Introduction

Modelling parallel systems

Linear Time Properties

Regular Properties

Linear Temporal Logic (LTL)

- syntax and semantics of LTL
- automata-based LTL model checking
- complexity of LTL model checking ⟵

Computation-Tree Logic

Equivalences and Abstraction

Complexity of LTL model checking

main steps of automata-based LTL model checking:

construction of an NBA $\mathcal{A}$ for $\neg\varphi$

persistence checking in the product $\mathcal{T} \otimes \mathcal{A}$

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The **LTL** model checking problem is **PSPACE**-complete

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we show

- just for fun: **coNP**-hardness
- **PSPACE**-completeness

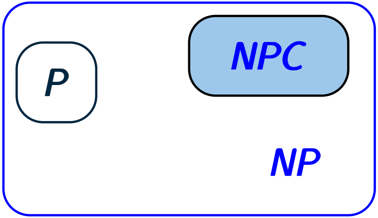
Recall: complexity classes

Complexity classes P, NP

P = class of decision problem solvable in deterministic polynomial time

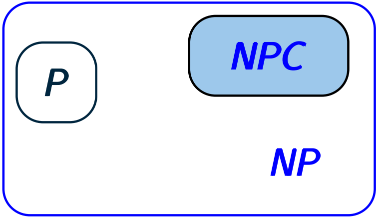
NP = class of decision problem solvable in nondeterministic polynomial time

Complexity classes *P*, *NP*



NPC = class of ***NP***-complete problems

Complexity classes P, NP



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(1) $L \in NP$

(2) L is NP-hard, i.e., $K \leq_{poly} L$ for all $K \in NP$

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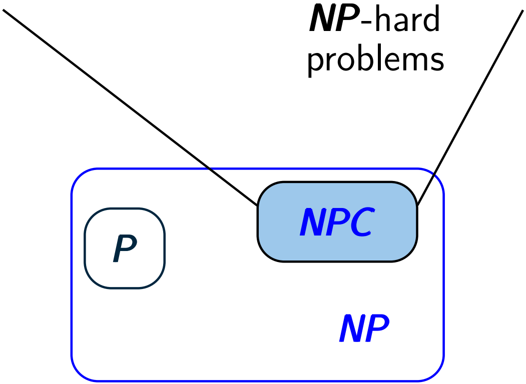

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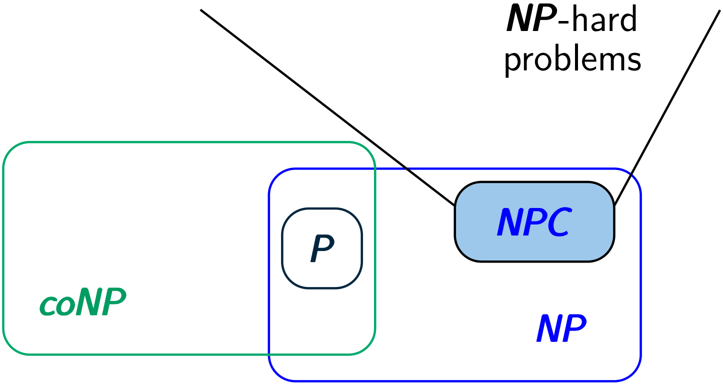
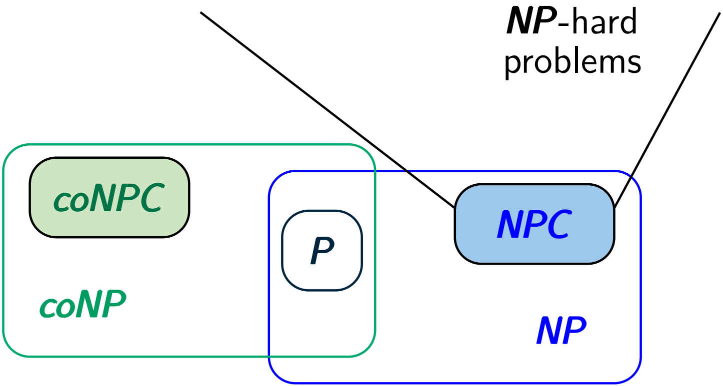


$$coNP \;=\; \{\, \overline{L} : L \in NP \,\}$$

$\uparrow$ complement of L



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(1) $L \in$ ***coNP***

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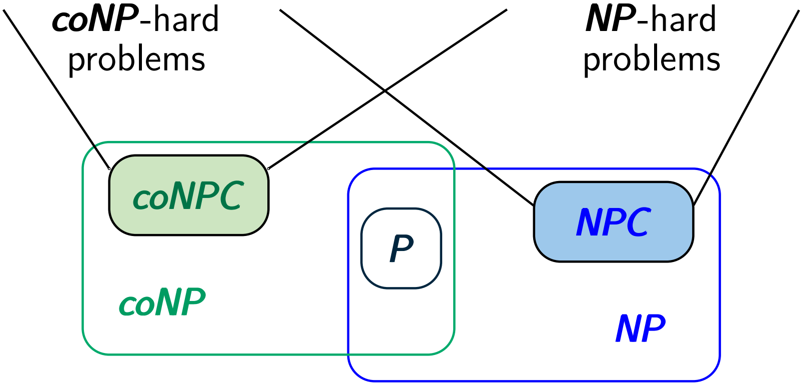

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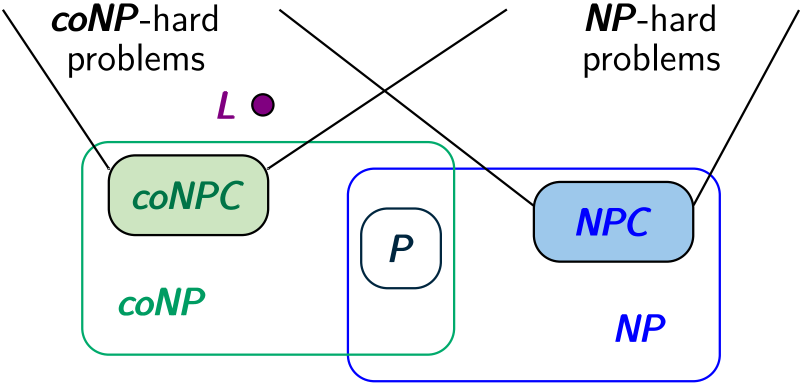

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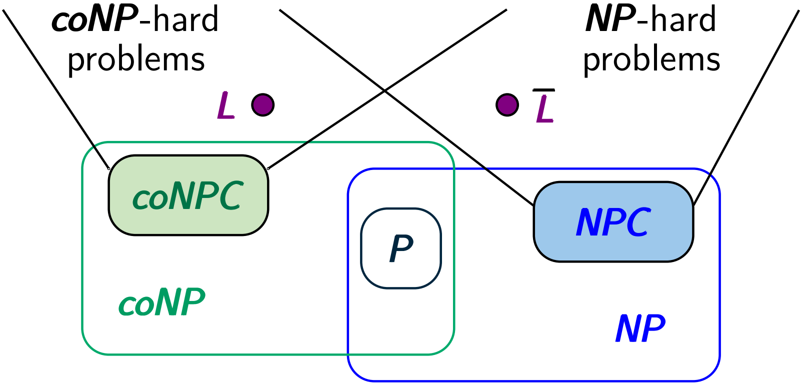

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***coNP*-hardness**

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proof by a polynomial reduction

$$HP \leq_{poly} \overline{LTL\text{-}MC}$$

HP: Hamilton path problem

$\overline{LTL\text{-}MC}$: complement of the **LTL** model checking problem

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Hamilton path problem ↗ HP

complement of the **LTL** model checking problem ↖ $\overline{LTL\text{-}MC}$

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NP-complete ↑ Hamilton path problem

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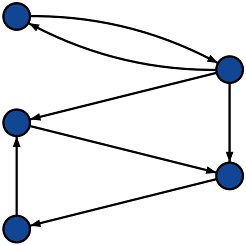

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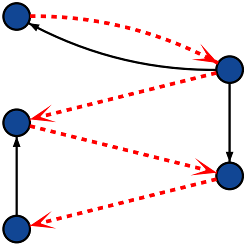

