Numerical simulations of Vlasov-Maxwell equations for laser plasma based on Poisson structure

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Abstract

In this paper, Poisson bracket is proposed for the laser-plasma interaction model introduced in [18], and it is proved by direct calculations that the Poisson bracket satisfies the Jacobi identity. Then a splitting method is proposed based on the decomposition (into three parts) of the Hamiltonian; in the quasi-relativistic case, the solutions of these three subsystems can be written out explicitly. Fourier spectral method and finite volume method are used in phase space discretization. The splitting is extended to the fully relativistic case combined with conservative splitting method. Finally, some numerical experiments are conducted to confirm good long time behavior of our schemes.

\textbf{Keywords:} laser-plasma interaction, Vlasov-Maxwell system, Poisson bracket, Hamiltonian splitting method.

1. Introduction

The system of Vlasov-Maxwell (VM) equations is an important model to describe the interactive dynamics of charged particles with electromagnetic field. In this paper, we focus on a reduced Vlasov-Maxwell (RVM) system which is introduced in [18, 25] to describe the interaction of laser and plasma.

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This laser-plasma (LP) model describes the action of a laser wave, called pump, penetrating into a plasma and heating it. During this process, the laser wave interacts with electrostatic waves and accelerates the electrons. In the plasma physics community, this model and its variants have been extensively studied.

The numerical approximation of this model or more generally of the Vlasov-Maxwell system is known to be very difficult since long term calculations are required including high dimensions in phase space and multi-scale phenomena. Several strategies have been proposed to solve numerically the Vlasov-Maxwell system. Among them, Particle in Cell (PIC) methods have been widely used. Its basic idea consists in approximating the plasma by a finite number of particles \([1, 3, 7, 14]\). The motion of these particles is computed by solving the characteristic equations forward in time, using time integrators such as Boris scheme \([4]\), and recently proposed structure-preserving numerical methods \([22, 23, 24, 34]\). The main advantage of PIC methods is its low computational cost. However numerical noise of PIC methods is too large to give a precise description of low density regions. Moreover, the numerical noise only decreases in \(1/\sqrt{N}\) when the number of particles \(N\) increases. With ever more powerful computers becoming available, methods relying on a discretization of the phase space (Grid based methods) have also been proposed. In some cases, for example, when particles in the tail of the distribution play an important role, or when the numerical noise becomes too evident, grid based methods are preferred. For instance, the semi-Lagrangian methods directly computes the distribution function on a grid of the phase space \([17, 30]\). The computation of the unknown is based on two steps: first, the characteristics equations have to be solved backward in time at each time step, and second, the unknown is updated by interpolating the value at the origin of the characteristics. There also exist conservative versions of the semi-Lagrangian schemes to deal with the preservation of the total mass (see \([12, 17]\), in which comparisons of different reconstructions methods are considered). Furthermore, one can also quote Discontinuous Galerkin methods for which energy-conserving algorithms can be derived (see \([10]\)).

In the cases of laser plasma interaction and Tokamak plasma, long time simulations of Vlasov-Maxwell models are required to describe plasma phenomena, such as turbulence and filamentation. In order to maintain long term accuracy, geometric methods proposed in \([16, 21, 29]\) for the purpose of preserving the underlying geometric structure of the dynamical system, such as symplectic structure, phase-space volume, and Poisson structure. As
kinetic models in plasma physics have very rich conservative properties, the idea to construct geometric integration methods for these systems has recently emerged [11, 15, 22, 32]. By preserving properties such as the Poisson structure of a Hamiltonian system, geometric integration methods usually generate numerical results with superior long-term behavior compared to standard methods, and are thus more suitable for large-scale, long-term simulations. Poisson structure could be considered as the generalization of symplectic structure. A lot of differential equations arising in physics, e.g., rigid body problem, magnetohydrodynamics (MHD) equations, and Korteweg-de Vries (KdV) equation, can be reformulated with a defined Poisson structure [27].

The laser-plasma model describing the interaction of laser and plasma, has attracted interest of many researches [2, 5, 6, 8, 13, 19, 20, 25, 31]. Existence of global weak solutions in non-relativistic case, characteristic solutions in quasi-relativistic case and mild solutions in relativistic case has been studied and proved in [5, 8]. Also, because of the lack of explicit expression of analytic solutions there are different strategies which have been developed to approximate the solutions of the laser-plasma model. One can quote Wavelet-MRA-based adaptive semi-Lagrangian method [2], WENO schemes [31], semi-Lagrangian method combined cubic spline interpolation [6, 25], for which the convergence analysis has been proved in [6], and forward semi-Lagrangian method which preserves charge [13].

In this paper, we present the Poisson bracket of the laser-plasma model and prove it satisfies the Jacobi identity, which to the best of our knowledge has not been given in literature before. Moreover, we analyze its Casimir function and corresponding conservative properties. Then a Hamiltonian splitting method based on a decomposition of the Hamiltonian into three parts is used to construct Poisson structure preserving schemes for the LP model. The laser-plasma model has at least three versions: non-relativistic, quasi-relativistic and relativistic cases according to the choice of the Lorentz factor. For the first two cases (non-relativistic and quasi-relativistic cases), implementing Hamiltonian splitting leads to three subsystems which can be solved exactly, and the exact solutions to subsystems can be combined with the grid-based approach in space the derive the fully numerical discretizations. As the implementation of splitting, the derived numerical scheme enjoys the following properties: (i) it is symplectic, i.e., it preserves the Hamiltonian structure of the LP system; (ii) it can be generalized to arbitrary high order in time by composition; (iii) it can be combined with
arbitrary phase space approximations. In the fully relativistic case however, the subsystems can not be solved exactly in time so that we propose a new splitting (which is actually not Hamiltonian in this case) inspired from the Hamiltonian splitting and for which we observe numerically significant improvements compared to the literature.

The idea of splitting technique used in this paper has similarities with [11, 26] in which a splitting is proposed for the Vlasov-Maxwell system. However, in this work, we deal with the laser-plasma model for which the corresponding bracket is presented and proved to satisfy the Jacobi identity. Moreover, the laser plasma system presents some additional difficulties. First, the Maxwell equations involve in this context second order spatial derivatives in space and nonlinear source terms; second, in the relativistic case, the Lorentz factor depends not only in the momentum but also in space which induces numerical difficulties since the transport in space can not be solved easily using spectral methods. Then, we use a conservative semi-Lagrangian method (PSM method) to handle this case.

The paper is organized as follows. In section 2 we introduce the LP system and its Poisson structure. In section 3, the Hamiltonian splitting method is applied to the LP system. In section 4, the phase space discretization of (quasi) relativistic LP system is described. Finally, in section 5, several numerical experiments are given to validate the long term behavior of our schemes.

2. The Hamiltonian structure of laser-plasma model

The main goal of this section is to introduce the laser-plasma model and its Hamiltonian structure. The model can be described by a reduced Vlasov-
Maxwell equations as

\[
\frac{\partial f}{\partial t} + \frac{p}{\gamma_1} \frac{\partial f}{\partial x} + \left( E_x - \frac{A_\perp}{\gamma_2} \cdot \frac{\partial A_\perp}{\partial x} \right) \frac{\partial f}{\partial p} = 0, \tag{1}
\]

\[
\frac{\partial A_\perp}{\partial t} = -E_\perp, \tag{2}
\]

\[
\frac{\partial E_x}{\partial t} = -\int_{\mathbb{R}} \frac{p}{\gamma_1} f(x, p, t) dp + \bar{J}(t), \tag{3}
\]

\[
\frac{\partial E_\perp}{\partial t} = -\frac{\partial^2 A_\perp}{\partial x^2} + A_\perp \int_{\mathbb{R}} \frac{1}{\gamma_2} f(x, p, t) dp, \tag{4}
\]

\[
\frac{\partial E_x}{\partial x} = \int_{\mathbb{R}} f(x, p, t) dp - 1, \text{ (Poisson equation)} \tag{5}
\]

where,

\[
\bar{J}(t) = \frac{1}{|\Omega|} \int_\Omega \int_{\mathbb{R}} \frac{p}{\gamma_1} f(x, p, t) dp dx.
\]

In this model, \( f(x, p, t) \) is the particle distribution function, \( x \in \Omega \subset \mathbb{R} \) is space variable, \( p \in \mathbb{R} \) is momentum variable, \( t \geq 0 \) denotes the time, and \( \cdot \) represents the scalar product in \( \mathbb{R}^2 \). Electric field and vector potential are denoted as \( E = (E_x, E_\perp) \), and \( A = (0, A_\perp) \) respectively, where \( E_x(x, t) \) is the longitudinal component along the \( x \) direction of propagation of electromagnetic wave, and \( (E_\perp(x, t), A_\perp(x, t)) = (E_y, A_y, A_z)(x, t) \in \mathbb{R}^2 \times \mathbb{R}^2 \) are the transverse electric field and vector potential. The factors \( \gamma_1, \gamma_2 \) are determined as follows:

- \( \gamma_1 = \gamma_2 = 1 \) in the nonrelativistic case,
- \( \gamma_1 = \sqrt{1 + p^2}, \gamma_2 = 1 \) in the quasi-relativistic case,
- \( \gamma_1 = \gamma_2 = \sqrt{1 + p^2 + |A_\perp|^2} \) in fully relativistic case.

