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On Timed Alternating Simulation for Concurrent Timed Games

Laura Bozzelli[1], **Axel Legay**[1], **Sophie Pinchinat**[1]
IRISA, Campus de Beaulieu, 35042 Rennes Cedex, FRANCE.

ABSTRACT. We address the problem of alternating simulation refinement for concurrent timed games (TG). We show that checking timed alternating simulation between TG is EXPTIME-complete, and provide a logical characterization of this preorder in terms of a meaningful fragment of a new logic, TAMTL*. TAMTL* is an action-based timed extension of standard alternating-time temporal logic ATL*, which allows to quantify on strategies where the designated player is not responsible for blocking time. While for full TAMTL*, model-checking TG is undecidable, we show that for its fragment TAMTL, corresponding to the timed version of ATL, the problem is instead in EXPTIME.

1 Introduction

Refinement preorders constitute the standard mathematical approach to formalize the relation between abstract and concrete versions of the same system. Intuitively, an implementation I refines an abstraction A when each behavior of I is allowed by A. Refinement usually comes together with a logical setting to formally express the requirements preserved by the preorder. The goal is to ensure that the properties proved about the abstract description continue to hold in the refined version (i.e., the implementation). This scenario may arise either because the design is being carried out in an incremental fashion, or because the system is too complex and an abstraction needs to be used to verify its properties.

In the design and analysis of reactive and distributed component-based systems, refinement usually refers to a single component, whose behavior depends on assumptions on its environment (the other components). In this context, traditional refinement preorders, like simulation [15], are inappropriate because they do not distinguish between the behaviors of the component and those of its environment; so that, refinement also restrict the environment behaviors. Recently, [5, 10, 8] have addressed this problem and succeeded in an elegant solution for finite-state systems based on the game paradigm: the system is modeled by a multi-player finite-state concurrent game, where at each step, the next state is determined by considering the "intersection" between the choices (behavioral options) made simultaneously and independently by all the players (the components). Thus, one can keep all assumptions about a component separated from those of its environment. In this framework, simulation refinement becomes *alternating simulation* [5], a preorder which exploits the game setting and is defined according to a designated player (component):* an implementation I refines an abstraction A of the same component whenever any possible behavioral option of I is allowed by A, and controvariantly, any possible behavioral option of the environment of A is allowed by the environment of I. In this way, the refinement restricts the component behaviors without restricting the permissible environment behaviors.

While classical simulation preserves universal fragments of standard branching temporal logics designed for closed systems such as CTL* [11], alternating simulation for a given

*or, more in general, w.r.t. any subset of players (coalitions) player preserves expressive fragments of *alternating-time temporal logics* designed for open systems such as ATL* [5, 4]. The latter is a convenient formalism for component-based systems modeled by finite-state concurrent games, where properties need to be guaranteed by a player irrespective of the behavior of the other players.

Our contribution. We address the problem of refinement for real-time component-based systems, agreeing on the crucial role of timed information in practical applications, *e.g.* in embedded–system applications. We extend the notion of alternating simulation refinement for finite-state concurrent games to the setting of (perfect-information) timed concurrent games (TG) with the element of surprise introduced in [9]. In this setting, at each step, players choose simultaneously and independently moves consisting of delayed actions: the move with the smallest delay is carried out and determines the next state (if the smallest delay is proposed by several players, then the move of one of them is chosen nondeterministically). Moreover, we propose the new logic TAMTL* as a language for specifying properties of timed component-based systems modeled by TG. TAMTL* is a real-time action-based extension of ATL*, in which the temporal operators correspond to those of the timed linear-time temporal logic MTL [13]. Differently from the known real-time extension of ATL*, namely TATL* [12], which is based on a dense-time continuous semantic (the system is observed at any point in time), we adopt a dense-time pointwise semantics (the system is observed through events) [17]. Furthermore and more importantly, we generalize the class of atomic formulas of MTL by introducing the notion of (*timed*) *multi-action constraint*. Intuitively, such constraints express requirements on the "observable" part of single steps along TG runs, i.e., the delay-action chosen by each player and the player which is selected in the current step. In this way we can directly express important properties such as the existence of reasonable strategies, that are strategies where the designated player is not responsible for blocking time progress. In TATL*, this is not directly possible: to express the above requirement we have to artificially extend the infinite labeled transition system (LTS) of the given TG in order to obtain another LTS that cannot be associated to any TG specification. Our main results are the following:

1. We show that checking timed alternating simulation between TG for a given player is EXPTIME-complete. The upper bound is proved by a non-trivial generalization of the region-abstraction approach used for checking timed simulation/bisimulation [7, 18]. The matching lower bound is shown by an easy and linear-time reduction from the problem of checking timed simulation, which is known to be EXPTIME-hard [14].
2. We provide a logical characterization of timed alternating simulation for a given player σ in terms of a meaningful fragment, σ-TAMTL^*_P, of TAMTL*, where strategy quantifiers are parameterized by σ and negation applies only to multi-action constraints. We show that a TG $\mathcal{A}$ is timed σ-simulated by a TG $\mathcal{B}$ precisely when each σ-TAMTL^*_P formula that holds in $\mathcal{A}$ also holds in $\mathcal{B}$. To the best of our knowledge, this is the first paper that provides a full logical characterization for a timed refinement preorder.
3. While for unrestricted TAMTL*, model checking TG is undecidable (since TAMTL* subsumes MTL over infinite words [16]), we show that for its fragment TAMTL, where each temporal operator is immediately preceded by a strategy quantifier, the problem is instead in EXPTIME. To do so, for each player σ, we associate to the given TG a region-abstraction finite-state turn-based game G_σ, and recursively reduce the prob-

lem to solving the games G_σ w.r.t. regular objectives. Compared to the TATL model checking algorithm in [12], our approach is direct and provides more insight on TG.

To simplify the presentation, we restrict our attention to the two-player case, but all results easily extend to the multi-player setting, where players play in coalitions. Due to lack of space, some proofs are omitted and can be found in Appendix.