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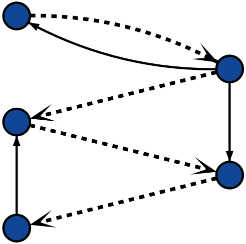

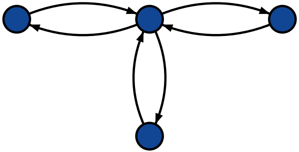

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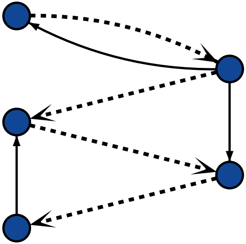

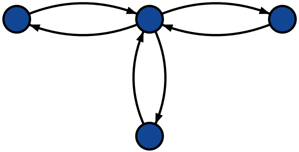

has no Hamilton path

HP is known to be ***NP***-complete

Polynomial reduction

$$HP \quad \leq_{poly} \quad \overline{\textit{LTL-MC}}$$

Hamilton path problem ↗ HP

↖ $\overline{\textit{LTL-MC}}$: complement of the **LTL** model checking problem

Polynomial reduction

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Hamilton path problem $\nearrow$ HP

$\overline{LTL\text{-}MC}$ $\nwarrow$ complement of the **LTL** model checking problem

finite directed graph G	$\xrightarrow{\text{poly time}}$	finite TS $\mathcal{T}$, LTL formula φ

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Hamilton path problem

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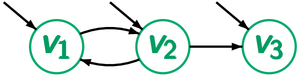
node-set V of G $\quad \widehat{=} \quad$ states of $\mathcal{T}$



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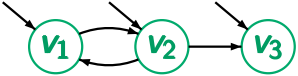

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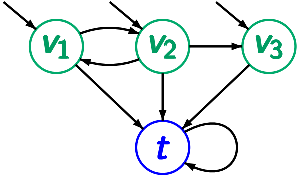
additional trap state t



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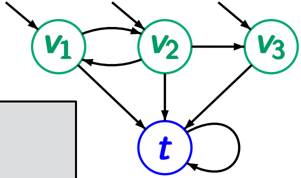

$\varphi = ?$

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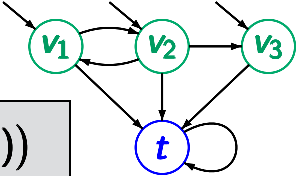

$$\varphi = \bigwedge_{v \in V} \left(\Diamond v \wedge \Box(v \rightarrow \bigcirc\Box\neg v) \right)$$

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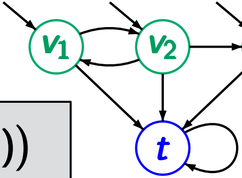

$$\varphi = \neg \bigwedge_{v \in V} \left(\Diamond v \wedge \Box(v \rightarrow \bigcirc \Box \neg v) \right)$$

Complexity of LTL model checking

We just saw:

The **LTL** model checking problem is ***coNP***-hard

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We now prove:

The **LTL** model checking problem is ***PSPACE***-complete

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  ↑ **DFS**-based analysis of the computation tree of an ***NP***-algorithm

The complexity class *PSPACE*

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  ↑

  DFS-based analysis of the computation tree of an *NP*-algorithm

  space requirements:
  recursion depth $\widehat{=}$ height of computation tree

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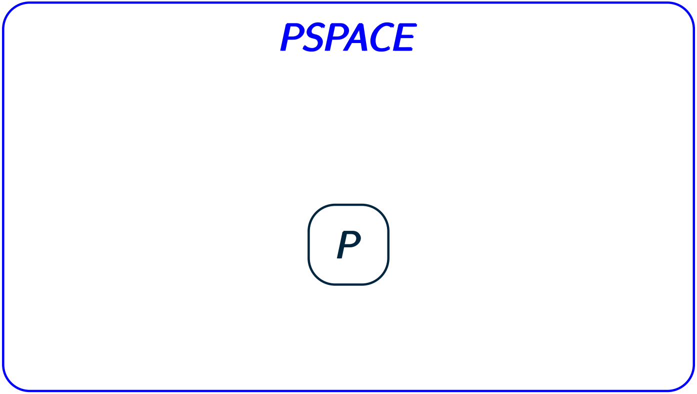
To prove $L \in$ *PSPACE* it suffices to provide a nondeterministic polynomially space-bounded algorithm for the complement $\overline{L}$ of L

Complexity classes *P*, *NP*, *coNP*, *PSPACE*



PSPACE = class of decision problems that are decidable in deterministic polynomial space

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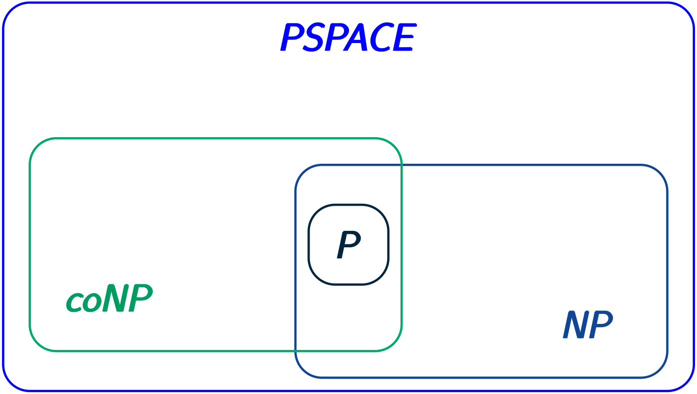

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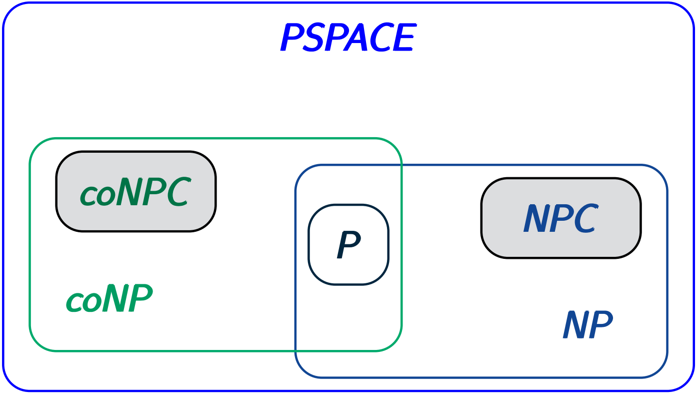

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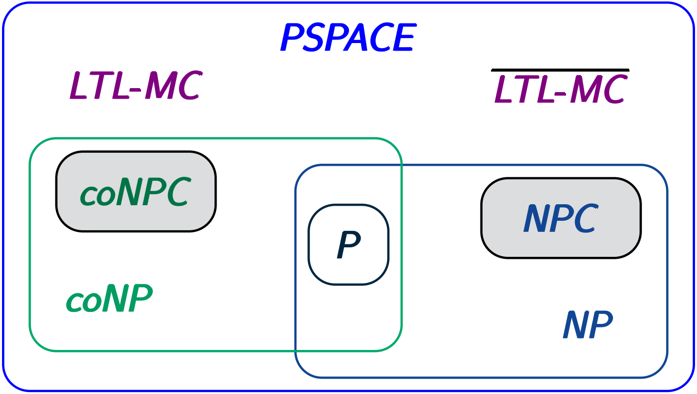

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PSPACE-completeness

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decision problem L is *PSPACE*-complete iff

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L is *PSPACE*-hard $\iff$ $\overline{L}$ is *PSPACE*-hard

$L \in \mathit{PSPACE}$ $\iff$ $\overline{L} \in \mathit{NPSPACE}$

PSPACE-completeness of *LTL-MC*

LTL-MC LTL model checking problem

"does $\pi \models \varphi$ hold for all paths π of $\mathcal{T}$ **?**"

$\overline{\textit{LTL-MC}}$ = complement of ***LTL-MC***

"does $\pi \not\models \varphi$ hold for some path π of $\mathcal{T}$ **?**"

