Beyond GALERKIN Projection by Using “Multi-space” Priors

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Let $\mathcal{H}$ be some Hilbert space with induced norm $\|\cdot\|$. We consider the problem of approximating the solutions of a parametric partial differential equation (PPDE), say $\mathcal{M} = \{\boldsymbol{h} : \mathrm{PDE}(\boldsymbol{h}, \theta) = 0 \text{ for some } \theta \in \Theta\}$, within a N-dimensional subspace $V_N \subset \mathcal{H}$. We consider a PPDE whose weak formulation takes the following form:

$$\text{find } \boldsymbol{h} \in \mathcal{H} \text{ such that } \ a_\theta(\boldsymbol{h}, \boldsymbol{h}') = b_\theta(\boldsymbol{h}') \quad \text{for } \boldsymbol{h}' \in \mathcal{H},$$

where $a_\theta(\cdot, \cdot)$ and $b_\theta(\cdot)$ are respectively some bilinear and linear forms.

The orthogonal projection (with respect to $\|\cdot\|$) of the elements of $\mathcal{M}$ onto V_N being usually too computationally-demanding, one standard option is to resort to Galerkin projection:

$$\text{find } \boldsymbol{h} \in V_N \text{ such that } \ a_\theta(\boldsymbol{h}, \boldsymbol{h}') = b_\theta(\boldsymbol{h}') \quad \text{for } \boldsymbol{h}' \in V_N.$$

The “quality” of the Galerkin approximation (its closeness to the true orthogonal projection) depends on the “conditioning” of the operator $a_\theta(\cdot, \cdot)$ and $b_\theta(\cdot)$ (*e.g.*, via their coercivity and continuity constants [1]). In some difficult case, Galerkin projection may thus leads to poor approximation results. In our work, we propose a simple way to improve Galerkin projections.

We consider the setup where V_N corresponds to the N-dimensional subspace computed via a reduced-basis method [1]. Now, while computing the subspace V_N, this type of methodology also generates a sequence of subspaces $\{V_i\}_{i=0}^N$ and some positive scalars $\{\epsilon_i\}_{i=0}^N$ such that

$$V_0 \subset V_1 \subset \ldots \subset V_N,$$

and

$$\sup_{\boldsymbol{h} \in \mathcal{M}} \mathrm{dist}(\boldsymbol{h}, V_i) \leq \epsilon_i.$$

The last inequality provides some information about $\mathcal{M}$ since it implies that the latter is included in the intersection of N (degenerate) ellipsoids, *i.e.*, $\mathcal{M} \subseteq \cap_{i=0}^N \{\boldsymbol{h} : \sup_{\boldsymbol{h} \in \mathcal{M}} \mathrm{dist}(\boldsymbol{h}, V_i) \leq \epsilon_i\}$.

In our work, we propose a new suboptimal projection method exploiting the fact that $\mathcal{M}$ is included in the intersection of a set of *known* ellipsoids. The proposed methodology boils down to the standard Galerkin projection when one single ellipsoid is considered. We provide both theoretical and empirical results showing that the proposed methodology clearly outperforms the standard Galerkin projection in some situations. Our derivations are based on the recent work by Binev *et al.* [2]

References

[1] A. Quarteroni, A. Manzoni and F. Negri *“Reduced Basis Method for Partial Differential Equation: An Introduction”*, Springer, Vol. 92, 2015.

[2] P. Binev, and *al.* *“Data assimilation in reduced modeling”*, SIAM/ASA Journal on Uncertainty Quantification, Vol. 5, 2017.