Related work. Refinement of real-time closed systems has been addressed in many papers (e.g. [3, 1, 18]), where systems are modeled by standard *timed automata* (TA) [3]. Timed language containment for TA is undecidable [3], while *timed simulation* [1, 18] between TA, which preserves the universal fragment of timed CTL (TCTL) [2], is EXPTIME-complete [18, 14]. For the open system setting, we are only aware of the recent work of Bulychev et al. [6], who propose timed simulation preorders for two-player timed games where partial observability is also taken into account. However, the games exploited there are asymmetric, which prevents a natural extension to the multi-player setting. Moreover, there are some significant restrictions on the model. For example, a player is enforced to play a discrete action if the invariant at the current location expires. Furthermore, their notion of preorder differs from ours in at least one crucial point: in their case, there is no interaction between the choices of opponent players in the underlying simulation game.

2 Preliminaries

2.1 Concurrent Timed Games

Let $\mathbb{R}_{\geq 0}$ be the set of non-negative reals and $\mathbb{Q}_{\geq 0}$ be the set of non-negative rational numbers. Fix a finite set of *clock* variables X. The set $C(X)$ of *clock constraints* (*over* X) is the set of boolean combinations of formulas of the form $x \sim c$, where $x \in X$, c is a natural number, and $\sim \in \{\leq, <\}$. A (*clock*) *valuation* (*over* X) is a function $v : X \to \mathbb{R}_{\geq 0}$ that maps every clock to a non-negative real number. Whether a valuation v *satisfies* a clock constraint $g \in C(X)$, denoted $v \models g$, is defined in a natural way. For $t \in \mathbb{R}_{\geq 0}$, the valuation $v + t$ is defined as $(v + t)(x) = v(x) + t$ for all $x \in X$. For $Y \subseteq X$, the valuation $v[Y := 0]$ is defined as $(v[Y := 0])(x) = 0$ if $x \in Y$ and $(v[Y := 0])(x) = v(x)$ otherwise.

DEFINITION 1.[3] *A* timed transition table (TT) *is a tuple* $\mathcal{T} = \langle Act, X, Q, \Delta, Inv\rangle$, *where* Act *is a finite set of actions,* Q *is a finite set of locations,* $\Delta \subseteq Q \times (Act \cup \{\bot\}) \times C(X) \times 2^X \times Q$ *is a finite* transition relation, *where* $\bot \notin Act$ *is the* null action, *and* $Inv : Q \to C(X)$ *maps each location to an invariant. We require that for each* $q \in Q$, *there is exactly one transition* $(q, \bot, g, Y, q')$ *from* q *associated with the null action; moreover,* $q' = q$, $g = true$, *and* $Y = \emptyset$.

A *state* of $\mathcal{T}$ is a pair (q, v) such that $q \in Q$, v is a valuation, and the invariant at location q is satisfied by v, i.e. $v \models Inv(q)$. The TT $\mathcal{T}$ induces an infinite-state labeled transition system (LTS) $[\![\mathcal{T}]\!] = \langle S, \to\rangle$ over the set of labels $\mathbb{R}_{\geq 0} \times (Act \cup \{\bot\}) \times \Delta$, where S is the set of $\mathcal{T}$-states, and the set of labeled edges $\to \subseteq S \times [\mathbb{R}_{\geq 0} \times (Act \cup \{\bot\}) \times \Delta] \times S$ is defined as: $(q, v) \xrightarrow{t,a,\delta} (q', v')$ *iff* $\delta \in \Delta$ is of the form $\delta = (q, a, g, Y, q')$ such that $v + t \models g \wedge Inv(q)$ and $v' = (v + t)[Y := 0]$. Note that if $(q, v) \xrightarrow{t,\bot,\delta} (q', v')$, then $q' = q$ and $v' = v + t$.

DEFINITION 2.[9] *A* (two-player concurrent) timed game (TG) *is a tuple* $\mathcal{A} = \langle \mathcal{T}, s^0, Act_0, Act_1\rangle$, *where* $\mathcal{T} = \langle Act, X, Q, \Delta, Inv\rangle$ *is a* TT, s^0 *is a designated initial state of* $\mathcal{T}$ *whose clock values are in* $\mathbb{Q}_{\geq 0}$, *and* $\{Act_0, Act_1\}$ *is a partition of* Act *with* $Act_0, Act_1 \neq \emptyset$.

A state of $\mathcal{A}$ is a state of $[\![\mathcal{T}]\!]$. For each $\sigma \in \{0,1\}$, let $Act_\sigma^\perp = Act_\sigma \cup \{\perp\}$. Intuitively, $Act_\sigma^\perp$ represents the set of actions for player σ. The set of moves $Mov_\mathcal{A}(\sigma)$ of player σ is given by $\mathbb{R}_{\geq 0} \times Act_\sigma^\perp \times \Delta$. For a state s, the set of *moves available to player σ in s*, written $Mov_\mathcal{A}(\sigma, s)$, is the set of moves $(t, a, \delta) \in Mov_\mathcal{A}(\sigma)$ such that $s \xrightarrow{t,a,\delta} s'$ for some state s', which is uniquely determined and is denoted by $Next_\mathcal{A}(s, \langle t, a, \delta \rangle)$. Observe that $Mov_\mathcal{A}(\sigma, s)$ is not empty since $(\perp, 0, (q, \perp, true, \emptyset, q)) \in Mov_\mathcal{A}(\sigma, s)$, where q is the location of s.

The timed game is intuitively played as follows. In each state s, each player σ chooses a move $(t, a, \delta) \in Mov_\mathcal{A}(\sigma, s)$ indicating that the player wants to play the transition δ associated with the action a after a delay of t time units. The null action $\perp$ signifies the player's intention to remain idle for the specified time delay. The move with the shorter proposed time delay determines the next state of the game; if both player propose the same delay, then one of the chosen moves occurs non-deterministically. An outcome of the game corresponds to an infinite path of $[\![\mathcal{T}]\!]$ augmented with additional information. Before formalizing these notions, we recall that in the standard definition of TG (see e.g. [9]) a move of a player just consists of a timed delay followed by an action. This because the underlying TT is assumed to be time-deterministic, i.e. for each $(t, a) \in \mathbb{R}_{\geq 0} \times Act$ and state s, there is at most one transition δ such that $s \xrightarrow{t,a,\delta} s'$. Here, we have removed this restriction. Thus, to uniquely determine the next state, a player has to specify also the transition to be taken.