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↑

$\exists$***LTL-MC*** existential LTL model checking problem
for $\mathcal{T}$ and LTL formula $\psi = \neg\varphi$

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show: ***∃LTL-MC*** $\in$ ***NPSPACE***

∃LTL-MC is ***PSPACE***-hard

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show: ***∃LTL-MC*** $\in$ ***NPSPACE*** $\Longrightarrow$ ***LTL-MC*** $\in$ ***PSPACE***

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Existential LTL model checking problem

given:	$\mathcal{T}$ be a finite transition system
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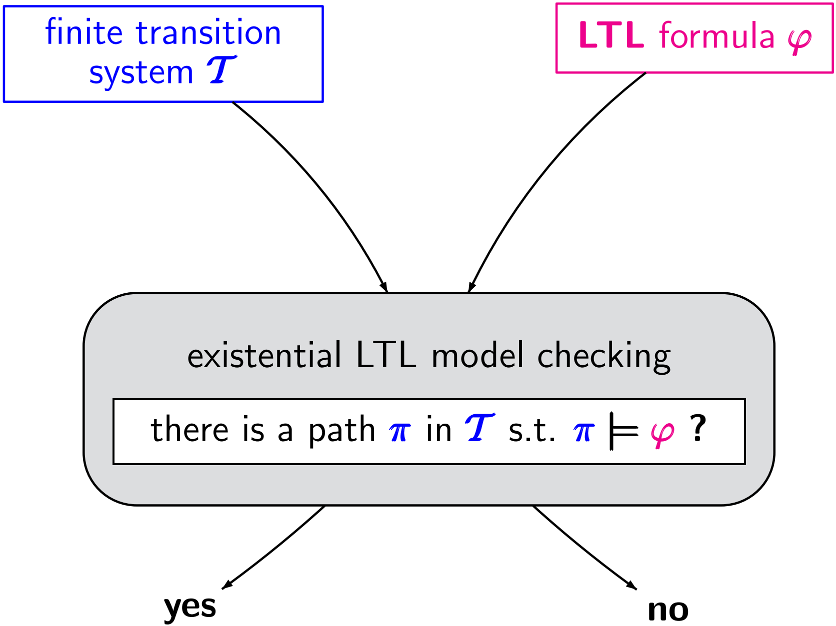
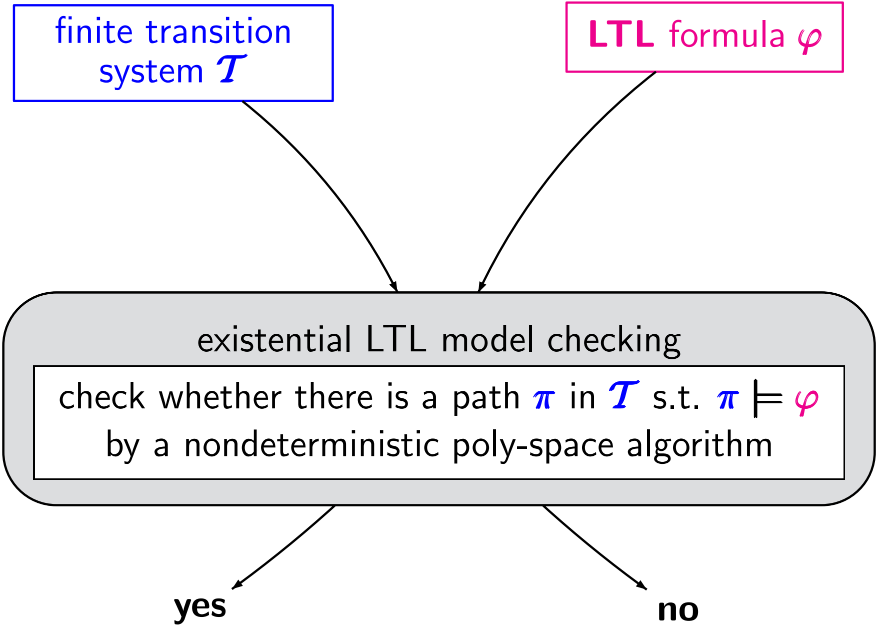
idea: use the **GNBA** $\mathcal{G}$ for φ (constructed by our LTL-2-GNBA algorithm)

Existential LTL model checking



Existential LTL model checking



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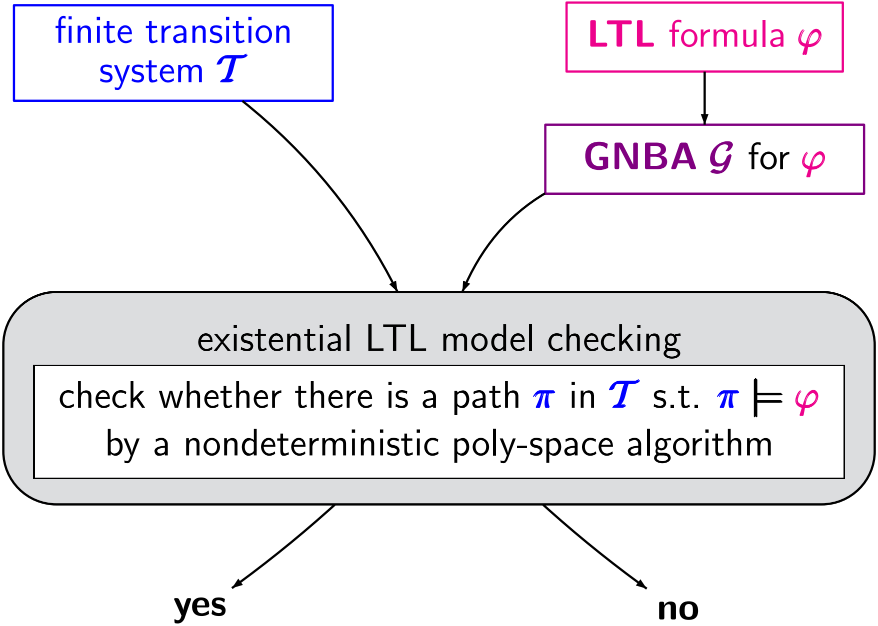

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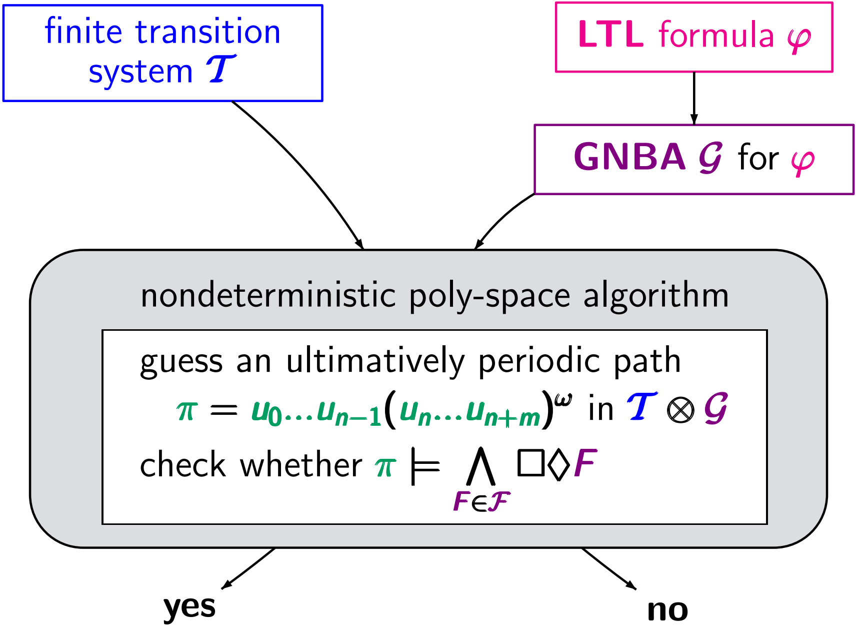

Recall: elementary formula-sets

closure $cl(\varphi)$:

- set of all subformulas of φ and their negations
- ψ and $\neg\neg\psi$ are identified

elementary formula-sets: subsets B of $cl(\varphi)$

- maximal consistent w.r.t. propositional logic
- locally consistent w.r.t. **U**