Finally, we impose that \( E_x \) is of zero average all the time, that is

\[
\int_\Omega E_x(x, t) dx = 0, \ \forall t \geq 0, \tag{6}
\]

which implies the presence of \( \bar{J} \) in (3).

For the LP system, we present the following bracket structure. Denote \( \mathcal{M} \) the infinite manifold \( \{ (f, E_x, E_\perp, A_\perp) \} \). For arbitrary two functionals \( \mathcal{F} \),
on $\mathcal{M}$, we define the Poisson bracket as,

$$\{\mathcal{F}, \mathcal{G}\} = \int_{\Omega \times \mathbb{R}} f \left[ \frac{\delta \mathcal{F}}{\delta f}, \frac{\delta \mathcal{G}}{\delta f} \right]_{xp} \, dx dp + \int_{\Omega \times \mathbb{R}} \left( \frac{\delta \mathcal{F}}{\delta E_x} \frac{\delta \mathcal{G}}{\delta p} - \frac{\delta \mathcal{G}}{\delta E_x} \frac{\delta \mathcal{F}}{\delta p} \right) \, dx dp$$

$$+ \int_{\Omega} \left( \frac{\delta \mathcal{G}}{\delta A_{\perp}} \cdot \frac{\delta \mathcal{F}}{\delta E_{\perp}} - \frac{\delta \mathcal{F}}{\delta A_{\perp}} \cdot \frac{\delta \mathcal{G}}{\delta E_{\perp}} \right) \, dx,$$

where $[\cdot, \cdot]_{xp}$ is the Lie bracket, i.e., $[h, g]_{xp} = \partial_x h \partial_p g - \partial_p h \partial_x g$ for any smooth function $h(x, p)$ and $g(x, p)$. The terms $\frac{\delta \mathcal{F}}{\delta f}$, $\frac{\delta \mathcal{F}}{\delta E_x}$, $\frac{\delta \mathcal{G}}{\delta E_{\perp}}$, and $\frac{\delta \mathcal{F}}{\delta A_{\perp}}$ denote the Fréchet derivatives of any smooth functional $\mathcal{F}$ with respect to $f$, $E_x$, $E_{\perp}$, and $A_{\perp}$ respectively. Associated with Poisson bracket, Casimir functional is defined below.

**Definition 2.1** [27] A smooth functional $\mathcal{C}$ defined on $\mathcal{M}$ is called a Casimir functional for the given Poisson bracket $\{\cdot, \cdot\}$ if for arbitrary smooth functional $\mathcal{G}$

$$\{\mathcal{C}, \mathcal{G}\} = 0.$$ 

In the quasi-relativistic case namely $\gamma_1 = \sqrt{1+p^2}$ and $\gamma_2 = 1$ case, we introduce the following Hamiltonian

$$\mathcal{H} = \int_{\Omega \times \mathbb{R}} \left( \sqrt{1+p^2} - 1 \right) f \, dx dp + \frac{1}{2} \int_{\Omega \times \mathbb{R}} |A_{\perp}|^2 f \, dx dp$$

$$+ \frac{1}{2} \int_{\Omega} (E_x^2 + |E_{\perp}|^2) dx + \frac{1}{2} \int_{\Omega} \left| \frac{\partial A_{\perp}}{\partial x} \right|^2 dx.$$ 

(8)

With the defined bracket (7) proved to be Poisson in the following proposition 2.1, and the Hamiltonian functional (8), the quasi-relativistic LP system (1)-(4) can be reformulated as a Poisson system

$$\dot{\mathcal{Z}} = \{\mathcal{Z}, \mathcal{H}\},$$

(9)

where $\mathcal{Z} = (f, E_x, E_{\perp}, A_{\perp})^T$. Here $f, E_x, E_{\perp}$, and $A_{\perp}$ are understood as functionals of $f, E_x, E_{\perp}$ and $A_{\perp}$, for example, $f(x_0, p_0, t) = \int_{\Omega \times \mathbb{R}} f(x, p, t) \delta(x-x_0, p-p_0) \, dp \, dx$. Non-relativistic and fully relativistic LP model can also be set in Poisson form with the corresponding Hamiltonians.

As follows, we prove the bracket structure defined by (7) is Poisson. To prove it, we need the following lemma from [28].

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^1Note that the Frechet derivative with respect to $E_x$ are calculated in consideration of the zero average variations (6), which implies the presence of the term $J$. 

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Lemma 2.1 (Bracket Theorem) To prove the Jacobi identity for generic brackets of the form \( \{ F, G \} = \langle \frac{\delta F}{\delta \psi}, J \frac{\delta G}{\delta \psi} \rangle \) with anti-adjoint operator \( J \), and \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \) inner product. One only needs to consider the explicit dependence of \( J \) with respect to \( \psi(x) \) when taking the functional derivative \( \delta \{ F, G \}/\delta \psi \).

Proposition 2.1 The bracket defined in (7) is a Poisson bracket.

Proof. First of all, to use the above Lemma, we reformulate our Poisson bracket as follows,
\[
\{ F, G \} = \langle \frac{\delta F}{\delta Z}, J \frac{\delta G}{\delta Z} \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the integral about the variables that the integrands depend on and \( J \) reads
\[
J = \begin{pmatrix}
-\frac{\partial f}{\partial p} & \frac{\partial f}{\partial x} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

It is easy to prove the bracket is real bilinear, skew symmetric, and satisfies Leibniz identity. Here we focus on the proof that bracket (7) satisfies Jacobi identity, i.e. for arbitrary functional \( F, G, H \) of \( f, E_x, E_\perp, A_\perp \), we shall prove
\[
\{ \{ F, G \}, H \} + \{ \{ G, H \}, F \} + \{ \{ H, F \}, G \} = 0. \tag{10}
\]
We denote the bracket \( \{ F, G \} \) by the sum of three parts, i.e
\[
\{ F, G \} = \{ F, G \}_{xp} + \{ F, G \}_{AE} + \{ F, G \}_{fE_x}, \tag{11}
\]
where
\[
\{ F, G \}_{xp} = \int_{\Omega \times \mathbb{R}} f \frac{\delta F}{\delta f} \frac{\delta G}{\delta f} \frac{\delta G}{\delta p} dxdp,
\]
\[
\{ F, G \}_{AE} = \int_{\Omega} \left( \frac{\delta G}{\delta A_\perp} \cdot \frac{\delta F}{\delta E_\perp} - \frac{\delta F}{\delta A_\perp} \cdot \frac{\delta G}{\delta E_\perp} \right) dx,
\]
\[
\{ F, G \}_{fE_x} = \int_{\Omega \times \mathbb{R}} \left( \frac{\delta F}{\delta E_x} \frac{\partial f}{\partial \delta G} \frac{\delta G}{\delta f} - \frac{\delta G}{\delta E_x} \frac{\partial f}{\partial \delta f} \right) dxdp.
\]
The Jacobi identity (10) then writes
\[
\{\{F, G\}, H\} + \text{cyc} \\
= \{\{F, G\}_{xp}, H\}_{xp} + \{\{F, G\}_{AE}, H\}_{AE} + \{\{F, G\}_{fE_{x}}, H\}_{fE_{x}} \\
+ \{\{F, G\}_{AE}, H\}_{xp} + \{\{F, G\}_{AE}, H\}_{AE} + \{\{F, G\}_{AE}, H\}_{fE_{x}} \\
+ \{\{F, G\}_{fE_{x}}, H\}_{xp} + \{\{F, G\}_{fE_{x}}, H\}_{AE} + \{\{F, G\}_{fE_{x}}, H\}_{fE_{x}} + \text{cyc},
\]
where, cyc is used to denote cyclic permutation, and the last 'cyc' above term contains 18 terms of nested subbrackets.

By Lemma 2.1, we only need consider the functional derivatives of \{F, G\} modulo the second derivative terms. In what follows \(\frac{\delta\{F, G\}}{\delta\psi}\) denotes the functional derivative modulo the second derivative terms. Equation (11) gives
\[
\frac{\delta\{F, G\}_{xp}}{\delta f} = 0, \quad \frac{\delta\{F, G\}_{xp}}{\delta E_{x}} = 0, \quad \frac{\delta\{F, G\}_{xp}}{\delta A_{\perp}} = 0,
\]
(12)
\[
\frac{\delta\{F, G\}_{AE}}{\delta f} = 0, \quad \frac{\delta\{F, G\}_{AE}}{\delta E_{x}} = 0, \quad \frac{\delta\{F, G\}_{AE}}{\delta A_{\perp}} = 0,
\]
(13)
\[
\frac{\delta\{F, G\}_{fE_{x}}}{\delta f} = -\frac{\delta F}{\delta E_{x}} \frac{\delta G}{\delta f} + \frac{\delta G}{\delta E_{x}} \frac{\delta F}{\delta f}, \quad \frac{\delta\{F, G\}_{fE_{x}}}{\delta E_{x}} = 0, \quad \frac{\delta\{F, G\}_{fE_{x}}}{\delta A_{\perp}} = 0.
\]
(14)
\[
\frac{\delta\{F, G\}_{fE_{x}}}{\delta E_{x}} = 0, \quad \frac{\delta\{F, G\}_{fE_{x}}}{\delta A_{\perp}} = 0, \quad \frac{\delta\{F, G\}_{fE_{x}}}{\delta E_{x}} = 0.
\]
(15)

The following assertions are immediate:

- Term 1 + cyc vanishes because \{F, G\}_{xp} corresponds to Lie Poisson bracket of the Vlasov-Poisson system (see [27]).
- Term 2 vanishes because of the second and third equations in Eq. (12).
- Terms 4, 5, 6 vanish because of Eq. (13).
- Term 8 vanishes because of the second and third equations in Eq. (14)-(15).
So the remaining terms are Terms 3, 7, and 9. We prove (see Appendix for the details) that Term 9 vanishes and the sum of Term 3 and Term 7 cancels out. We then conclude that the Jacobi identity is satisfied.