For moves $(t_0, a_0, \delta_0) \in Mov_\mathcal{A}(0, s)$ and $(t_1, a_1, \delta_1) \in Mov_\mathcal{A}(1, s)$, the *joint destination move*, written $JDM(\langle t_0, a_0, \delta_0 \rangle, \langle t_1, a_1, \delta_1 \rangle)$, is $\{\langle t_0, a_0, \delta_0 \rangle, \langle t_1, a_1, \delta_1 \rangle\}$ if $t_0 = t_1$, and the singleton $\{\langle t_k, a_k, \delta_k \rangle\}$ for the unique $k \in \{0,1\}$ such that $t_k < t_{1-k}$ otherwise.

A run of $\mathcal{A}$ is a finite or infinite sequence $\pi = s_0, \langle m_1^0, m_1^1, \sigma_1 \rangle, s_1, \langle m_2^0, m_2^1, \sigma_2 \rangle, s_2, \ldots$ such that for any k, $s_k \in S$, $m_{k+1}^0 \in Mov_\mathcal{A}(0, s_k)$, $m_{k+1}^1 \in Mov_\mathcal{A}(1, s_k)$, $\sigma_{k+1} \in \{0,1\}$, $m_{k+1}^{\sigma_{k+1}} \in JDM(m_{k+1}^0, m_{k+1}^1)$, and $s_{k+1} = Next_\mathcal{A}(s_k, m_{k+1}^{\sigma_{k+1}})$. For each k, we denote by π^k the suffix-run of π starting from state s_k, and by $\pi[0,k]$ the prefix-run of π leading to state s_k. The *duration* $DUR(\pi)$ of π is the sum of timestamps of the selected moves $m_{k+1}^{\sigma_{k+1}}$ along π. An infinite run π is *divergent* if $DUR(\pi) = +\infty$. Let *FRuns* be the set of finite runs of $\mathcal{A}$. For $\pi \in FRuns$, we denote by $last(\pi)$ the last state of π. A *strategy* f_σ for player $\sigma \in \{0,1\}$ is a mapping $f_\sigma : FRuns \rightarrow Mov_\mathcal{A}(\sigma)$ assigning to each finite run π a move to be proposed by player σ at $last(\pi)$ such that $f_\sigma(\pi) \in Mov_\mathcal{A}(\sigma, last(\pi))$. For each state s, the set of *outcomes of strategy f_σ from s*, $Outcomes_\mathcal{A}(\sigma, s, f_\sigma)$, is the set of all *infinite* runs $s_0, \langle m_1^0, m_1^1, \sigma_1 \rangle, s_1, \langle m_2^0, m_2^1, \sigma_2 \rangle, s_2 \ldots$ such that $s_0 = s$, and for each $k \geq 0$, $f_\sigma(s_0, \langle m_1^0, m_1^1, \sigma_1 \rangle, s_1, \ldots s_k) = m_{k+1}^\sigma$. Let $\pi = s_0, \langle m_1^0, m_1^1, \sigma_1 \rangle, s_1, \langle m_2^0, m_2^1, \sigma_2 \rangle, s_2 \ldots$ with $m_k^j = (t_k^j, a_k^j, \delta_k^j)$ (for each $j = 0,1$ and $k \geq 1$). The *trace of* π, written $trace(\pi)$, is $\langle (t_1^0, a_1^0), (t_1^1, a_1^1), \sigma_1 \rangle, \langle (t_2^0, a_2^0), (t_2^1, a_2^1), \sigma_2 \rangle, \ldots$

We are also interested in strategies f_σ of player $\sigma \in \{0,1\}$ which are physically meaningful, i.e. such that player σ is not responsible for blocking time progress [9]. For each player $\sigma \in \{0,1\}$, let $Blameless_\sigma$ be the set of infinite runs $\pi = s_0, \langle m_1^0, m_1^1, \sigma_1 \rangle, s_1, \ldots$ such that player σ is responsible only for finitely many steps, i.e. such that there is $k \geq 1$ so that for all $j \geq k$, $\sigma_j = 1 - \sigma$. Note that $Blameless_\sigma$ does not distinguish between runs which have the same trace. A strategy f_σ for player σ is *reasonable* in a state s iff for all runs π in $Outcomes_\mathcal{A}(\sigma, s, f_\sigma)$, either π is divergent or $\pi \in Blameless_\sigma$. Note that our notion of reasonable strategy is slightly different from that given in [9]. However, it is easy to show that the two notions are equivalent (for details, see Appendix A).

2.2 The logic TAMTL*

In this subsection, we introduce a real-time action-based extension of the alternating-time temporal logic ATL* [4], called TAMTL*, based on a dense-time pointwise semantics.
Fix two nonempty and disjoint sets of actions Act_0 and Act_1. A (*timed*) *multi-action* over (Act_0, Act_1) is a triple $\theta = \langle (t_0, a_0), (t_1, a_1), \sigma \rangle$, where $(t_i, a_i) \in \mathbb{R}_{\geq 0} \times Act_i^{\perp}$ for $i = 0, 1$, $\sigma \in \{0, 1\}$, and $t_\sigma \leq t_0, t_1$. Note that the traces of runs in TG on (Act_0, Act_1) are sequence of multi-actions. A (*timed*) *multi-action constraint* χ is a triple $\chi = \langle (a'_0, \sim_0 c_0), (a'_1, \sim_1 c_1), \sigma' \rangle$, where $\sigma' \in \{0, 1\}$ and for $i = 0, 1$, $a'_i \in Act_i^{\perp}$, $\sim_i \in \{=, <, \leq, >, \geq\}$, and $c_i \in \mathbb{Q}_{\geq 0}$. The above multi-action θ *satisfies* χ, written $\theta \models \chi$, iff $\sigma = \sigma'$ and for $i = 0, 1$, $a_i = a'_i$ and $t_i \sim_i c_i$.
The sets of *state formulas* φ and *path formulas* ψ of TAMTL* over (Act_0, Act_1) are defined as:

$$\varphi := true \mid \neg\, \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle\langle \sigma \rangle\rangle \psi \mid \langle\langle \sigma \rangle\rangle_{re} \psi$$
$$\psi := \chi \mid \varphi \mid \neg\, \psi \mid \psi \wedge \psi \mid \psi \vee \psi \mid \psi\, \mathcal{U}_I\, \psi \mid \psi \widetilde{\mathcal{U}}_I \psi$$

