2\pi} \sum_{i,j=1}^{2} \partial_{\xi_i} \int_0^{2\pi} [(I - \Pi)A_i(t, \tau, \xi)]L^{-1} [(I - \Pi)(A_j(t, \tau, \xi) \partial_{\xi_j} G(t, \xi))] d\tau
\]

\[
= \nabla_\xi \cdot (D\nabla_\xi G),
\]

where \( D \) is a 2 \times 2 diffusion matrix of components

\[
D_{i,j} = \frac{1}{2\pi} \int_0^{2\pi} (I - \Pi)A_iL^{-1}[(I - \Pi)A_j]d\tau, \quad i, j = 1, \ldots, 2.
\]

Diagonal terms vanish since \( L \) is antisymmetric:

\[
D_{i,i} = \frac{1}{2\pi} \int_0^{2\pi} (I - \Pi)A_iL^{-1}[(I - \Pi)A_i]d\tau = -\frac{1}{2\pi} \int_0^{2\pi} L^{-1}[(I - \Pi)A_i](I - \Pi)A_i d\tau = 0, \quad i = 1, 2.
\]

For extra-diagonal terms, we have for \( i \neq j \)

\[
D_{i,j} = \frac{1}{2\pi} \int_0^{2\pi} (I - \Pi)A_iL^{-1}[(I - \Pi)A_j]d\tau
\]

\[
= -\frac{1}{2\pi} \int_0^{2\pi} L^{-1}[(I - \Pi)A_i](I - \Pi)A_j d\tau
\]

\[
= -D_{j,i}.
\]

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Hence, the diffusion term $\nabla_\xi \cdot (D \nabla_\xi G)$ can be simplified

$$\nabla_\xi \cdot (D \nabla_\xi G) = \sum_{i,j=1}^{2} \partial_{\xi_i} (D_{i,j} \partial_{\xi_j} G)$$

$$= \sum_{i,j=1}^{2} \left[ (\partial_{\xi_i} D_{i,j}) \partial_{\xi_j} G + D_{i,j} \partial_{\xi_i} \partial_{\xi_j} G \right]$$

$$= \sum_{i,j=1}^{2} \left[ (\partial_{\xi_i} D_{i,j}) \partial_{\xi_j} G \right] + D : \nabla^2_\xi G$$

$$= (\nabla_\xi \cdot D) \cdot \nabla_\xi G + D : \nabla^2_\xi G,$$

where we used the notation $(\nabla_\xi \cdot D)_j = \sum_{i=1}^{2} \partial_{\xi_i} D_{i,j}$. But, using the properties of anti-symmetry of $D$ and symmetry of $\nabla^2_\xi G$ (whose components are $(\nabla^2_\xi G)_{i,j} = \partial_{\xi_i} \partial_{\xi_j} G$), the last term $D : \nabla^2_\xi G$ then vanishes since

$$D : \nabla^2_\xi G = \sum_{i,j=1}^{2} D_{i,j} \partial_{\xi_i} \partial_{\xi_j} G = \sum_{i,j=1}^{2} D_{j,i} \partial_{\xi_j} \partial_{\xi_i} G = -\sum_{i,j=1}^{2} D_{i,j} \partial_{\xi_i} \partial_{\xi_j} G = -D : \nabla^2_\xi G.$$

An important consequence is that the first order model does not include second order derivatives

$$\partial_t G - \Pi [A(t, \tau, \xi)] \cdot \nabla_\xi G - \varepsilon (\nabla_\xi \cdot D) \cdot \nabla_\xi G = 0. \tag{5.2}$$

First, we saw that the first order model is a transport model. We can prove the following proposition

**Proposition 5.2** The equation (5.2) has a vector field which is divergence free.

**Proof.** We have already proved that $\nabla_\xi \cdot A = 0$ so that the same is true for $\Pi (A)$. It remains to verify that $\nabla_\xi \cdot (\nabla_\xi \cdot D) = 0$. Since

$$\nabla_\xi \cdot (\nabla_\xi \cdot D) = \sum_{i,j=1}^{2} \partial_{\xi_i} \partial_{\xi_j} D_{j,i} = \nabla^2_\xi : D,$$

and since the contracted product of an antisymmetric matrix with a symmetric one vanishes, the corrective term is divergence free and the proposition is proved. \hfill \blacksquare

**Hamiltonian structure**

Since (1.6) has an Hamiltonian $\mathcal{H}(t, x, v) = \frac{1}{2} (x^2 + v^2) + \phi(t, x)$ where $\phi$ is the electric potential is related to the electric field $E_{tot}(t, x) = -\partial_\xi \phi(t, x)$, the model (4.1) has also
an Hamiltonian $H(t,\tau,\xi) = \phi(t,\xi_1 \cos \tau + \xi_2 \sin \tau)$. Then, the model (4.1) on $F(t,\tau,\xi)$ can be written as

$$\partial_t F - \{H,F\} = -\frac{1}{\varepsilon} \partial_\tau F, \quad (5.3)$$

with $\{H,F\} = \partial_{\xi_1} H \partial_{\xi_2} F - \partial_{\xi_2} H \partial_{\xi_1} F$.

**Proposition 5.3** If the model (5.3) has an Hamiltonian $H$, then the limit model has the Hamiltonian $(\Pi H)$ and the first order model has the Hamiltonian $(\Pi H) + \varepsilon D_{1,2}$.

**Proof.** We derived the averaged model on $G$ from (5.3), with the notations $A_1 = -\partial_{\xi_2} H$ and $A_2 = \partial_{\xi_1} H$

$$\partial_t G - (\Pi A) \cdot \nabla_\xi G - \varepsilon (\nabla_\xi \cdot D) \cdot \nabla_\xi G = 0,$$

or

$$\partial_t G - \{\Pi H, G\} - \varepsilon (\nabla_\xi \cdot D) \cdot \nabla_\xi G.$$

Let us focus on the corrective term $(\nabla_\xi \cdot D) \cdot \nabla_\xi G$ The matrix $D$ in the first order corrective term writes

$$D = \begin{pmatrix} 0 & D_{1,2} \\ -D_{1,2} & 0 \end{pmatrix},$$

so that the first order term is

$$(\nabla_\xi \cdot D) \cdot \nabla_\xi G = (\partial_{\xi_2} D_{1,2} \partial_{\xi_1} G + \partial_{\xi_1} D_{2,1} \partial_{\xi_2} G) = (\partial_{\xi_2} D_{1,2} \partial_{\xi_1} G - \partial_{\xi_1} D_{2,1} \partial_{\xi_2} G) = -\{D_{1,2}, G\}.$$

The Hamiltonian of the first order model is then $H_\varepsilon = (\Pi H) + \varepsilon D_{1,2}$ which enables to re-write the first order model as

$$\partial_t G - \{H_\varepsilon, G\} = 0.$$

**References**