Proposition 2.2 The functional

\[ C_{x_0}(Z) = \int_{\Omega} \frac{\partial E}{\partial x} \delta(x-x_0) dx dp - \int_{\Omega \times \mathbb{R}} f \delta(x-x_0) dx dp, \quad x_0 \in \Omega \]

is a Casimir functional of the Poisson system (9), which indicates that the Poisson equation (5) is satisfied by the solution of LP system (1)-(4).

**Proof.** According to the above definition 2.1, we need prove \( \{ C_{x_0}(Z), G(Z) \} = 0 \) for arbitrary smooth functional \( G(Z) \). First we have the following variational derivatives:

\[ \frac{\delta C_{x_0}(Z)}{\delta f} = -\delta(x-x_0), \quad \frac{\delta C_{x_0}(Z)}{\delta E_{\perp}} = \delta A_{\perp}(Z) = 0. \]

Inserting the above variational derivatives into (7) gives

\[ \{ C_{x_0}(Z), G(Z) \} = \int_{\Omega \times \mathbb{R}} \frac{\partial \delta(x-x_0)}{\partial x} \left( -f \frac{\partial G}{\partial p} \frac{\delta f}{\delta p} - \frac{\partial f}{\partial p} \frac{\delta G}{\delta f} \right) dx dp + \int_{\Omega \times \mathbb{R}} \frac{\partial G}{\partial E_{x}} \frac{\partial f}{\partial p} \delta(x-x_0) dx dp. \]

With integration by parts, the first term above is 0. As \( f \) is compactly supported in \( p \) direction, the second term is also 0. In this way \( \{ C_{x_0}(Z), G(Z) \} = 0 \), i.e, \( C_{x_0} \) is a Casimir functional. This proves the proposition.

Based on the Poisson structure of the LP system, in the following sections we construct the Poisson structure preserving schemes for the (quasi) relativistic LP system.

3. Time discretizations by Hamiltonian splitting

In this section, we consider the time discretization for (quasi) relativistic case by Hamiltonian splitting similar to [11, 26].

\footnote{Note that \( \delta \) is a generalized function, and the \( \frac{\partial \delta}{\partial x}(x-x_0) \) below represents the derivatives of generalized function.}
3.1. Quasi-relativistic case

For quasi-relativistic case, the Hamiltonian (8) is split into three parts:

\[ H = H_E + H_A + H_p, \]

where \( H_p, H_A \) and \( H_E \) are given by

\[
H_E = \frac{1}{2} \int_{\Omega} (E_x^2 + |E_\perp|^2) dx, \tag{16}
\]

\[
H_A = \frac{1}{2} \int_{\Omega \times \mathbb{R}} |A_\perp|^2 f dx dp + \frac{1}{2} \int_{\Omega} \left| \frac{\partial A_\perp}{\partial x} \right|^2 dx, \tag{17}
\]

\[
H_p = \int_{\Omega \times \mathbb{R}} \left( \sqrt{1 + p^2} - 1 \right) f dx dp. \tag{18}
\]

With the notations introduced in the above section, the Poisson system (9) can be expressed as

\[
\dot{Z} = \{Z, H_E\} + \{Z, H_A\} + \{Z, H_p\}. \]

In the following, we explain that the three subsystems

\[
\dot{Z} = \{Z, H_E\}, \quad \dot{Z} = \{Z, H_A\}, \quad \dot{Z} = \{Z, H_p\}
\]

can be solved exactly.

**The subsystem corresponding to** \( H_E \) **is**

\[
\frac{\partial f}{\partial t} + E_x \frac{\partial f}{\partial p} = 0, \quad \frac{\partial E_x}{\partial t} = 0, \quad \frac{\partial E_\perp}{\partial t} = 0, \quad \frac{\partial A_\perp}{\partial t} = -E_\perp. \tag{19}
\]

For given initial values \((f_0(x, p), E_{x0}(x), E_{\perp0}(x), A_{\perp0}(x))\) at time \( t = 0 \), the solution at time \( t \) can be written explicitly,

\[
f(x, p, t) = f_0(x, p - tE_{x0}(x)), \quad E_x(x, t) = E_{x0}(x),
\]

\[
E_\perp(x, t) = E_{\perp0}(x), \quad A_\perp(x, t) = A_{\perp0}(x) - tE_{\perp0}(x).
\]

Moreover, if \( E_{x0} \) and \( f_0 \) satisfy the Poisson equation (5), then it holds true for any later time; this can be easily seen by the fact that \( E(x, t) \) is constant in time and \( \int_{\mathbb{R}} f(x, p, t) dp = \int_{\mathbb{R}} f_0(x, p) dp \).
The subsystem corresponding to $H_A$ is

$$\frac{\partial f}{\partial t} - A_\perp \cdot \frac{\partial A_\perp}{\partial x} \frac{\partial f}{\partial p} = 0, \quad \frac{\partial E_x}{\partial t} = 0, \quad (20)$$

$$\frac{\partial E_\perp}{\partial t} = -\frac{\partial^2 A_\perp}{\partial x^2} + A_\perp \int f(x, p, t) dp, \quad \frac{\partial A_\perp}{\partial t} = 0. \quad (21)$$

With initial values $(f_0(x, p), E_{x0}(x), E_{\perp0}(x), A_{\perp0}(x))$ at time $t = 0$, the solution at time $t$ is as follows,

$$f(x, p, t) = f_0 \left( x + t A_{\perp0}(x) \cdot \frac{\partial A_{\perp0}(x)}{\partial x} \right), \quad E_x(x, t) = E_{x0}(x),$$

$$E_\perp(x, t) = E_{\perp0}(x) - t \frac{\partial^2 A_{\perp0}(x)}{\partial x^2} + t A_{\perp0}(x) \int f_0(x, p) dp, \quad A_\perp(x, t) = A_{\perp0}(x).$$

Indeed, integrating the first equation in (20) about $p$ leads to

$$\int f(x, p, t) dp = \int \frac{\partial f}{\partial t} dp - \int \frac{\partial A_\perp}{\partial x} \frac{\partial f}{\partial p} dp = 0,$$

so that the right hand side of the equation on $E_\perp$ does not depend on time and it can be integrated in time exactly. Another consequence of the density preservation is that the Poisson equation is propagated along this step.

The subsystem corresponding to $H_p$ is

$$\frac{\partial f}{\partial t} + \frac{p}{\gamma} \frac{\partial f}{\partial x} = 0, \quad \frac{\partial E_x}{\partial t} = -\int \frac{p}{\gamma} f(x, p, t) dp + \bar{J}(t), \quad \frac{\partial E_\perp}{\partial t} = 0, \quad \frac{\partial A_\perp}{\partial t} = 0. \quad (22)$$

With initial values $(f_0(x, p), E_{x0}(x), E_{\perp0}(x), A_{\perp0}(x))$ at time $t = 0$, the solution is as follows,

$$f(x, p, t) = f_0 \left( x - t \frac{p}{\gamma}, p \right), \quad E_\perp(x, t) = E_{\perp0}(x), \quad A_\perp(x, t) = A_{\perp0}(x),$$

$$E_x(x, t) = E_{x0}(x) - \int_0^t \int \frac{p}{\gamma} f(x, p, \tau) dp d\tau + \int_0^t \bar{J}(\tau) d\tau.$$

One can prove that the Poisson equation is preserved along this step assuming it is satisfied initially, i.e. $\frac{\partial E_{\perp0}(x)}{\partial x} = \int f_0(x, p) dp - 1$. Indeed, by directly
differentiating $E_x(t)$ with respect to $x$, we have,

$$
\frac{\partial E_x(x,t)}{\partial x} = \frac{\partial E_{x0}(x)}{\partial x} - \int_0^t \int_\mathbb{R} \frac{p \partial f(x,p,\tau)}{\partial x} dp d\tau \\
= \frac{\partial E_{x0}(x)}{\partial x} + \int_0^t \int_\mathbb{R} \frac{\partial f(x,p,\tau)}{\partial \tau} dp d\tau \text{ (refer the first equation in (22))} \\
= \frac{\partial E_{x0}(x)}{\partial x} + \int_\mathbb{R} f(x,p,t) dp - \int_\mathbb{R} f_0(x,p) dp = \int_\mathbb{R} f(x,p,t) dp - 1.
$$

3.2. Fully relativistic case

In this subsection, we intend to extend the above approach to the relativistic case. In this case, we will see that the Hamiltonian splitting method gives subsystem that cannot be solved explicitly (see appendix B). Thus, we generalize the Hamiltonian splitting used in quasi-relativistic case by combining with the conservative splitting. The derived subsystems (still denoted by $\mathcal{H}_E, \mathcal{H}_A, \mathcal{H}_p$) are listed as follows.