For $\varphi = a \, \mathsf{U}(\neg a \wedge b)$, the elementary sets are:

$\{\ a,\ \ b, \neg(\neg a \wedge b), \varphi\}$ $\{\ a,\ \ b, \neg(\neg a \wedge b), \neg\varphi\}$

$\{\ a, \neg b, \neg(\neg a \wedge b), \varphi\}$ $\{\ a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

$\{\neg a,\ \ b,\ \ \neg a \wedge b\ , \varphi\}$ $\{\neg a, \neg b, \neg(\neg a \wedge b), \neg\varphi\}$

Recall: GNBA for LTL-formula φ

$$\mathcal{G} = (Q, 2^{AP}, \delta, Q_0, \mathcal{F})$$

state space: $Q = \{B \subseteq cl(\varphi) : B \text{ is elementary}\}$

initial states: $Q_0 = \{B \in Q : \varphi \in B\}$

transition relation: for $B \in Q$ and $A \in 2^{AP}$:

if $A \neq B \cap AP$ then $\delta(B, A) = \varnothing$

if $A = B \cap AP$ then $\delta(B, A) =$ set of all $B' \in Q$ s.t.

$$\bigcirc\psi \in B \quad \text{iff} \quad \psi \in B'$$

$$\psi_1 \mathsf{U} \psi_2 \in B \quad \text{iff} \quad (\psi_2 \in B) \vee (\psi_1 \in B \wedge \psi_1 \mathsf{U} \psi_2 \in B')$$

acceptance set $\mathcal{F} = \{F_{\psi_1 \mathsf{U} \psi_2} : \psi_1 \mathsf{U} \psi_2 \in cl(\varphi)\}$

where $F_{\psi_1 \mathsf{U} \psi_2} = \{B \in Q : \psi_1 \mathsf{U} \psi_2 \notin B \vee \psi_2 \in B\}$

Criterion for existential LTL properties

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$$(1) \quad \langle s_n, B_{n+1} \rangle = \langle s_{n+m}, B_{n+m+1} \rangle$$

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(2) whenever $\psi_1 \mathsf{U} \psi_2 \in B_{n+1} \cup \ldots \cup B_{n+m}$ then $\psi_2 \in B_{n+1} \cup \ldots \cup B_{n+m}$

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NPSPACE-algorithm for $\exists$*LTL-MC*

The existential LTL model checking problem

given: finite TS $\mathcal{T}$, LTL formula φ
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- guess nondeterministically an ultimatively periodic path $\pi = u_0\, u_1 \ldots u_{n-1}(u_n \ldots u_{n+m})^\omega$ of $\mathcal{T} \otimes \mathcal{G}$

  ↑ GNBA for φ obtained by our LTL-2-GNBA algorithm

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- check whether the guessed path meets the acceptance condition of $\mathcal{G}$

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guess two natural numbers $n, m \leq k$ s.t. $m \geq 1$
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If so then return "**yes**". Otherwise return "**no**".

PSPACE-completeness of $\exists$*LTL-MC*

We saw that:

The existential LTL model checking problem

given: finite TS $\mathcal{T}$, LTL formula φ

question: is there a path π in $\mathcal{T}$ with $\pi \models \varphi$ **?**

belongs to ***NPSPACE***

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It remains to prove the *PSPACE*-hardness

PSPACE-hardness of ∃*LTL-MC*

we show that for all problems $K \in PSPACE$:

$$K \leq_{poly} \exists LTL\text{-}MC$$

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Let

- $\mathcal{M}$ be a deterministic Turing machine (DTM) that decides K,
- P a polynomial

such that $\mathcal{M}$ started with an input word w visits at most $P(|w|)$ tape cells

PSPACE-hardness of $\exists$*LTL-MC*

we show that for all problems $K \in \mathit{PSPACE}$:

$$K \leq_{poly} \exists \mathit{LTL\text{-}MC}$$

Given DTM $\mathcal{M}$ that decides K with polynomial space bound $P(n)$, provide a polynomial reduction:

input word w for $\mathcal{M}$	poly time $\longrightarrow$	TS $\mathcal{T}$ LTL-formula φ

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Given DTM $\mathcal{M}$ that decides K with polynomial space bound $P(n)$, provide a polynomial reduction:

input word w for $\mathcal{M}$	$\xrightarrow{\text{poly time}}$	TS $\mathcal{T}$ LTL-formula φ
$\mathcal{M}$ accepts w, i.e., $w \in K$	iff	there is path π of $\mathcal{T}$ with $\pi \models \varphi$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$

DTM $\mathcal{M}$ visits at the most the tape cells $1, 2, \ldots, P(n)$ for inputs of length n (where P is a polynomial)

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\$	A_1	$\ldots$	A_{n-1}	A_n	$\sqcup$	$\sqcup$	$\ldots$	$\sqcup$	$\ldots$
1	2		n	$n+1$					

initial tape configuration for input $w = A_1 A_2 \ldots A_n$

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⊔ $\widehat{=}$ blank symbol of $\mathcal{M}$

$ $\widehat{=}$ symbol for the left border of the tape

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$\$$	A_1	...	A_{n-1}	A_n	$\sqcup$	$\sqcup$	...	$\sqcup$	...
1	2		n	$n+1$				$P(n)$	↑ not visited

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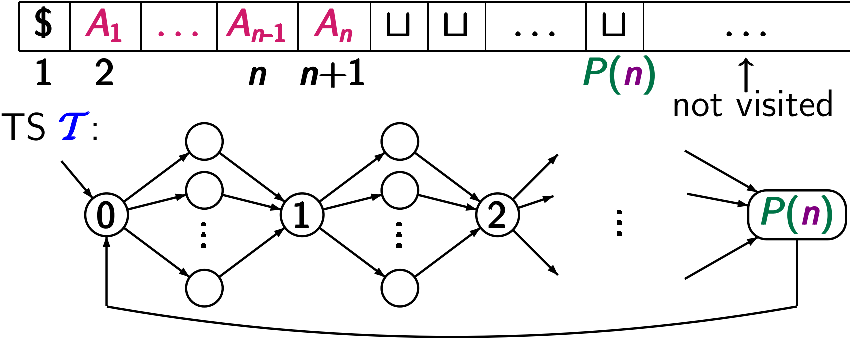

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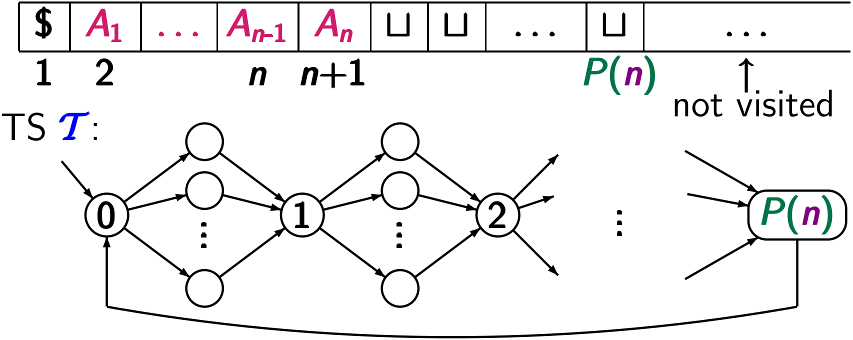

states of $\mathcal{T}$: $0, 1, \ldots, P(n)$,

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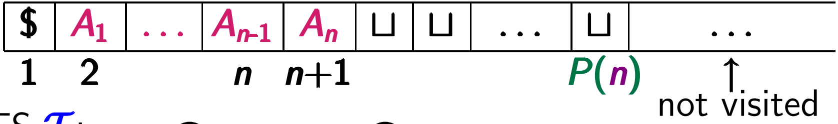
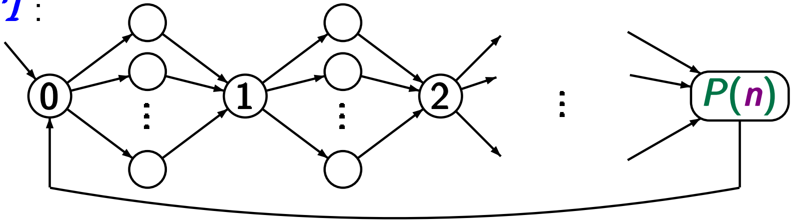