where $\sigma \in \{0, 1\}$, $\langle\langle \sigma \rangle\rangle$ and $\langle\langle \sigma \rangle\rangle_{re}$ are *strategy quantifiers*, where, intuitively, $\langle\langle \sigma \rangle\rangle_{re}$ is restricted to σ-reasonable strategies, χ is a multi-action constraint, $\mathcal{U}_I$ is the *constrained strict until operator*, where I is an interval with bounds in $\mathbb{Q}_{\geq 0} \cup \{+\infty\}$, and $\widetilde{\mathcal{U}}_I$ is the dual of $\mathcal{U}_I$.
The set of state formulas φ forms the language TAMTL*. TAMTL* is interpreted over states of TG. Let $\mathcal{A}$ be a TG over (Act_0, Act_1), s be a state of $\mathcal{A}$, and π be an infinite run of $\mathcal{A}$. For a state formula φ and a path formula ψ, the satisfaction relations $(\mathcal{A}, s) \models \varphi$ and $(\mathcal{A}, \pi) \models \psi$, meaning that φ holds at state s and ψ holds along the run π in $\mathcal{A}$, respectively, are defined by induction as follows (we omit the rules for boolean connectives):

$(\mathcal{A}, s) \models \langle\langle \sigma \rangle\rangle \psi$	iff	there is a strategy f of player σ such that for all $\pi \in Outcomes_{\mathcal{A}}(\sigma, s, f)$, $(\mathcal{A}, \pi) \models \psi$
$(\mathcal{A}, s) \models \langle\langle \sigma \rangle\rangle_{re} \psi$	iff	there is a *reasonable* strategy f of player σ such that for all $\pi \in Outcomes_{\mathcal{A}}(\sigma, s, f)$, $(\mathcal{A}, \pi) \models \psi$ if π is divergent
$(\mathcal{A}, \pi) \models \chi$	iff	$trace(\pi) = \theta_0, \theta_1, \ldots$ and $\theta_0 \models \chi$
$(\mathcal{A}, \pi) \models \varphi$	iff	$\pi = s_0, \ldots$ and $(\mathcal{A}, s_0) \models \varphi$
$(\mathcal{A}, \pi) \models \psi_1 \mathcal{U}_I \psi_2$	iff	there is $i > 0$ such that $DUR(\pi[0, i]) \in I$, $(\mathcal{A}, \pi^i) \models \psi_2$, and $(\mathcal{A}, \pi^k) \models \psi_1$ for all $0 < k < i$
$(\mathcal{A}, \pi) \models \psi_1 \widetilde{\mathcal{U}}_I \psi_2$	iff	$(\mathcal{A}, \pi) \models \neg((\neg\psi_1) \mathcal{U}_I (\neg\psi_2))$

We write $\mathcal{A} \models \varphi$ to mean that $(\mathcal{A}, s^0) \models \varphi$ for the initial state s^0 of $\mathcal{A}$. We use some standard shortcuts: $\Diamond_I \psi := true\, \mathcal{U}_I\, \psi$ (*eventually*), $\Box_I \psi := \neg \Diamond_I \neg\psi$ (*always*), and $\bigcirc_I \psi := (\neg true)\, \mathcal{U}_I\, \psi$ (*next*). We omit the subscript I when $I = \mathbb{R}_{\geq 0}$. We denote by TAMTL the fragment of TAMTL* consisting of the formulas in which every temporal operator is immediately preceded by a strategy quantifier. Moreover, for $\sigma \in \{0, 1\}$, let σ-TAMTL$^*_P$ be the fragment of TAMTL* (not closed under negation) in which all strategy quantifiers are parameterized by σ, and negation is applied only to multi-action constraints. Intuitively, σ-TAMTL$^*_P$ formulas describe behaviors that player σ can enforce no matter what player $1 - \sigma$ does. Note that since TG are not determined [9], the dual σ-$\widetilde{\text{TAMTL}}^*_P$ of σ-TAMTL$^*_P$ does not correspond to $(1 - \sigma)$-TAMTL$^*_P$. By the equivalence below, it follows that in fact $\langle\langle \sigma \rangle\rangle_{re}$ is a *derivate operator* in TAMTL* and also in σ-TAMTL$^*_P$. However, this does not hold for the logic TAMTL.

$$\langle\langle \sigma \rangle\rangle_{re} \psi \equiv \langle\langle \sigma \rangle\rangle \big(((\Box \Diamond_{[1,\infty[}\, true) \rightarrow \psi) \wedge (\neg(\Box \Diamond_{[1,\infty[}\, true) \rightarrow \psi_{blameless_\sigma}) \big)$$
$$\psi_{blameless_\sigma} := \Diamond \Box \big(\bigvee_{a_0 \in Act_0^{\perp}} \bigvee_{a_1 \in Act_1^{\perp}} \langle (a_0, \geq 0), (a_1, \geq 0), 1 - \sigma \rangle \big)$$

Example 3. *Let $Act_0 = \{a\}$ and $Act_1 = \{b\}$. The 0-TAMTL$^*_P$ formula $\langle\langle 0\rangle\rangle_{re} \Box (\langle(a, \geq 0), (b, \geq 0), 0\rangle \rightarrow \Diamond_{[1,1]}\langle(a, \geq 0), (b, \geq 0), 1\rangle)$ requires that player 0 has a reasonable strategy ensuring that along every its divergent outcome, every a-event (i.e., the action a is selected in the current step) is followed one time unit later by a b-event.*

3 Timed Alternating Simulation

In this section, we introduce the notion of timed alternating simulation between TG which generalizes alternating simulation between finite-state concurrent games [5].

Fix two *comparable* TG $\mathcal{A} = \langle \mathcal{T}_\mathcal{A}, s_0^\mathcal{A}, Act_0^\mathcal{A}, Act_1^\mathcal{A}\rangle$ and $\mathcal{B} = \langle \mathcal{T}_\mathcal{B}, s_0^\mathcal{B}, Act_0^\mathcal{B}, Act_1^\mathcal{B}\rangle$, i.e. such that $Act_0^\mathcal{A} = Act_0^\mathcal{B}$ and $Act_1^\mathcal{A} = Act_1^\mathcal{B}$. Let $S_\mathcal{A}$ (resp., $S_\mathcal{B}$) be the set of states of $\mathcal{A}$ (resp., $\mathcal{B}$).