The subsystem corresponding to $\mathcal{H}_E$ is the same as in quasi-relativistic case:

$$
\frac{\partial f}{\partial t} + E_x \frac{\partial f}{\partial p} = 0, \quad \frac{\partial E_x}{\partial t} = 0, \quad \frac{\partial E_\perp}{\partial t} = 0, \quad \frac{\partial A_\perp}{\partial t} = -E_\perp.
$$

The subsystem corresponding to $\mathcal{H}_A$ is

$$
\frac{\partial f}{\partial t} - \frac{\partial}{\partial p} \left( \frac{1}{\gamma} A_\perp \cdot \frac{\partial A_\perp}{\partial x} f \right) = 0, \\
\frac{\partial E_\perp}{\partial t} = -\frac{\partial^2 A_\perp}{\partial x^2} + A_\perp \int_\mathbb{R} \frac{1}{\gamma} f(x,p,t) dp, \quad \frac{\partial E_x}{\partial t} = 0, \quad \frac{\partial A_\perp}{\partial t} = 0.
$$

where we recall that here $\gamma = \sqrt{1 + p^2 + |A_\perp|^2}$ now depends on $p$ (and still on $t$ and $x$).

The subsystem corresponding to $\mathcal{H}_p$ is:

$$
\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} \left( \frac{p}{\gamma} f \right) = 0, \quad \frac{\partial E_\perp}{\partial t} = 0 \\
\frac{\partial E_x}{\partial t} = -\int_\mathbb{R} \frac{p}{\gamma} f(x,p,t) dp + \tilde{J}(t), \quad \frac{\partial A_\perp}{\partial t} = 0.
$$
Remark For relativistic LP system (1) - (4), we note that the 2D advective field \( \left( \frac{p}{\gamma}, E_x - \frac{A_\perp}{\gamma} \cdot \frac{\partial A_\perp}{\partial x} \right) \) is divergence free, i.e.

\[
\nabla_x \left( E_x - \frac{A_\perp}{\gamma} \cdot \frac{\partial A_\perp}{\partial x} \right) = -\left( \frac{p}{\gamma^3} \right) A_\perp \cdot \frac{\partial A_\perp}{\partial x} + \left( \frac{p}{\gamma^3} \right) A_\perp \cdot \frac{\partial A_\perp}{\partial x} = 0.
\]

So the following two kinds of form of Vlasov equation (1) are equivalent.

\[
\begin{align*}
\frac{\partial f}{\partial t} + \frac{p}{\gamma} \frac{\partial f}{\partial x} + \left( E_x - \frac{A_\perp}{\gamma} \cdot \frac{\partial A_\perp}{\partial x} \right) \frac{\partial f}{\partial p} &= 0, \quad \text{(advective form)} \\
\frac{\partial f}{\partial t} + \nabla_x \left( E_x - \frac{A_\perp}{\gamma} \cdot \frac{\partial A_\perp}{\partial x} \right) f &= 0. \quad \text{(conservative form)}
\end{align*}
\]

Note that we use conservative splitting to handle with Vlasov equation. As in the above remark, at the continuous level, the 2D Vlasov forms (conservative and non conservative) are equivalent. However, when one wants to use splitting method, one has to split in the conservative form (see [25]). And the advective splitting (see the appendix C) may lead to bad conservation of the total mass when long time simulations are conducted. We compare the numerical behaviors of conservative and advective splitting in section 5.

4. Phase space discretization

In this section, we focus on the space phase discretization coupled with the splitting methods presented above.

We assume that the system is periodic in \( x \) direction with period \( L \) and the phase space is \((x, p) \in [0, L] \times \mathbb{R}\). The computational domain is \([0, L] \times [-P_L, P_R]\), where \( P_L, P_R \) is chosen large enough to guarantee that \( f \) is sufficiently small at \( p = -P_L \) and \( p = P_R \). The mesh is as follows:

\[
\begin{align*}
x_j &= (j - 1)\Delta x, \quad j = 1, \ldots, M, \quad \Delta x = L/M, \\
p_{\ell - 1/2} &= (\ell - 1)\Delta p - P_L, \quad \ell = 1, \ldots, N, \quad \Delta p = (P_R + P_L)/N.
\end{align*}
\]

4.1. Quasi-relativistic case

Phase space representation of the unknown. Firstly, we present the spatial discretization for discretizing \( E_x, E_\perp, A_\perp \). Let us detail the way we approximate \( E_x \) (the same strategy will be used for \( E_\perp \) and \( A_\perp \)). Denoting
\( E_{x,j}(t) = E_x(x_j,t) \), we use the spectral Fourier expansion to approximate \( E_x \) gives as \( E_x \) in the \( x \)-direction is periodic,

\[
E_{x,j}(t) = \sum_{k=-M/2+1}^{M/2} \hat{E}_{x,k}(t)e^{2\pi ijk/M}.
\] (24)

with \( j = 1, \ldots, M \).

To discretize the distribution function \( f \) in phase space, we use a spectral representation in space direction whereas a finite-volume discretization defined on the \( p \)-grid with the grid size \( \Delta p \) is used. Using the notation \( f_{j,\ell} \) to denote the average of \( f(x_j, p, t) \) over a cell \( P_\ell = [p_{\ell-\frac{1}{2}}, p_{\ell+\frac{1}{2}}] \) with \( p_\ell = (\ell - \frac{1}{2})\Delta p - P_L \) being the midpoint of the control cell \( P_\ell (\ell = 1, 2, \ldots, N) \). Explicitly, it can be expressed as,

\[
f_{j,\ell}(t) := \frac{1}{\Delta p} \int_{P_\ell} f(x_j, p, t) dp.
\]

Then, the Fourier expansion is performed in the \( x \)-variable to get

\[
f_{j,\ell}(t) = \sum_{k=-M/2+1}^{M/2} \hat{f}_{k,\ell}(t)e^{2\pi ijk/M}, \quad j = 1, \ldots, M.
\]

To evaluate the value of \( f \) off-grid in \( p \) direction, we need to reconstruct a continuous function by using \( \hat{f}_{j,\ell} \). Here, considering the accuracy and efficiency, the so-called Parabolic Spline Method (PSM) introduced in [33] which is mass conserving is used. It has been pointed out in [12, 33] that using PSM for \( f \) is equivalent to use the widely-used cubic splines for the primitive function of \( f \). For the presentation, we consider a simple advection problem in \( p \) direction satisfied by \( f(x, p, t) \)

\[
\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial p} = 0.
\]

One uses the following identity,

\[
f_\ell(x, t) := \frac{1}{\Delta p} \int_{p_{\ell-\frac{1}{2}}}^{p_{\ell+\frac{1}{2}}} f(x, p, t) dp = \frac{1}{\Delta p} \int_{p_{\ell-\frac{1}{2}}}^{p_{\ell+\frac{1}{2}} - at} f_0(x, p) dp.
\] (25)
If one assumes that $p_{\ell+\frac{1}{2}}-at \in [p_{\ell-\frac{1}{2}}, p_{\ell+\frac{1}{2}}]$ (this restriction is done for clarity, the method works without this assumption), one has

$$f_{\ell}(x,t) = \frac{1}{\Delta p} \int_{p_{\ell-\frac{1}{2}}-at}^{p_{\ell+\frac{1}{2}}-at} f(x,p,0)dp + f_{\ell}(x,0) \left[ \frac{1}{\Delta p} \int_{p_{\ell+\frac{1}{2}}}^{p_{\ell+\frac{1}{2}}-at} f(x,p,0)dp \right] (26)$$

Then, since $f_{\ell}(x,0)$ is known for all $\ell = 1, \ldots, N$, one can reconstruct a piecewise polynomial function $f(x,p,0)$ using the averages $f_{\ell}(x,0)$. Here, a PSM reconstruction is used (see [12, 33] for more details). Evaluating now the above expression at $x = x_j$, we get

$$f_{j,\ell}(t) = \frac{1}{\Delta p} \int_{p_{\ell-\frac{1}{2}}-at}^{p_{\ell+\frac{1}{2}}-at} f(x_j,p,0)dp + f_{j,\ell}(0) \left[ \frac{1}{\Delta p} \int_{p_{\ell+\frac{1}{2}}}^{p_{\ell+\frac{1}{2}}-at} f(x_j,p,0)dp \right] (27)$$

where again, $f(x_j,p,0)$ is reconstructed by using PSM for known initial integral averages $f_{j,\ell}(0)$.

Now, the way we solve the subsystems $\mathcal{H}_E$, $\mathcal{H}_A$ and $\mathcal{H}_p$ in the quasi-relativistic case is given as follows.