TS $\mathcal{T}$:



states of $\mathcal{T}$: $0, 1, \ldots, P(n)$, $\langle q, A, i \rangle$, $\langle *, A, i \rangle$

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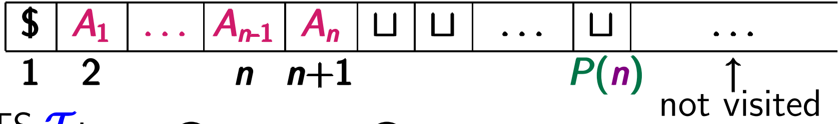

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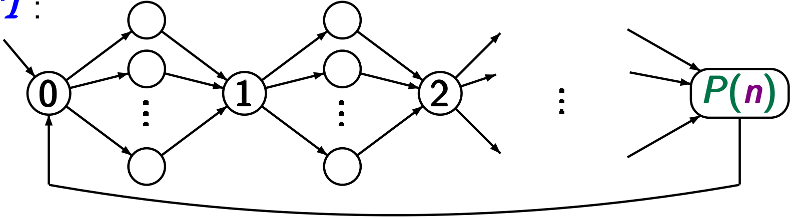

states of $\mathcal{T}$: $0, 1, \ldots, P(n)$, $\langle q, A, i \rangle$, $\langle *, A, i \rangle$
where q is a state of $\mathcal{M}$, A a tape symbol, $1 \leq i \leq P(n)$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$

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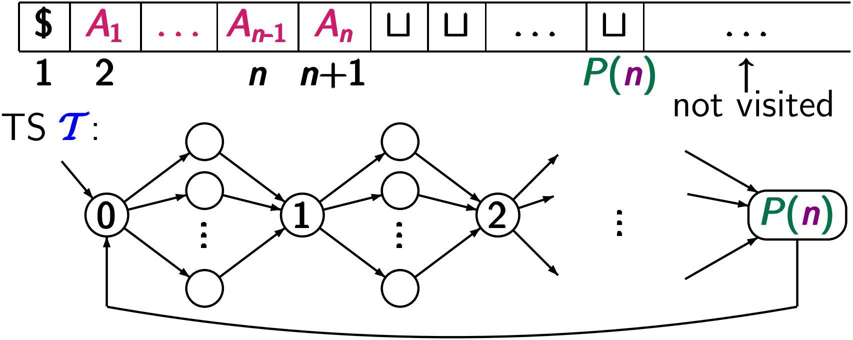
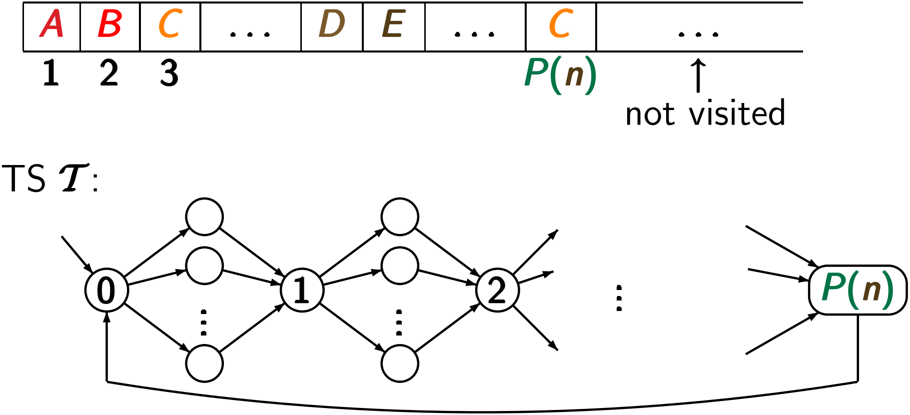
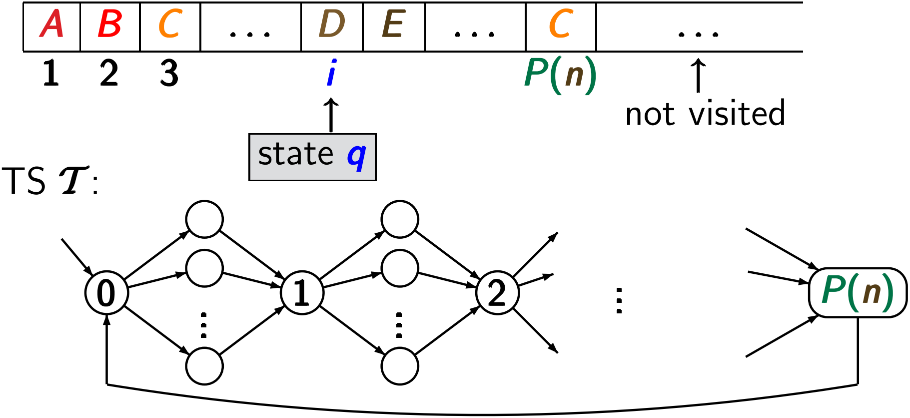


idea: TS $\mathcal{T}$ encodes each configuration of $\mathcal{M}$ by a path fragment from state 0 to state $P(n)$

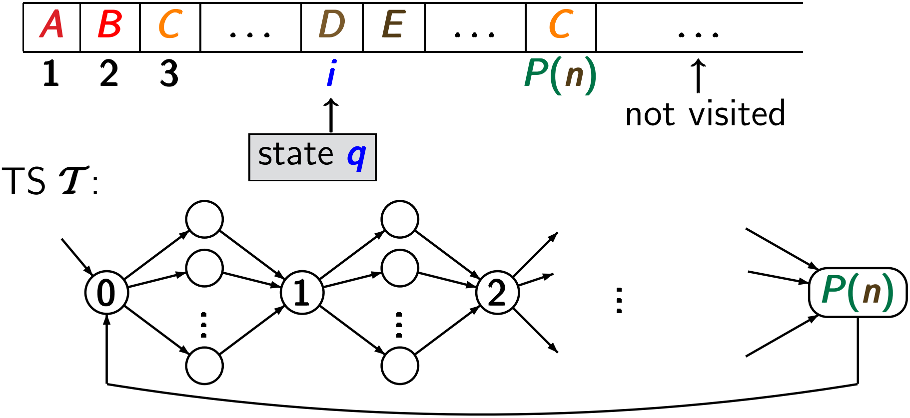
Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$



Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$

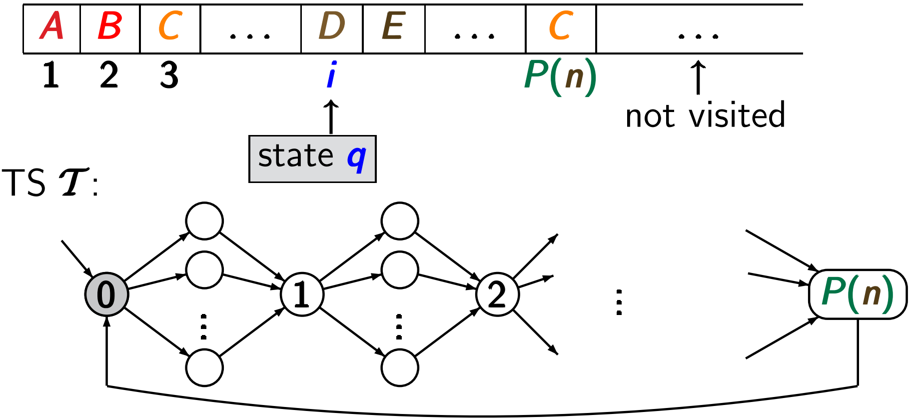


Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$



suppose $\delta(q, D) = (p, B, +1)$

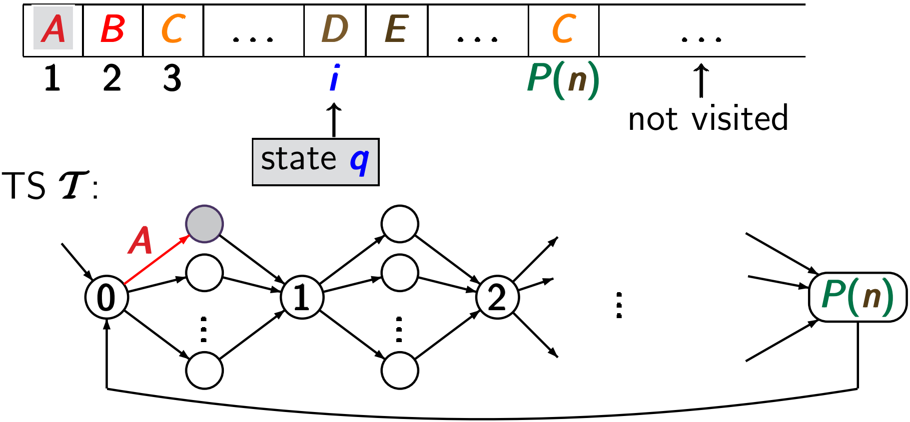
Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$