Definition 4. *For a player $\sigma \in \{0,1\}$, a relation $H \subseteq S_\mathcal{A} \times S_\mathcal{B}$ is a* timed alternating simulation for player σ from $\mathcal{A}$ to $\mathcal{B}$ *iff for all $(s_\mathcal{A}, s_\mathcal{B}) \in H$, the following holds:*
- *for every move $m_\sigma^\mathcal{A} = (t, a, \delta_\mathcal{A}) \in Mov_\mathcal{A}(\sigma, s_\mathcal{A})$, there is a matching move $m_\sigma^\mathcal{B} = (t, a, \delta_\mathcal{B}) \in Mov_\mathcal{B}(\sigma, s_\mathcal{B})$ such that for every move $m_{1-\sigma}^\mathcal{B} = (t', b, \delta'_\mathcal{B}) \in Mov_\mathcal{B}(1-\sigma, s_\mathcal{B})$, there is a matching move $m_{1-\sigma}^\mathcal{A} = (t', b, \delta'_\mathcal{A}) \in Mov_\mathcal{A}(1-\sigma, s_\mathcal{A})$ so that for all $i = 0, 1$,*

$$m_i^\mathcal{A} \in JDM(m_0^\mathcal{A}, m_1^\mathcal{A}) \ \ \textit{implies} \ \ (Next_\mathcal{A}(s_\mathcal{A}, m_i^\mathcal{A}), Next_\mathcal{B}(s_\mathcal{B}, m_i^\mathcal{B})) \in H$$

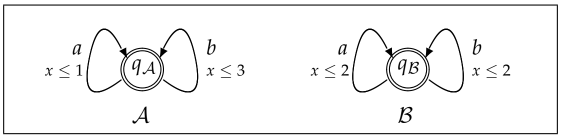
Note that $m_i^\mathcal{B} \in JDM(m_0^\mathcal{B}, m_1^\mathcal{B})$. If there is a timed alternating simulation H for player σ from $\mathcal{A}$ to $\mathcal{B}$ such that $(s_0^\mathcal{A}, s_0^\mathcal{B}) \in H$, we say that $\mathcal{B}$ *timed σ-simulates* $\mathcal{A}$, and we write $\mathcal{A} \preceq_\sigma \mathcal{B}$. Note that $\preceq_\sigma$ is a preorder on TG. We can give a game-theoretic interpretation of timed alternating simulation for a player $\sigma \in \{0,1\}$. Consider the following two-player *turn-based* game whose set of *main* positions is $S_\mathcal{A} \times S_\mathcal{B}$. The initial position is $(s_0^\mathcal{A}, s_0^\mathcal{B})$. Each round consists of five steps as follows. Assume that the current main position is $(s_\mathcal{A}, s_\mathcal{B})$. Then:
1. The antagonist chooses a move $m_\sigma^\mathcal{A} = (t, a, \delta_\mathcal{A}) \in Mov_\mathcal{A}(\sigma, s_\mathcal{A})$ of player σ in $\mathcal{A}$ available at state $s_\mathcal{A}$, and moves to position $p_1 = (s_\mathcal{A}, s_\mathcal{B}, m_\sigma^\mathcal{A})$.
2. The protagonist, from p_1, chooses a matching move $m_\sigma^\mathcal{B} = (t, a, \delta_\mathcal{B}) \in Mov_\mathcal{B}(\sigma, s_\mathcal{B})$ of player σ in $\mathcal{B}$ available at state $s_\mathcal{B}$, and moves to position $p_2 = (s_\mathcal{A}, s_\mathcal{B}, m_\sigma^\mathcal{A}, m_\sigma^\mathcal{B})$.
3. The antagonist, from p_2, chooses a move $m_{1-\sigma}^\mathcal{B} = (t', b, \delta'_\mathcal{B}) \in Mov_\mathcal{B}(1-\sigma, s_\mathcal{B})$ of player $1-\sigma$ in $\mathcal{B}$ available at state $s_\mathcal{B}$, and moves to position $p_3 = (s_\mathcal{A}, s_\mathcal{B}, m_\sigma^\mathcal{A}, m_\sigma^\mathcal{B}, m_{1-\sigma}^\mathcal{B})$.
4. The protagonist, from p_3, chooses a matching move $m_{1-\sigma}^\mathcal{A} = (t', b, \delta'_\mathcal{A}) \in Mov_\mathcal{A}(\sigma, s_\mathcal{A})$ of player $1-\sigma$ in $\mathcal{A}$ available at state $s_\mathcal{A}$, and moves to $p_4 = (s_\mathcal{A}, s_\mathcal{B}, m_\sigma^\mathcal{A}, m_\sigma^\mathcal{B}, m_{1-\sigma}^\mathcal{B}, m_{1-\sigma}^\mathcal{A})$.
5. The antagonist, from position p_4, chooses $i = 0, 1$ such that $m_i^\mathcal{A} \in JDM(m_0^\mathcal{A}, m_1^\mathcal{A})$, and moves to the main position $(Next_\mathcal{A}(s_\mathcal{A}, m_i^\mathcal{A}), Next_\mathcal{B}(s_\mathcal{B}, m_i^\mathcal{B}))$.

If the game proceeds ad infinitum, then the protagonist wins. Otherwise, the game reaches a position from which the protagonist cannot choose in steps 2 or 4 above a matching move, and the antagonist wins. It easily follows that $\mathcal{B}$ timed σ-simulates $\mathcal{A}$ iff the protagonist has a winning strategy. Note that for each $\sigma \in \{0,1\}$, we have a different turn-based game.

Intuitively, $\mathcal{B}$ timed σ-simulates $\mathcal{A}$ iff player σ is more powerful in game $\mathcal{B}$ than in game $\mathcal{A}$, i.e. each behavior that player σ can induce in $\mathcal{A}$, it can also induce in $\mathcal{B}$. The following lemma, whose proof is in Appendix B, formalizes this intuition. Let $H \subseteq S_\mathcal{A} \times S_\mathcal{B}$. For a run π of $\mathcal{A}$ and a run π' of $\mathcal{B}$ having the same length, we write $H(\pi, \pi')$ to mean that for each prefix-run $\pi[0,k]$ of π, $(last(\pi[0,k]), last(\pi'[0,k])) \in H$.