The subsystem corresponding to $\mathcal{H}_E$ is

$$\frac{\partial f}{\partial t} + E_x \frac{\partial f}{\partial p} = 0, \quad \frac{\partial E_x}{\partial t} = 0, \quad \frac{\partial E_\perp}{\partial t} = 0, \quad \frac{\partial A_\perp}{\partial t} = -E_\perp,$$

with initial condition $f(x,p,0), E_x(x,0), E_\perp(x,0), A_\perp(x,0)$. The second and third equations give $E_x(x,t) = E_x(x,0)$ and $E_\perp(x,t) = E_\perp(x,0)$. For the fourth equation, since the right hand side does not depend on time, we have $A_\perp(x,t) = A_\perp(x,0) - tE_\perp(x,0)$. In terms of the Fourier coefficients, the vector potential evolve according to,

$$A_{\perp k}(t) = A_{\perp k}(0) - t\hat{E}_{\perp k}(0).$$

The distribution $f$ is advected with constant velocity and is advanced in time using the PSM method described above in the previous subsection. The subsystem corresponding to $\mathcal{H}_A$ is

$$\frac{\partial f}{\partial t} - \frac{\partial A_\perp}{\partial x} \frac{\partial f}{\partial p} = 0, \quad \frac{\partial E_x}{\partial t} = 0, \quad \frac{\partial E_\perp}{\partial t} = -\frac{\partial^2 A_\perp}{\partial x^2} + A_\perp \int f(x,p,t)dp, \quad \frac{\partial A_\perp}{\partial t} = 0,$$

with initial condition $f(x,p,0), E_x(x,0), E_\perp(x,0), A_\perp(x,0)$. The Fourier coefficients of $E_\perp$ are advanced in time following,

$$\hat{E}_{\perp k}(t) = \hat{E}_{\perp k}(0) - t \left( \frac{2\pi i k}{L} \right)^2 A_{\perp k}(0) + \sum_{\ell=1}^{N} \sum_{k_1+k_2=k_1, k_1=M/2+1, \ldots, M/2, k_2=M/2+1, \ldots, M/2} \Delta p A_{\perp k_1}(0)f_{k_2,\ell}(0).$$
while \( E_x \) and \( A_\perp \) stay constant in time. The distribution function is advected with constant velocity \( tA_\perp(x_j,0) \cdot \frac{\partial A_\perp}{\partial x}(x_j,0) \) in \( p \) direction, which can be done using PSM (see (25)).

The subsystem corresponding to \( \mathcal{H}_p \) is

\[
\frac{\partial f}{\partial t} + \frac{p}{\gamma} \frac{\partial f}{\partial x} = 0, \quad \frac{\partial E_x}{\partial t} = - \int_{\mathbb{R}} \frac{p}{\gamma} f(x,p,t) dp + \tilde{J}(t), \quad \frac{\partial E_\perp}{\partial t} = 0, \quad \frac{\partial A_\perp}{\partial t} = 0,
\]

where \( \gamma = \sqrt{1 + p^2} \) since we deal with the quasi-relativistic case here. First, we compute distribution function, for which we have a spectral expansion in space, then

\[
\hat{f}_{k,\ell}(t) = \hat{f}_{k,\ell}(0) \exp \left( -\frac{2\pi i}{L} k \tau \frac{p_\ell}{\sqrt{1 + p_\ell^2}} \right), \quad k = -M/2+1, \ldots, M/2, \ell = 1, \ldots, N.
\]

The computation of \( E_x \) is as follows.

When \( k = 0 \),

\[
\hat{E}_{x,0}(t) = \frac{1}{L} \int_{\Omega} E(x,t) dx = 0, \quad (28)
\]

when \( k \neq 0 \),

\[
\hat{E}_{x,k}(t) = \hat{E}_{x,k}(0) - \Delta p \int_{0}^{t} \sum_{\ell=1}^{N} \hat{f}_{k,\ell}(\tau) \frac{p_\ell}{\sqrt{1 + p_\ell^2}} d\tau,
\]

\[
= \hat{E}_{x,k}(0) - \Delta p \int_{0}^{t} \sum_{\ell=1}^{N} \hat{f}_{k,\ell}(0) \exp \left( -\frac{2\pi i}{L} k \tau \frac{p_\ell}{\sqrt{1 + p_\ell^2}} \right) \frac{p_\ell}{\sqrt{1 + p_\ell^2}} d\tau,
\]

\[
= \hat{E}_{x,k}(0) + \Delta p \frac{L}{2\pi i k} \sum_{\ell=1}^{N} \hat{f}_{k,\ell}(0) \left[ \exp \left( -\frac{2\pi i}{L} k \tau \frac{p_\ell}{\sqrt{1 + p_\ell^2}} \right) - 1 \right]. \quad (29)
\]

From the above calculations, we can prove that the Poisson equation is satisfied at time \( t \) if it is satisfied at time 0, i.e.

\[
\frac{2\pi i k}{L} \hat{E}_{x,k}(0) = \Delta p \sum_{\ell=1}^{N} \hat{f}_{k,\ell}(0)
\]
for $k \neq 0$. Indeed, inserting the latter relation in (29), we get
\[
\hat{E}_{x,k}(t) = \frac{L}{2\pi i k} \left[ \Delta p \sum_{\ell=1}^{N} \hat{f}_{k,\ell}(0) \right] + \Delta p \frac{L}{2\pi i k} \sum_{\ell=1}^{N} \hat{f}_{k,\ell}(0) \left[ \exp \left( -i \frac{2\pi}{L} kt \frac{p_{\ell}}{\sqrt{1 + p_{\ell}^2}} \right) - 1 \right]
\]
\[
= \Delta p \frac{L}{2\pi i k} \sum_{\ell=1}^{N} \hat{f}_{k,\ell}(0) \exp \left( -i \frac{2\pi}{L} kt \frac{p_{\ell}}{\sqrt{1 + p_{\ell}^2}} \right) \]
\[
= \Delta p \frac{L}{2\pi i k} \sum_{\ell=1}^{N} \hat{f}_{k,\ell}(t)
\]
which is a spectral version of the Poisson equation at time $t$. The case $k = 0$
does not pose any additional difficulties as the average of the $E_x$ is imposed
to be zero by total mass conservation.

4.2. Fully relativistic case

The discretizations of $f, E_x, E_\perp$ and $A_\perp$ are as follows,
\[
f_{j,\ell} = \frac{1}{\Delta x \Delta p} \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{p_{\ell - 1/2}}^{p_{\ell + 1/2}} f(x,p) dx dp,
\]
\[
E_{x,j} = E_x(x_j), \quad E_{\perp,j} = E_\perp(x_j), \quad A_{\perp,j} = A_\perp(x_j).
\]

Different from the quasi-relativistic case, we use PSM method in $x$ and $p$
direction to solve each subsystem below.

The subsystem corresponding to $\mathcal{H}_A$ is
\[
\frac{\partial f}{\partial t} - \frac{\partial}{\partial p} \left( \frac{1}{\gamma} A_\perp : \frac{\partial A_\perp}{\partial x} f \right) = 0, \tag{30}
\]
\[
\frac{\partial E_\perp}{\partial t} = - \frac{\partial^2 A_\perp}{\partial x^2} + A_\perp \int_\mathbb{R} \frac{1}{\gamma} f(x,p,t) dp, \quad \frac{\partial E_x}{\partial t} = 0, \quad \frac{\partial A_\perp}{\partial t} = 0.
\]

where we recall that here $\gamma = \sqrt{1 + p^2 + |A_\perp|^2}$ now depends on $p$ (and still
on $t$ and $x$). Hence, the characteristic equation corresponding to the equation
for $f$ writes ($s \in [0,t]$)
\[
x_j(s) = 0, \quad \dot{p}_{\ell+1/2}(s) = -\frac{1}{\sqrt{1 + p_{\ell+1/2}^2(s) + |A_\perp|^2(x_j,0)}} A_\perp(x_j,0) \cdot \frac{\partial A_\perp(x_j,0)}{\partial x}, \quad \tag{31}
\]
\[
x_j(t) = x_j, \quad p_{t+1/2}(t) = p_{t+1/2}.
\]
Actually we can integrate the above ODE exactly. Indeed, we can use the following formula:

$$
p(s) \sqrt{1 + p^2(s) + |A_{\perp}|^2(x_j, 0)} = \frac{2}{3} \frac{d}{ds} (1 + p^2(s) + |A_{\perp}|^2(x_j, 0))^{3/2}
$$

so that we have

$$
\frac{d}{ds} (1 + p^2(s) + |A_{\perp}|^2(x_j, 0))^{3/2} = -\frac{3}{2} t A_{\perp}(x_j, 0) \frac{\partial A_{\perp}(x_j, 0)}{\partial x}
$$

and we can obtain the following expression of the feet of the characteristics

$$
p_{\ell+1/2}(0) = \pm \left[ (1 + p^2_{\ell+1/2} + |A_{\perp,j}(0)|^2)^{3/2} + \frac{3t}{2} A_{\perp,j}(0) \frac{\partial A_{\perp,j}(0)}{\partial x} \right]^{1/2} - 1 - |A_{\perp,j}(0)|^{1/2}
$$

(32)

the sign above can be determined by the sign of \( \dot{p}_{\ell+1/2}(s) \) in (31). Then, to solve (30), the numerical unknown \( f_{j,\ell}(t) \) is obtained from the value of \( f_{j,\ell}(0) \) thanks to

$$
f_{j,\ell}(t) = \frac{1}{\Delta x} \int_{p_{\ell-1/2}(0)}^{p_{\ell+1/2}(0)} f_j(p, 0) dp,
$$

where \( p_{\ell+1/2}(0) \) are given by (32) and \( f_j(p, 0) \) is reconstructed from the average values \( f_{j,\ell}(0) \) by using PSM. To solve numerically the equation on \( E_{\perp} \), we use a trapezoidal integration (with respect to time \( t \)) and it comes

$$
E_{\perp}(x_j, t) = E_{\perp}(x_j, 0) - t \frac{\partial A_{\perp}}{\partial x^2}(x_j, 0) + t A_{\perp}(x_j, 0) \frac{\Delta p}{2} \sum_{\ell}^{N} \frac{1}{\gamma_{j,\ell}} \left( f_{j,\ell}(t) + f_{j,\ell}(0) \right)
$$

where \( \gamma_{j,\ell} = \sqrt{1 + p^2_{\ell} + |A_{\perp}|^2(x_j, 0)} \).