0

path fragment for the configuration $ABC\ldots\, q\, D\ldots C$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$



$0 \ \langle *, A, 1 \rangle$

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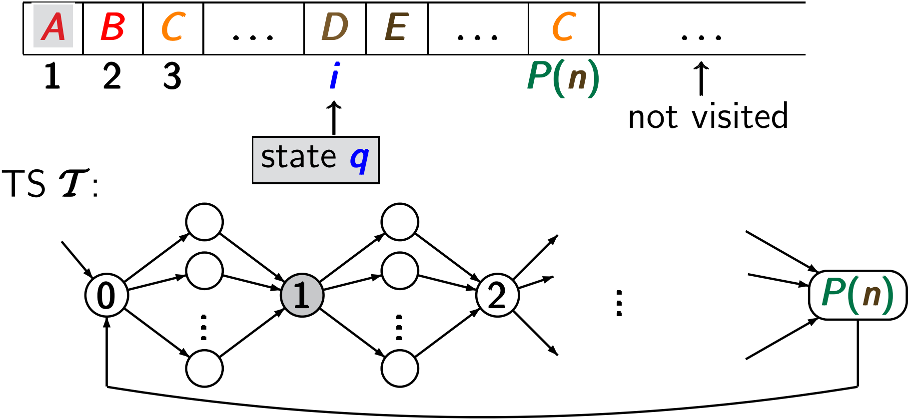

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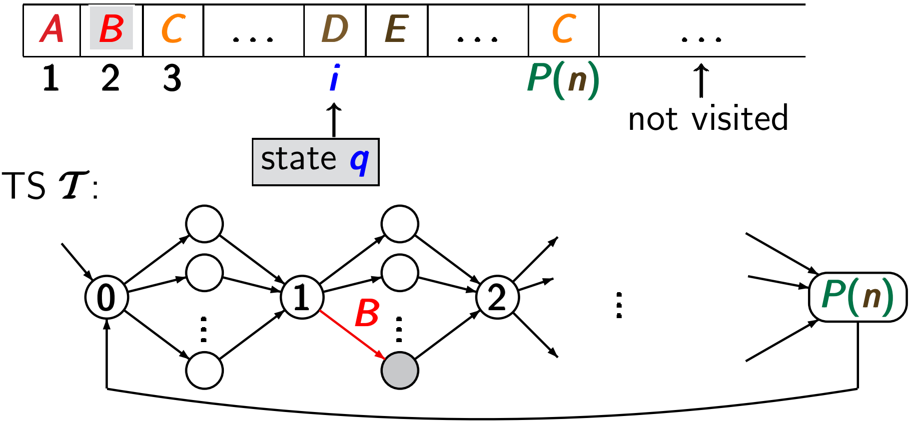

$0 \ \langle *, A, 1 \rangle \ 1 \ \langle *, B, 2 \rangle$

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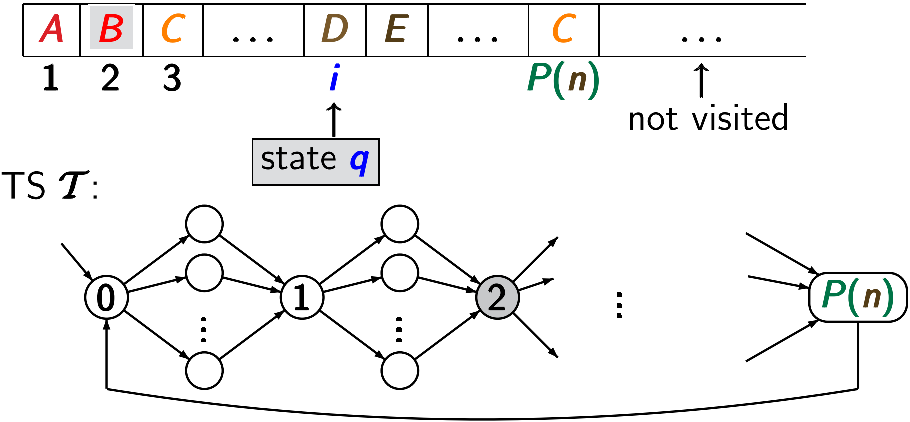

$0 \ \langle *, A, 1 \rangle \ 1 \ \langle *, B, 2 \rangle \ 2$

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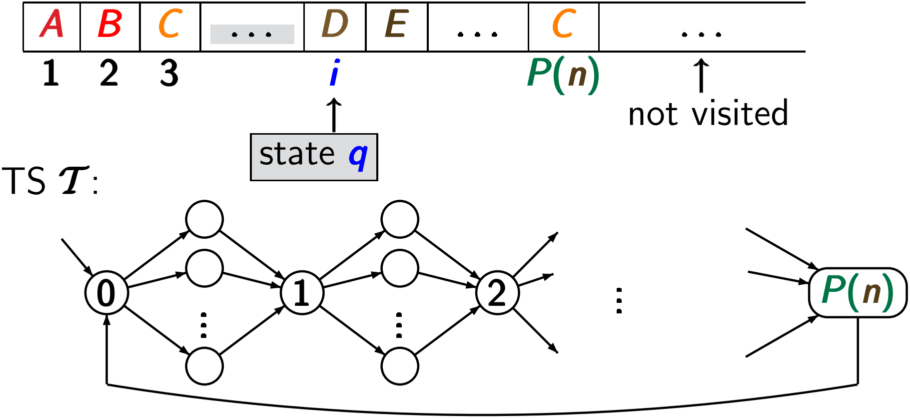

$0 \; \langle *, A, 1 \rangle \; 1 \; \langle *, B, 2 \rangle \; 2 \; \ldots \; (i{-}1)$

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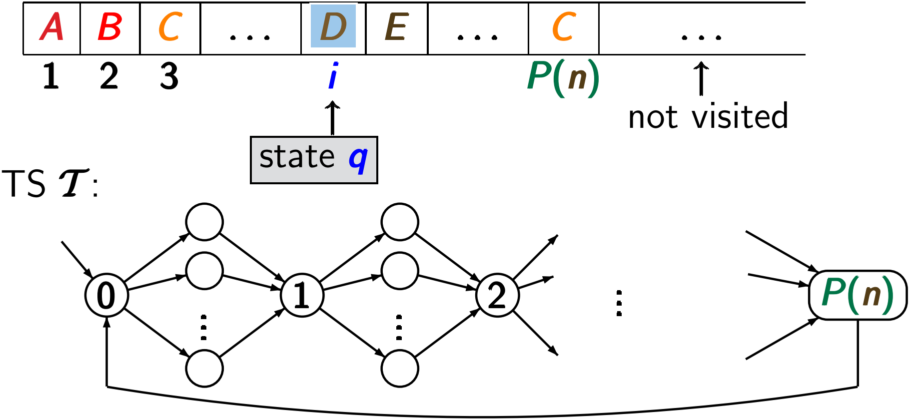