LEMMA 5. *Let H be a timed alternating simulation for player $\sigma \in \{0,1\}$ from $\mathcal{A}$ to $\mathcal{B}$. Then, for all $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in H$ and strategy $f_{\mathcal{A}}$ of player σ in $\mathcal{A}$, there exists a strategy $f_{\mathcal{B}}$ of player σ in $\mathcal{B}$ such that for every run $\pi_{\mathcal{B}} \in Outcomes_{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}})$, there exists a run $\pi_{\mathcal{A}} \in Outcomes_{\mathcal{A}}(\sigma, s_{\mathcal{A}}, f_{\mathcal{A}})$ so that $H(\pi_{\mathcal{A}}, \pi_{\mathcal{B}})$ and $trace(\pi_{\mathcal{A}}) = trace(\pi_{\mathcal{B}})$.*

EXAMPLE 6. *The figure depicts two simple TG $\mathcal{A}$ and $\mathcal{B}$ with $Act_0 = \{a\}$ and $Act_1 = \{b\}$. Let $s^0_{\mathcal{A}} = (q_{\mathcal{A}}, v)$ and $s^0_{\mathcal{B}} = (q_{\mathcal{B}}, v)$ be the initial states of $\mathcal{A}$ and B, where v is any valuation with $v(x) \leq 1$. It easily follows that $\mathcal{B}$ timed 0-simulates $\mathcal{A}$ and $\mathcal{A}$ timed 1-simulates B, but the vice versa of each of two conditions does not hold. Moreover, note that there exists no (standard) timed simulation from $\mathcal{A}$ to $\mathcal{B}$ and vice versa (w.r.t. the given initial states).*



3.1 Checking Timed Alternating Simulation

In this subsection, we show that for given comparable TG $\mathcal{A}$ and $\mathcal{B}$, and player $\sigma \in \{0,1\}$, checking whether $\mathcal{A} \preceq_\sigma \mathcal{B}$ is decidable via a suitable *region* abstraction, and the check can be done in exponential time. Our approach generalizes those proposed in [7, 18] for checking timed bisimulation and timed simulation. Fix two comparable TG $\mathcal{A} = \langle \mathcal{T}_{\mathcal{A}}, s^{\mathcal{A}}_0, Act_0, Act_1 \rangle$ and $\mathcal{B} = \langle \mathcal{T}_{\mathcal{B}}, s^{\mathcal{B}}_0, Act_0, Act_1 \rangle$. Let $S_{\mathcal{A}}$ (resp., $S_{\mathcal{B}}$) be the set of states of $\mathcal{A}$ (resp., $\mathcal{B}$), and let $X_{\mathcal{A}}$ (resp., $X_{\mathcal{B}}$) be the set of clocks of $\mathcal{A}$ (resp., $\mathcal{B}$). W.l.o.g. we can assume that $X_{\mathcal{A}} \cap X_{\mathcal{B}} = \emptyset$.

Region equivalence [3]: we denote by K_{max} the largest constant occurring in the clock constraints of $\mathcal{A}$ and $\mathcal{B}$. Given a clock valuation $v_{\mathcal{A}}$ over $X_{\mathcal{A}}$ and a clock valuation $v_{\mathcal{B}}$ over $X_{\mathcal{B}}$, the clock valuation $v_{\mathcal{A}} \| v_{\mathcal{B}}$ over $X_{\mathcal{A}} \cup X_{\mathcal{B}}$ is defined in the obvious way (recall that $X_{\mathcal{A}} \cap X_{\mathcal{B}} = \emptyset$). For $t \in \mathbb{R}_{\geq 0}$, $\lfloor t \rfloor$ denotes the integral part of t and $fract(t)$ denotes its fractional part. The *region equivalence relation over* $S_{\mathcal{A}} \times S_{\mathcal{B}}$, written $\approx_{\mathcal{A}\|\mathcal{B}}$, is defined as follows: $((q_{\mathcal{A}}, v_{\mathcal{A}}), (q_{\mathcal{B}}, v_{\mathcal{B}})) \approx_{\mathcal{A}\|\mathcal{B}} ((q'_{\mathcal{A}}, v'_{\mathcal{A}}), (q'_{\mathcal{B}}, v'_{\mathcal{B}}))$ iff $q_{\mathcal{A}} = q'_{\mathcal{A}}$, $q_{\mathcal{B}} = q'_{\mathcal{B}}$, and for each $x \in X_{\mathcal{A}} \cup X_{\mathcal{B}}$, either both $(v_{\mathcal{A}} \| v_{\mathcal{B}})(x), (v'_{\mathcal{A}} \| v'_{\mathcal{B}})(x) > K_{max}$, or the following holds:

- $\lfloor (v_{\mathcal{A}} \| v_{\mathcal{B}})(x) \rfloor = \lfloor (v'_{\mathcal{A}} \| v'_{\mathcal{B}})(x) \rfloor$ and $fract((v_{\mathcal{A}} \| v_{\mathcal{B}})(x)) = 0$ iff $fract((v'_{\mathcal{A}} \| v'_{\mathcal{B}})(x)) = 0$;
- for each $y \in X_{\mathcal{A}} \cup X_{\mathcal{B}}$ s.t. $(v_{\mathcal{A}} \| v_{\mathcal{B}})(y) \leq K_{max}$, $fract((v_{\mathcal{A}} \| v_{\mathcal{B}})(x)) \leq fract((v_{\mathcal{A}} \| v_{\mathcal{B}})(y))$ iff $fract((v'_{\mathcal{A}} \| v'_{\mathcal{B}})(x)) \leq fract((v'_{\mathcal{A}} \| v'_{\mathcal{B}})(y))$ (ordering of the fractional parts).

Let $Reg_{\mathcal{A}\|\mathcal{B}}$ be the set of equivalence classes of $\approx_{\mathcal{A}\|\mathcal{B}}$, called *regions*. By [3], $Reg_{\mathcal{A}\|\mathcal{B}}$ is finite and its size is singly exponential in the sizes of $\mathcal{A}$ and $\mathcal{B}$.