The subsystem corresponding to \( H_p \) is

$$
\begin{align*}
\frac{\partial f}{\partial t} + \frac{\partial}{\partial x} \left( p \frac{\partial f}{\partial x} \right) = 0, & \quad \frac{\partial E_{\perp}}{\partial t} = 0, \\
\frac{\partial E_{\perp}}{\partial t} &= - \int_{R}^{p} \frac{f(x, p, t)}{\gamma} dp + \bar{J}(t), & \quad \frac{\partial A_{\perp}}{\partial t} = 0.
\end{align*}
$$

We have to solve the characteristic equation backward in time \( (s \in [0, t]) \)

$$
\dot{x}_{j+1/2}(s) = \frac{p_{\ell}}{\sqrt{1 + p^2_{\ell} + |A_{\perp}|^2(x_{j+1/2}(s), 0)}}, \quad x_{j+1/2}(t) = x_{j+1/2},
$$

(33)

As we use PSM is \( x \) direction, \( f_{j,\ell} \) is solved

$$
f_{j,\ell}(t) = \frac{1}{\Delta x} \int_{x_{j-1/2}(0)}^{x_{j+1/2}(0)} f_{\ell}(x, 0) dx,
$$

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where $x_{j\pm 1/2}(0)$ are obtained by solving (33) numerically (a fourth Runge-Kutta method is used) and $f_\ell(x,0)$ is reconstructed from the average values $f_{j,\ell}(0)$ by using PSM.

To compute $E_x(x_j, t)$, we also use a trapezoidal time integration

$$E_x(x_j, t) = E_x(x_j, 0) - \frac{t}{2} \sum_{\ell=1}^{N} \frac{p_\ell}{\gamma_{j,\ell}} (f_{j,\ell}(t) + f_{j,\ell}(0)) + \frac{t}{2L} (\bar{J}(t) + \bar{J}(0)),$$

where $\bar{J}(s) = \Delta p \Delta x \sum_{j=1}^{M} \sum_{\ell=1}^{N} \frac{p_\ell}{\gamma_{j,\ell}} f_{j,\ell}(s)$.

The subsystem corresponding to $\mathcal{H}_E$ is the same as in the quasi-relativistic case and the same numerical strategy is used.

5. Numerical experiments

In this section, numerical simulations in both the quasi-relativistic and relativistic cases are conducted using the splitting methods presented above. Moreover, comparisons with standard methods (from [25]) are also referred.

5.1. Quasi-relativistic case (QR)

For quasi-relativistic LP system, we consider the so-called parametric instability test case. Parametric instabilities are a common wave-wave interaction. They arise when a nonlinearity such as a pressure gradient couples waves. In plasmas, the parametrically instability happens when electromagnetic waves propagate through a plasma layer (see [2, 25]). In this test, the initial distribution function is a homogeneous Maxwellian with a small perturbation:

$$f(x,p) = \frac{1}{\sqrt{2\pi T}} \exp \left(-\frac{p^2}{2T}\right)(1 + a \cos(kx)),$$

where $T = 3/511$ denotes the temperature, $a = 0.001$ is the perturbation amplitude, $k = 1/\sqrt{2}$ is wave number, and a circularly polarized electromagnetic field is taken as follows [6],

$$E_x = \frac{a}{k} \sin(kx), \quad E_y = E_0 \cos(kx), \quad E_z = E_0 \sin(kx),$$

$$A_y = -E_0 \sin(kx), \quad A_z = E_0 \cos(kx),$$

with quiver momentum $E_0 = \sqrt{3}$. 
We use Lie (first order) and Strang (second order) splitting for Hamiltonian splitting, i.e.,

\[ Z_{\Delta t} = \phi_{H_p,\Delta t} \circ \phi_{H_E,\Delta t} \circ \phi_{H_A,\Delta t} Z(0), \]
\[ Z_{\Delta t} = \phi_{H_p,\Delta t/2} \circ \phi_{H_E,\Delta t/2} \circ \phi_{H_A,\Delta t} \circ \phi_{H_E,\Delta t/2} \circ \phi_{H_p,\Delta t/2} Z(0), \]

where \( \phi_{H_E,t}, \phi_{H_A,t} \) and \( \phi_{H_p,t} \) are the solution mapping of the subsystems corresponding to \( \mathcal{H}_E \), \( \mathcal{H}_A \), and \( \mathcal{H}_p \) which can be solved exactly.

![Figure 1: QR case: time evolution of distribution function at \( t = 10, 24 \). Left: Lie Hamiltonian splitting method. Right: method in [6].](image)

We take the results from [6] as a comparison, in which semi-Lagrangian method of order 2 in time is considered with two dimensional interpolation by cubic splines. For these three schemes (Lie, Strang and 2D-interpolation), the numerical parameters are chosen as follows: \( \Delta t = 0.02 \), \( (x, p) \in [0, 2\pi/k] \times \)
$[-\pi/k, \pi/k]$, $M = N = 128$. First, we compare in Figs. 1-3 the results from [6] and those obtained by the Lie Hamiltonian splitting. In Fig. 1, the contour of distribution function $f$ by the two methods gives qualitatively the same results. It shows that as time goes on, the two vortices appear and separate. However, if one looks at the relative total energy (where the total energy is given by (8)) and at the total charge $\rho(t) = \int_{\Omega \times \mathbb{R}} f(x, p, t) dp dx$, one can observe some significant differences.

![Figure 2: QR case: time evolution of relative charge error and relative total energy computed by the Lie Hamiltonian splitting method.](image)

Indeed, the classical method [6] leads to a drift of the error when the instability starts (around $t \approx 22$), whereas the Lie Hamiltonian splitting methods give very good results; the charge is preserved up to machine precision and the relative total energy error oscillates around a constant for very large times which is a typical behavior of symplectic schemes. In Fig. 4 and 5, the results obtained by the Strang Hamiltonian splitting are displayed. The contour of $f$ are the same as those obtained by the above two methods. However, the relative total energy error is much smaller for Strang splitting than Lie splitting ($10^{-4}$ versus $2 \times 10^{-3}$) which makes this method
very attractive for this test.

5.2. Fully relativistic case (R)

For the relativistic case, we check the numerical method by computing the vortices induced by the relativistic parametric instability generated by an ultra-intense pump wave in a periodic box. We use the following initial values,

\[
f(x, p) = \frac{1}{\sqrt{2\pi T}} \exp \left( -\frac{p^2}{2T} \right), \quad (T = 3/511),
\]

\[
E_x = 0, \ E_y = E_0 \cos(kx), \ E_z = E_0 \sin(kx),
\]

\[
A_y = -E_0 \sin(kx), \ A_z = E_0 \cos(kx), \quad (k = 1/\sqrt{2}, \ E_0 = \sqrt{3}),
\]

which is the same as the initial values of quasi-relativistic case, except that no perturbation is considered for the distribution function. The phase space domain is \([0, 2\pi/k] \times [-\pi/k, \pi/k]\), the number of grid points is \(M = N = 128\),
and the time step is 0.01. We use the conservative and advective splitting method to solve this problem. Here the Strang splitting method is used which can be written as

$$Z_{\Delta t} = \phi_{H_E, \Delta t/2} \circ \phi_{H_A, \Delta t/2} \circ \phi_{H_p, \Delta t} \circ \phi_{H_A, \Delta t/2} \circ \phi_{H_E, \Delta t/2} Z(0),$$

where $Z_{\Delta t} = (f(t), E_x(t), E_{\perp}(t), A_{\perp}(t))$.