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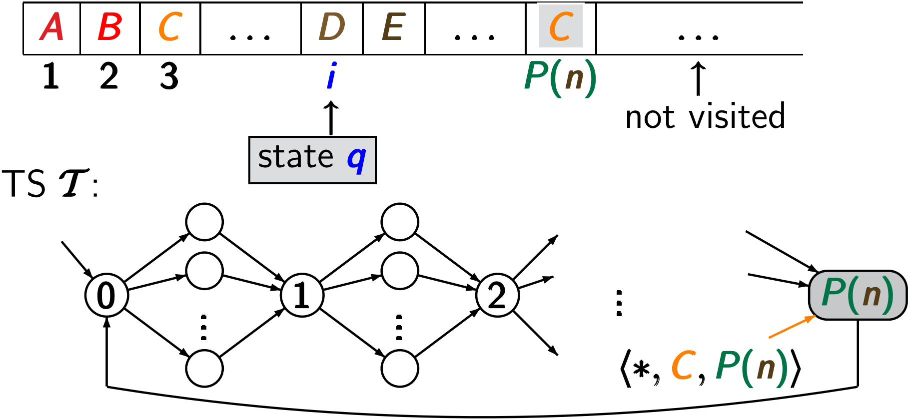

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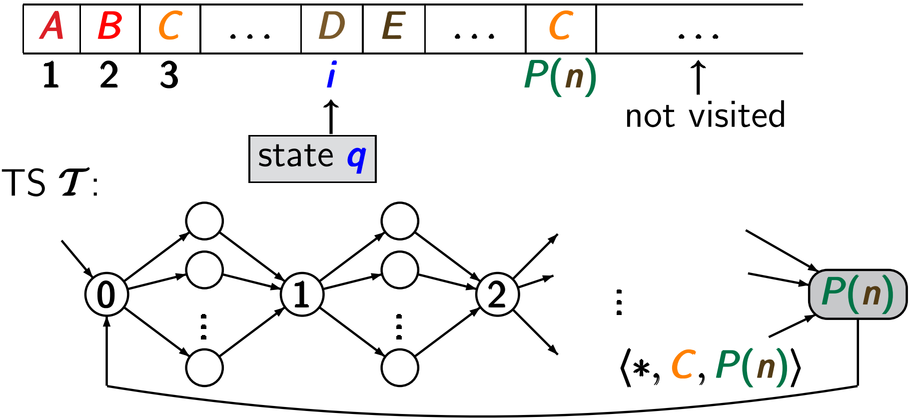

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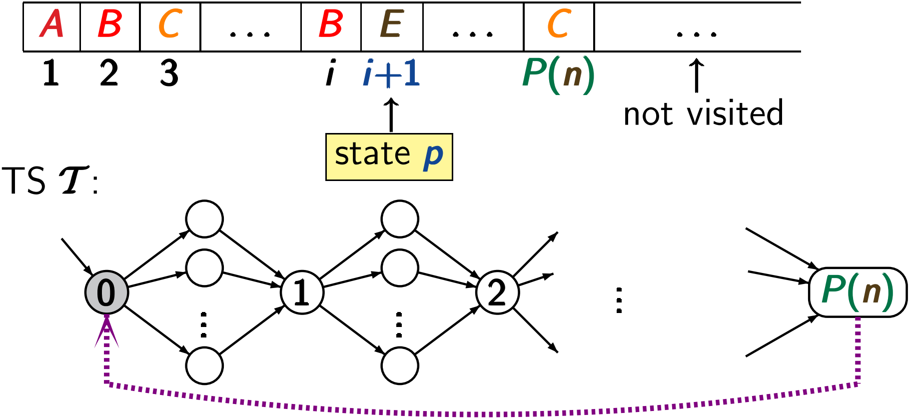

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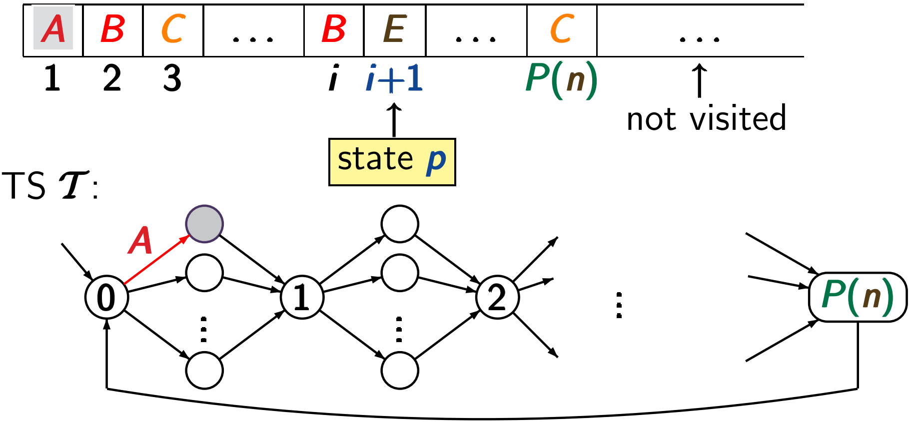

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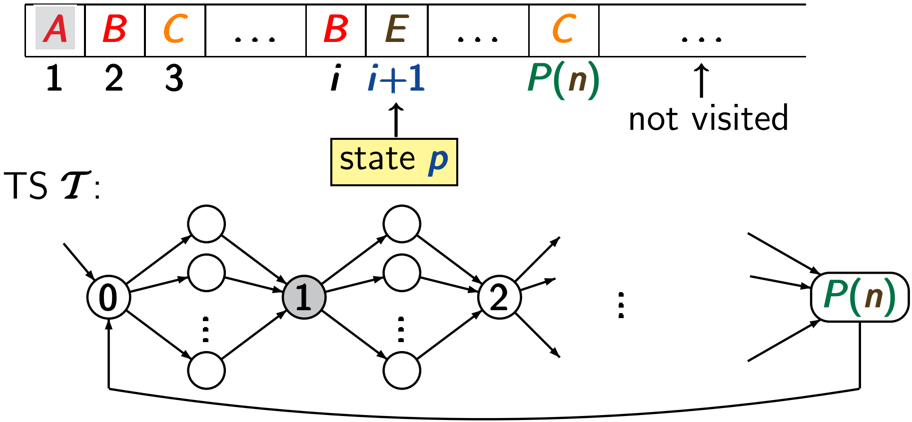

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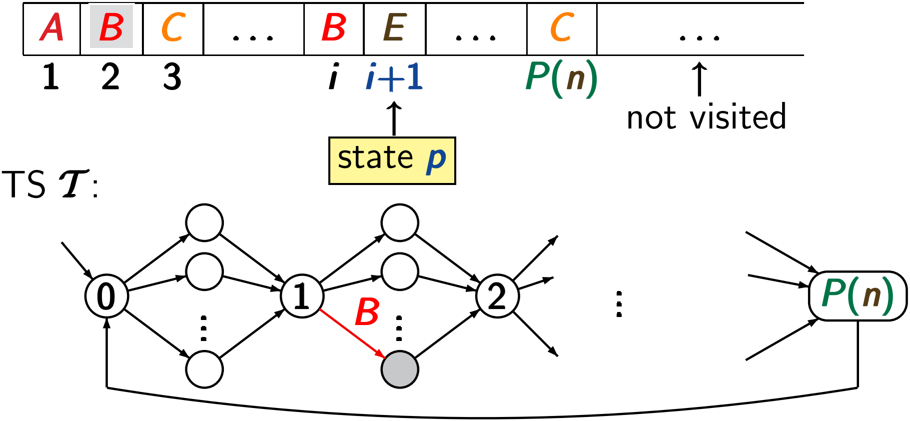