Finite Sampling of $\mathbb{R}_{\geq 0}$: let $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$ and $x_1, \ldots, x_n$ be the clocks in $X_{\mathcal{A}} \cup X_{\mathcal{B}}$ whose values $t_1, \ldots, t_n$ in $(s_{\mathcal{A}}, s_{\mathcal{B}})$ are not greater than K_{max}. Assume w.l.o.g. that $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_n$, where $\tau_i = fract(t_i)$ for $1 \leq i \leq n$. Let $\tau_0 = 0$, $\tau_{n+1} = 1$, and $min(s_{\mathcal{A}}, s_{\mathcal{B}}) = min\{\lfloor t_1 \rfloor, \ldots, \lfloor t_n \rfloor, K_{max}\}$. We consider the following *finite* set of real numbers:

$$Times(s_{\mathcal{A}}, s_{\mathcal{B}}) = \{h - \tfrac{1}{2}(\tau_i + \tau_{i+1}) \mid i = 0, \ldots, n \text{ and } h = 1, \ldots, K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})\} \cup$$
$$\{h - \tau_i \mid i = 1, \ldots, n \text{ and } h = 1, \ldots, K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})\} \cup \{0, \ldots, K_{max} + 1 - min(s_{\mathcal{A}}, s_{\mathcal{B}})\}\}$$

Thus, $Times(s_{\mathcal{A}}, s_{\mathcal{B}})$ consists of the integers in $\{0, \ldots, K_{max} + 1 - min(s_{\mathcal{A}}, s_{\mathcal{B}})\}$ plus the distances between the points p and the integers $1, \ldots, K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})$, where p is either a fractional part τ_j or the mid-point of some interval $[\tau_i, \tau_{i+1}]$ with $0 \leq i \leq n$. Intuitively, the *distance* d between a mid-point $\frac{1}{2}(\tau_i + \tau_{i+1})$ and an integer $h = 1, \ldots, K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})$

is used as a representative for all timestamps t such that $h - \tau_{i+1} < t < h - \tau_i$ (formally, $(v_\mathcal{A} \| v_\mathcal{B}) + t \approx_{\mathcal{A}\|\mathcal{B}} (v_\mathcal{A} \| v_\mathcal{B}) + d$, where $v_\mathcal{A}$ and $v_\mathcal{B}$ are the clock valuations of $s_\mathcal{A}$ and $s_\mathcal{B}$).

Checking if $\mathcal{A} \preceq_\sigma \mathcal{B}$ for $\sigma \in \{0,1\}$: let H_σ^{max} be the maximal timed alternating simulation for player σ from $\mathcal{A}$ to $\mathcal{B}$.[†] We show that H_σ^{max} is a union of regions and 'corresponds' to the greatest fixpoint of a computable monotone operator defined on the powerset of $Reg_{\mathcal{A}\|\mathcal{B}}$.

DEFINITION 7.*[Goodness] Let $\Gamma \subseteq Reg_{\mathcal{A}\|\mathcal{B}}$ be a set of regions and let $R \in \Gamma$. We say that R* is good in Γ w.r.t. player $\sigma \in \{0,1\}$ iff *there is $(s_\mathcal{A}, s_\mathcal{B}) \in R$ such that:*

1. *for every move $m_\sigma^\mathcal{A} = (t, a, \delta_\mathcal{A}) \in Mov_\mathcal{A}(\sigma, s_\mathcal{A})$ with $\underline{t \in Times(s_\mathcal{A}, s_\mathcal{B})}$, there is a matching move $m_\sigma^\mathcal{B} = (t, a, \delta_\mathcal{B}) \in Mov_\mathcal{B}(\sigma, s_\mathcal{B})$ such that for every move $m_{1-\sigma}^\mathcal{B} = (t', b, \delta'_\mathcal{B}) \in Mov_\mathcal{B}(1-\sigma, s_\mathcal{B})$ with $\underline{t' \in Times(s_\mathcal{A}, s_\mathcal{B})}$, there is a matching move $m_{1-\sigma}^\mathcal{A} = (t', b, \delta'_\mathcal{A}) \in Mov_\mathcal{A}(1-\sigma, s_\mathcal{A})$ so that for all $i = 0,1$ with $m_i^\mathcal{A} \in JDM(m_0^\mathcal{A}, m_1^\mathcal{A})$,*
$$(Next_\mathcal{A}(s_\mathcal{A}, m_i^\mathcal{A}), Next_\mathcal{B}(s_\mathcal{B}, m_i^\mathcal{B})) \in R_i \text{ for some } R_i \in \Gamma$$

For $\sigma \in \{0,1\}$, let $\Omega_\sigma : 2^{Reg_{\mathcal{A}\|\mathcal{B}}} \rightarrow 2^{Reg_{\mathcal{A}\|\mathcal{B}}}$ be the monotone operator defined as follows: $\Omega_\sigma(\Gamma) = \{R \in \Gamma \mid R \text{ is good in } \Gamma \text{ w.r.t. player } \sigma\}$. We show that Ω_σ is computable and $H_\sigma^{max} = \bigcup_{R \in \Gamma_{max}} R$, where Γ_{max} is the greatest fixpoint of Ω_σ. For this, we need the following crucial technical result whose proof is in Appendix C.

LEMMA 8. *Let $\Gamma \subseteq Reg_{\mathcal{A}\|\mathcal{B}}$ be a set of regions and $R \in \Gamma$ such that R is good in Γ w.r.t. player $\sigma \in \{0,1\}$. Then, Condition 1 in Definition 7 holds* for each $(s_\mathcal{A}, s_\mathcal{B}) \in R$, *and additionally the constraint that the timestamps have to be chosen in $Times(s_\mathcal{A}, s_\mathcal{B})$ can be relaxed.*

Let $H \subseteq S_\mathcal{A} \times S_\mathcal{B}$ be a timed alternating simulation for player $\sigma \in \{0,1\}$ from $\mathcal{A}$ to $\mathcal{B}$. We denote by $\Gamma_H \subseteq Reg_{\mathcal{A}\|\mathcal{B}}$ the set $\Gamma_H = \{R \in Reg_{\mathcal{A}\|\mathcal{B}} \mid R \cap H \neq \emptyset\}$. Evidently, Γ_H is a fixpoint of Ω_σ. Thus, by Lemma 8, we obtain the following results.