For the conservative splitting, we firstly display the electron distribution function at four different times. We can see from Fig. 6 that two vertices appear around $t = 75$ which start to roll themselves and then become very complicated and fine structures appear that can not be captured by the mesh. However, these results are in very good agreement with the results obtained in [6]. The time evolution of relative charge error and relative total
energy error (given by (36)) are then presented for this numerical test. Fig. 7 illustrates that the Strang splitting scheme (even if it is not Hamiltonian) has a very good behavior regarding the conservation of these two invariants. In particular, the errors are much smaller than the errors presented in [6]. Finally, the time history of second mode of $E_x$ and $E_y$ is displayed in Fig. 8. The numerical resolution of the dispersion equation (see [18]) gives a growth rate of $\gamma = 0.409$ for the second mode of $E_x$ and a growth rate of $\gamma = 0.32$ for the second mode of $E_y$. From Fig. 8, we observe that our numerical results are in very good agreement with analytic results in the linear phase. To end up this numerical section, we performed some numerical results considering the advective formulation (see in the appendix C) of the splitting considered here. We want to emphasize (as already discussed in [12, 25]) that the conservative form of the equations (and then conservative numerical methods) has to be considered. Then, in the following results, semi-Lagrangian method is used combined cubic spline reconstruction in both $x$ and $p$ directions. In Figs. 9-11, the time history of the charge and energy relative errors are displayed.
Figure 6: R case: time evolution of distribution function computed by Strang splitting for conservative splitting at $t = 75, 78, 81, 90$.

We can see these errors are larger than their counterparts in the conservative case, which confirms the superiority of conservative splitting.

6. Conclusions

In this work, the Poisson structure of the LP system is investigated. Based on this structure, a Hamiltonian splitting method is given to numerically solve this system in the quasi-relativistic case. Fourier spectral and finite volume methods are used in phase space discretization. The scheme is applied in parametric instability, in which our scheme shows very good long term conservation of charge and energy. The splitting method is extended into relativistic case in two kinds of forms: advective form and conservative form.
Figure 7: R case: time evolution of relative charge error and relative energy error computed by Strang splitting for conservative splitting.

Figure 8: R case: time evolution of second mode of $E_x$ and $E_y$ computed by Strang splitting for conservative splitting.
Figure 9: R case: time evolution of distribution function computed by Strang splitting for advective splitting at $t = 127, 130, 132, 150$.

Figure 10: R case: time evolution of second mode of $E_x$ and $E_y$ computed by Strang splitting for advective splitting.
Figure 11: R case: time evolution of relative charge error and relative energy error computed by Strang splitting for advective splitting.
For the latter form, conservative method is used in phase space discretization to preserve charge exactly.

There are several perspectives of this work can be envisaged. First, application of this method to more complex laser plasma numerical experiments can be conducted as in [2]. Second, more careful phase space discretizations can be considered to construct fully discrete energy preserving and Poisson structure preserving methods.

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7. Appendix

Appendix A. In this Appendix A, we detail the end of the proof of Proposition 2.1. Indeed, one has to prove that the sum of terms 3, 7 and 9 vanish. Concerning Term 3, by using the first and fourth equations in Eq. (12), we have,

\[
\text{Term 3 + cyc} = \int_{\Omega \times \mathbb{R}} \frac{\delta H}{\delta E_x} \frac{\partial f}{\partial p} \left[ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right]_{xp} dxdp + \text{cyc}. 
\]
For Term 9, the first and last equations in Eq. (15) leads to

\[
\text{Term } 9 + \text{cyc} = \int_{\Omega \times \mathbb{R}} -\frac{\delta H}{\delta E_x} \frac{\partial f}{\partial p} \left( -\frac{\delta F}{\delta E_x} \frac{\partial G}{\delta p} \frac{\delta f}{\delta p} + \frac{\delta G}{\delta E_x} \frac{\partial \delta F}{\partial \delta f} \right) dxdp + \text{cyc}
\]

\[
= \int_{\Omega \times \mathbb{R}} \left( \frac{\delta F}{\delta E_x} \frac{\partial \delta G}{\partial \delta f} \frac{\delta f}{\delta p} + \frac{\delta G}{\delta E_x} \frac{\partial \delta f}{\partial \delta f} \right) \left( -\frac{\partial f}{\partial p} \frac{\partial \delta H}{\partial \delta E_x} \right) dxdp
\]

\[
+ \int_{\Omega \times \mathbb{R}} \left( \frac{\delta G}{\delta E_x} \frac{\partial \delta H}{\partial \delta f} \frac{\delta f}{\delta p} + \frac{\delta H}{\delta E_x} \frac{\partial \delta f}{\partial \delta f} \right) \left( -\frac{\partial f}{\partial p} \frac{\partial \delta F}{\partial \delta E_x} \right) dxdp
\]

\[
+ \int_{\Omega \times \mathbb{R}} \left( \frac{\delta H}{\delta E_x} \frac{\partial \delta F}{\partial \delta f} \frac{\delta f}{\delta p} + \frac{\delta F}{\delta E_x} \frac{\partial \delta f}{\partial \delta f} \right) \left( -\frac{\partial f}{\partial p} \frac{\partial \delta G}{\partial \delta E_x} \right) dxdp = 0.
\]

In the above identity, the terms 9a and 9d cancel, but also 9b and 9e, and the same is true for 9c and 9f, which enables us to conclude that the above term vanishes.
We finally focus on Term 7:

\[ \text{Term 7 + cyc} = \int_{\Omega \times \mathbb{R}} f \left[ \frac{\delta \{F, G\} f_{E x}}{\delta f}, \frac{\delta H}{\delta f} \right]_{xp} + \text{cyc} \]

\[ = \int_{\Omega \times \mathbb{R}} f \frac{\partial}{\partial x} \left( -\frac{\delta F}{\delta E_x} \frac{\partial}{\partial f} + \frac{\delta G}{\delta E_x} \frac{\partial}{\partial p} \right) \frac{\partial}{\partial p} \frac{\delta f}{\delta f} \, dx dp \]

\[ - \int_{\Omega \times \mathbb{R}} f \frac{\partial}{\partial p} \left( -\frac{\delta F}{\delta E_x} \frac{\partial}{\delta f} + \frac{\delta G}{\delta E_x} \frac{\partial}{\delta p} \right) \frac{\partial}{\partial x} \frac{\delta f}{\delta f} \, dx dp + \text{cyc} \]

\[ = \int_{\Omega \times \mathbb{R}} \left( -\frac{\delta F}{\delta E_x} \frac{\partial}{\partial p} \frac{\delta f}{\delta f} + \frac{\delta G}{\delta E_x} \frac{\partial}{\partial p} \frac{\delta f}{\delta f} \right) \left( \frac{\partial f}{\partial p} \frac{\delta H}{\partial f} - \frac{\partial f}{\partial x} \frac{\delta H}{\partial f} \right) \, dx dp \]

\[ + \int_{\Omega \times \mathbb{R}} \left( -\frac{\delta G}{\delta E_x} \frac{\partial}{\partial p} \frac{\delta f}{\delta f} + \frac{\delta H}{\delta E_x} \frac{\partial}{\partial p} \frac{\delta f}{\delta f} \right) \left( \frac{\partial f}{\partial p} \frac{\delta F}{\partial f} - \frac{\partial f}{\partial x} \frac{\delta F}{\partial f} \right) \, dx dp \]

\[ + \int_{\Omega \times \mathbb{R}} \left( -\frac{\delta H}{\delta E_x} \frac{\partial}{\partial p} \frac{\delta f}{\delta f} + \frac{\delta F}{\delta E_x} \frac{\partial}{\partial p} \frac{\delta f}{\delta f} \right) \left( \frac{\partial f}{\partial p} \frac{\delta G}{\partial f} - \frac{\partial f}{\partial x} \frac{\delta G}{\partial f} \right) \, dx dp \]

\[ = \int_{\Omega \times \mathbb{R}} \left( -\frac{\delta F}{\delta E_x} \frac{\partial}{\partial p} \frac{\delta f}{\delta f} + \frac{\delta G}{\delta E_x} \frac{\partial}{\partial p} \frac{\delta f}{\delta f} \right) \left( \frac{\partial f}{\partial p} \frac{\delta H}{\partial f} - \frac{\partial f}{\partial x} \frac{\delta H}{\partial f} \right) \, dx dp + \text{cyc}. \]

In the above calculations, we have first performed some integrations by parts (in \(x\) and \(p\)) and used the fact that the sum of Terms 7a, 7b, 7c is zero.