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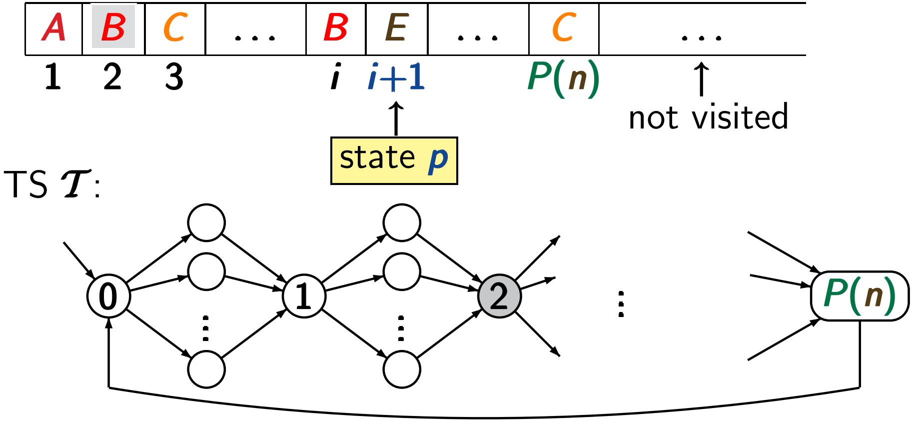

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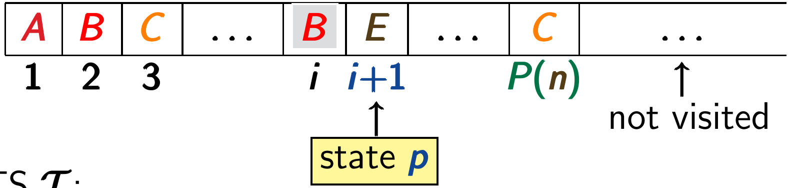

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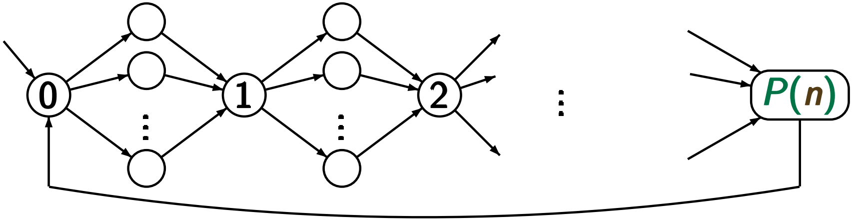

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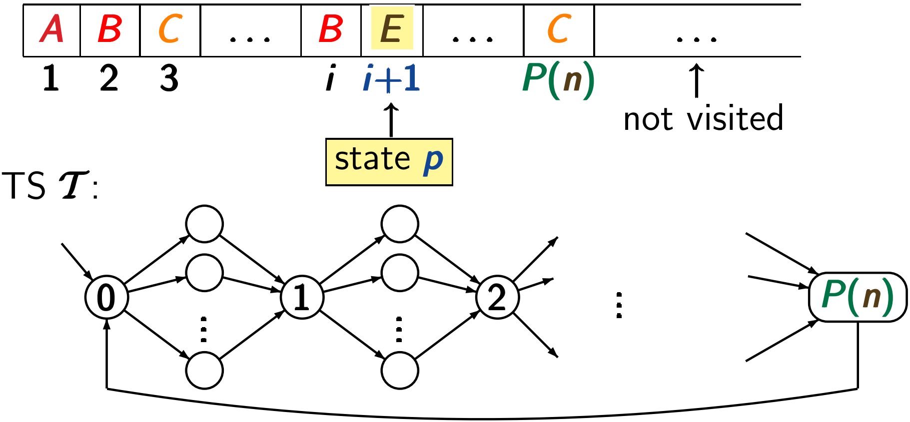

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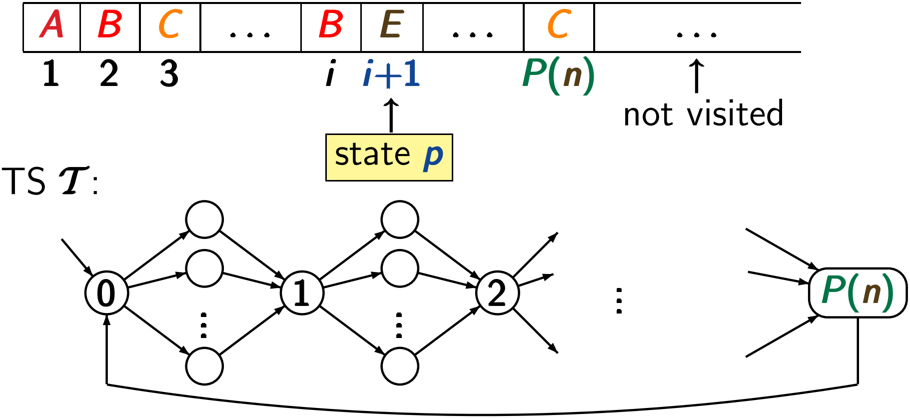

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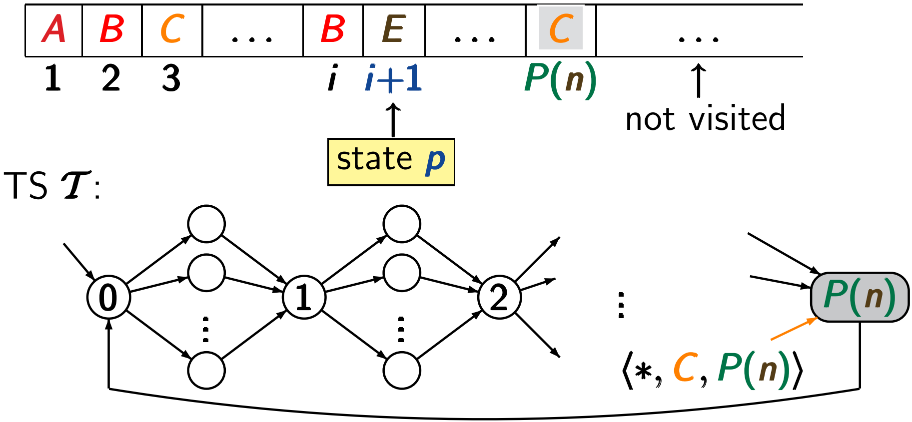

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Let $\mathcal{M}$ be a DTM with polynomial space bound $P(n)$

- state space Q
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- input alphabet $\Sigma \subseteq \Gamma$
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input word w for $\mathcal{M}$	$\xrightarrow{\text{poly time}}$	TS $\mathcal{T}$ LTL-formula φ
$\mathcal{M}$ accepts w, i.e., $w \in K$	iff	there is path π of $\mathcal{T}$ with $\pi \models \varphi$

Polynomial reduction $w \mapsto (\mathcal{T}, \varphi)$

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transitions:

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LTL formula $\varphi \stackrel{\text{def}}{=} \varphi^w_{start} \wedge \varphi_\delta \wedge \varphi_{conf} \wedge \varphi_{accept}$

Complexity of LTL model checking problem

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We saw that:

The existential **LTL** model checking problem

given: finite TS $\mathcal{T}$, LTL formula φ

question: is there a path π in $\mathcal{T}$ with $\pi \models \varphi$ **?**

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As ***PSPACE*** = ***coPSPACE*** we get:

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additionally we proved ***coNP***-hardness
using an LTL-encoding of the Hamilton-path problem

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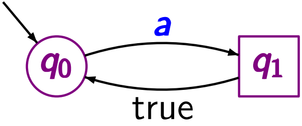

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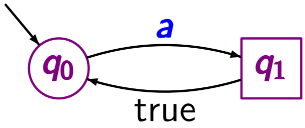

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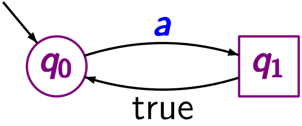

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$\sigma \;=\; \{a\}\{a\}\{a\}\varnothing\{a\}^\omega \;\not\models\; \varphi$, but $\sigma \in \mathcal{L}_\omega(\mathcal{A})$

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examples:	$\Diamond\Box a \wedge \Box\Diamond\neg a$	unsatisfiable
	$a \,\mathsf{U}\, b \wedge \Box\neg b$	unsatisfiable
	$\Diamond\Box a \wedge a \,\mathsf{U}(\Box b)$	satisfiable

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