Since Term 9 is zero, we have to sum up Term 3 given by (34) and Term 7.
given by (35). It leads to

$$
\text{Term 3 + cyc + Term 7 + cyc} = \int_{\Omega \times R} \left( \frac{\partial}{\partial x} \frac{\delta F}{\delta f} \frac{\partial}{\partial p} \frac{\delta G}{\delta f} - \frac{\partial}{\partial p} \frac{\delta F}{\delta x} \frac{\partial}{\partial f} \frac{\delta G}{\delta f} \right) \left( - \frac{\partial f}{\partial p} \frac{\delta H}{\delta E} \right) dx dp + \text{cyc}
$$

$$
+ \int_{\Omega \times R} \left( - \frac{\delta F}{\delta E} \frac{\partial}{\partial p} \frac{\delta G}{\delta f} + \frac{\delta G}{\delta E} \frac{\partial}{\partial p} \frac{\delta F}{\delta f} \right) \frac{\partial f}{\partial p} \frac{\partial}{\partial x} \frac{\delta H}{\delta f} dx dp + \text{cyc}
$$

$$
= \int_{\Omega \times R} \left( \frac{\partial F}{\partial f} \frac{\partial G}{\partial f} - \frac{\partial F}{\partial p} \frac{\partial G}{\partial f} \right) \left( - \frac{\partial f}{\partial p} \frac{\delta H}{\delta E} \right) dx dp
$$

$$
+ \int_{\Omega \times R} \left( \frac{\partial G}{\partial f} \frac{\partial H}{\partial f} - \frac{\partial G}{\partial p} \frac{\partial H}{\delta f} \right) \left( - \frac{\partial f}{\partial p} \frac{\delta F}{\delta E} \right) dx dp
$$

$$
= \int_{\Omega \times R} \left( \frac{\partial H}{\partial f} \frac{\partial F}{\delta f} - \frac{\partial H}{\partial p} \frac{\partial F}{\partial x} \right) \left( - \frac{\partial f}{\partial p} \frac{\delta G}{\delta E} \right) dx dp
$$

$$
+ \int_{\Omega \times R} \left( - \frac{\delta F}{\delta E} \frac{\partial}{\partial p} \frac{\delta G}{\delta f} + \frac{\delta G}{\delta E} \frac{\partial}{\partial p} \frac{\delta F}{\delta f} \right) \frac{\partial f}{\partial p} \frac{\partial}{\partial x} \frac{\delta H}{\delta f} dx dp + \text{cyc}
$$

$$
+ \int_{\Omega \times R} \left( - \frac{\delta G}{\delta E} \frac{\partial}{\partial p} \frac{\delta H}{\delta f} + \frac{\delta H}{\delta E} \frac{\partial}{\partial p} \frac{\delta G}{\delta f} \right) \frac{\partial f}{\partial p} \frac{\partial}{\partial x} \frac{\delta F}{\delta f} dx dp
$$

$$
+ \int_{\Omega \times R} \left( - \frac{\delta H}{\delta E} \frac{\partial}{\partial p} \frac{\delta F}{\delta f} + \frac{\delta F}{\delta E} \frac{\partial}{\partial p} \frac{\delta H}{\delta f} \right) \frac{\partial f}{\partial p} \frac{\partial}{\partial x} \frac{\delta G}{\delta f} dx dp = 0.
$$

In the above identity, the terms labelled by the same letter vanish which enables to prove that the Jacobi identity is satisfied.

**Appendix B.** In this Appendix B, we list the three subsystems Hamiltonian splitting method gives and detail the derivation of subsystem $H_p$. 

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For relativistic case, $\gamma = \sqrt{1 + p^2 + |A_\perp|^2}$, and the Hamiltonian writes
\[ H = H_p + H_E + H_A, \]
where $H_p$, $H_A$ and $H_E$ are given by
\[ H_p = \int_{\Omega \times \mathbb{R}} \left( \sqrt{1 + p^2 + |A_\perp|^2} - 1 \right) f dx dp, \]
\[ H_A = \frac{1}{2} \int_{\Omega} \left| \frac{\partial A_\perp}{\partial x} \right|^2 dx, \]
\[ H_E = \frac{1}{2} \int_{\Omega} \left( E_x^2 + |E_\perp|^2 \right) dx. \]
The subsystem corresponding to $H_E$ is the same as in the quasi-relativistic case but the subsystem corresponding to $H_p$ now writes
\[ \frac{\partial f}{\partial t} + p \frac{\partial f}{\gamma \partial x} - \frac{1}{\gamma} A_\perp \cdot \frac{\partial A_\perp}{\partial x} \frac{\partial f}{\partial p} = 0, \]
\[ \frac{\partial E_x}{\partial t} = - \int_{\mathbb{R}} p f(x, p, t) dp + \bar{J}(t), \]
\[ \frac{\partial E_\perp}{\partial t} = \frac{A_\perp}{\gamma} \int_{\mathbb{R}} f(x, p, t) dp, \frac{\partial A_\perp}{\partial t} = 0. \]
We can see that we cannot solve $H_p$ explicitly.

Finally, the subsystem corresponding to $H_A$ is
\[ \frac{\partial f}{\partial t} = 0, \frac{\partial E_x}{\partial t} = 0, \frac{\partial E_\perp}{\partial t} = - \frac{\partial^2 A_\perp}{\partial x^2}, \frac{\partial A_\perp}{\partial t} = 0. \]

For the details of derivation of $H_p$, as $H_p = \int_{\Omega \times \mathbb{R}} \left( \sqrt{1 + p^2 + |A_\perp|^2} - 1 \right) f dx dp$, we have the following variational derivatives,
\[ \frac{\delta H_p}{\delta f} = \sqrt{1 + p^2 + |A_\perp|^2} - 1, \frac{\delta H_p}{\delta E_x} = 0, \frac{\delta H_p}{\delta E_\perp} = 0, \frac{\delta H_p}{\delta A_\perp} = A_\perp \int_{\Omega \times \mathbb{R}} \frac{1}{\gamma} f dp. \]
For the evolution of $f$, we denote $F = \int_{\Omega \times \mathbb{R}} f \delta(x - x_0, p - p_0) dx dp$, and then
\[ \frac{\delta F}{\delta f} = \delta(x - x_0, p - p_0), \frac{\delta F}{\delta E_x} = 0, \frac{\delta F}{\delta E_\perp} = 0, \frac{\delta F}{\delta A_\perp} = 0. \] Inserting the variational derivatives of $F$ and $H_p$ into our Poisson bracket, we have
\[ \{ F, H_p \} = - \frac{p_0}{\sqrt{1 + p_0^2 + |A_\perp(x_0, t)|^2}} \frac{\partial f}{\partial x}(x_0, p_0, t) + \frac{A_\perp(x_0, t)}{\gamma} \frac{\partial A_\perp(x_0, t)}{\partial x} \frac{\partial f}{\partial p}(x_0, p_0, t). \]
As \( \frac{df}{dt} = \{F, \mathcal{H}_p\} \), \( F = f(x_0, p_0, t) \), and \( x_0, p_0 \) are arbitrary points in \( \Omega \times \mathbb{R} \), evolution of \( f \) can be expressed as

\[
\frac{\partial f}{\partial t} = \{f, \mathcal{H}_p\} = -\frac{p}{\gamma} \frac{\partial f}{\partial x} + \frac{1}{\gamma} A_\perp \cdot \frac{\partial A_\perp}{\partial x} \frac{\partial f}{\partial p}.
\]

In the same way, the evolution of \( E_\perp \) and \( A_\perp \) is

\[
\frac{\partial E_\perp}{\partial t} = \frac{A_\perp}{\gamma} \int_{\mathbb{R}} f(x, p, t) dp, \quad \frac{\partial A_\perp}{\partial t} = 0.
\]

For the evolution of \( E_x \), we denote \( F = \int_{\Omega} E_x \delta(x - x_0) dx \). Then we have \( \frac{\delta F}{\delta f} = 0, \frac{\delta F}{\delta E_x} = \delta(x - x_0) \frac{1}{L}, \frac{\delta F}{\delta A_\perp} = \mathbf{0}, \frac{\delta F}{\delta p} = \mathbf{0} \). \( \frac{\delta F}{\delta E_x} = \delta(x - x_0) \frac{1}{L} \) comes from the constraint that \( \int_{\Omega} E_x dx = 0 \). Following the above procedures, we have

\[
\frac{\partial E_x}{\partial t} = -\int_{\mathbb{R}} p f(x, p, t) dp + \bar{J}(t).
\]

**Appendix C.** Adveective splitting for fully relativistic case is as follows.

The subsystem corresponding to \( \mathcal{H}_E \) is:

\[
\frac{\partial f}{\partial t} + E_x \frac{\partial f}{\partial p} = 0, \quad \frac{\partial E_x}{\partial t} = 0, \quad \frac{\partial E_\perp}{\partial t} = 0, \quad \frac{\partial A_\perp}{\partial t} = -E_\perp.
\]

The subsystem corresponding to \( \mathcal{H}_A \) is

\[
\frac{\partial f}{\partial t} - \frac{1}{\gamma} A_\perp \cdot \frac{\partial A_\perp}{\partial x} \frac{\partial f}{\partial p} = 0,
\]

\[
\frac{\partial E_\perp}{\partial t} = -\frac{\partial^2 A_\perp}{\partial x^2} + A_\perp \int_{\mathbb{R}} \frac{1}{\gamma} f(x, p, t) dp, \quad \frac{\partial E_x}{\partial t} = 0, \quad \frac{\partial A_\perp}{\partial t} = 0.
\]

The subsystem corresponding to \( \mathcal{H}_p \) is:

\[
\frac{\partial f}{\partial t} + \frac{p}{\gamma} \frac{\partial f}{\partial x} = 0, \quad \frac{\partial E_\perp}{\partial t} = 0
\]

\[
\frac{\partial E_x}{\partial t} = -\int_{\mathbb{R}} p f(x, p, t) dp + \bar{J}(t), \quad \frac{\partial A_\perp}{\partial t} = 0.
\]

**References**