COROLLARY 9. *If $H \subseteq S_\mathcal{A} \times S_\mathcal{B}$ is a timed alternating simulation for player $\sigma \in \{0,1\}$ from $\mathcal{A}$ to $\mathcal{B}$, then $\bigcup_{R \in \Gamma_H} R$ is a timed alternating simulation for player $\sigma \in \{0,1\}$ from $\mathcal{A}$ to $\mathcal{B}$.*

COROLLARY 10. *Let $\Gamma \subseteq Reg_{\mathcal{A}\|\mathcal{B}}$ be a set of regions and $\sigma \in \{0,1\}$. Then, $\Omega_\sigma(\Gamma) = \Gamma$* iff *$\bigcup_{R \in \Gamma} R$ is a timed alternating simulation for player σ from $\mathcal{A}$ to $\mathcal{B}$.*

Fix $\sigma \in \{0,1\}$. By Corollary 9, H_σ^{max} is a union of regions in $Reg_{\mathcal{A}\|\mathcal{B}}$, and by Corollary 10, $H_\sigma^{max} = \bigcup_{R \in \Gamma_{max}} R$, where Γ_{max} is the greatest fixpoint of Ω_σ. Note that Γ_{max} can be obtained by iterative applications of Ω_σ starting with $\Gamma_0 = Reg_{\mathcal{A}\|\mathcal{B}}$. There can be at most $|Reg_{\mathcal{A}\|\mathcal{B}}|$ many iterations. Moreover, by Lemma 8, Condition 1 in Definition 7 is independent on what representative is chosen for the given class of equivalence. Since $|Times(s_\mathcal{A}, s_\mathcal{B})|$ for $(s_\mathcal{A}, s_\mathcal{B}) \in S_\mathcal{A} \times S_\mathcal{B}$ and $|Reg_{\mathcal{A}\|\mathcal{B}}|$ are singly exponential in the sizes of $\mathcal{A}$ and $\mathcal{B}$, if follows that $\Omega_\sigma(\Gamma)$ for given $\Gamma \subseteq Reg_{\mathcal{A}\|\mathcal{B}}$ can be computed in single exponential time in the sizes of $\mathcal{A}$ and $\mathcal{B}$. Since $\mathcal{A} \preceq_\sigma \mathcal{B}$ iff $(s_0^\mathcal{A}, s_0^\mathcal{B}) \in H_\sigma^{max}$, checking whether $\mathcal{A} \preceq_\sigma \mathcal{B}$ is in EXPTIME. We can show that the problem is also EXPTIME-hard by a straightforward and linear reduction (see Appendix D) from the problem of checking timed simulation between TT without invariants, which is known to be EXPTIME-hard [14]. Thus, we obtain the following result.

THEOREM 11. *Given two comparable TG $\mathcal{A}$ and $\mathcal{B}$ and player $\sigma \in \{0,1\}$, the problem of checking whether $\mathcal{A} \preceq_\sigma \mathcal{B}$ is* EXPTIME*-complete.*

[†] note that H_σ^{max} exists since the union of timed alternating simulations is still a timed alternating simulation

3.2 Logical characterization of timed alternating simulation

In this subsection, we give a logical characterization of timed alternating simulation for a given player $\sigma \in \{0,1\}$ in terms of the fragment σ-TAMTL^*_P of TAMTL^*.

THEOREM 12. *Let $\mathcal{A}$ and $\mathcal{B}$ be two TG over (Act_0, Act_1) and $\sigma \in \{0,1\}$. Then, $\mathcal{A} \preceq_\sigma \mathcal{B}$* if and only if *for every σ-TAMTL^*_P formula φ, $\mathcal{A} \models \varphi$ implies $\mathcal{B} \models \varphi$. Hence, $\mathcal{A} \preceq_\sigma \mathcal{B}$* if and only if *for every σ-$\widetilde{\mathsf{TAMTL}^*_P}$ formula $\widetilde{\varphi}$, $\mathcal{B} \models \widetilde{\varphi}$ implies $\mathcal{A} \models \widetilde{\varphi}$.*

Sketched proof. For the direct implication ($\Rightarrow$), it suffices to show that for all path formulas ψ and state formulas φ of σ-TAMTL^*_P, the following holds: if H is a timed alternating simulation for player σ from $\mathcal{A}$ to B, then

1. for all $(s_\mathcal{A}, s_\mathcal{B}) \in H$, $(\mathcal{A}, s_\mathcal{A}) \models \varphi$ implies $(\mathcal{B}, s_\mathcal{B}) \models \varphi$.
2. for all infinite runs $\pi_\mathcal{A}$ of $\mathcal{A}$ and $\pi_\mathcal{B}$ of $\mathcal{B}$ s.t. $H(\pi_\mathcal{A}, \pi_\mathcal{B})$ and $trace(\pi_\mathcal{A}) = trace(\pi_\mathcal{B})$, $(\mathcal{A}, \pi_\mathcal{A}) \models \psi$ implies $(\mathcal{B}, \pi_\mathcal{B}) \models \psi$.

The proof is by induction on the structure of formulas. The non-trivial case is that of state formulas of the form $\langle\langle\sigma\rangle\rangle\psi$ (recall that $\langle\langle\sigma\rangle\rangle_{re}$ is a derivate operator in σ-TAMTL^*_P). Assume that $(s_\mathcal{A}, s_\mathcal{B}) \in H$ and $(\mathcal{A}, s_\mathcal{A}) \models \langle\langle\sigma\rangle\rangle\psi$. Thus, there is a strategy $f_\mathcal{A}$ of player σ in $\mathcal{A}$ such that for each outcome $\pi_\mathcal{A}$ of $f_\mathcal{A}$ from $s_\mathcal{A}$, $(\mathcal{A}, \pi_\mathcal{A}) \models \psi$. Since $(s_\mathcal{A}, s_\mathcal{B}) \in H$, by Lemma 5, there is a strategy $f_\mathcal{B}$ of player σ in $\mathcal{B}$ such that for each outcome $\pi_\mathcal{B}$ of $f_\mathcal{B}$ from $s_\mathcal{B}$, there is an outcome $\pi_\mathcal{A}$ of $f_\mathcal{A}$ from $s_\mathcal{A}$ so that $H(\pi_\mathcal{A}, \pi_\mathcal{B})$ and $trace(\pi_\mathcal{A}) = trace(\pi_\mathcal{B})$. By ind. hyp., Property 2 holds for the path formula ψ. Hence, evidently, the result follows.

For the converse implication ($\Leftarrow$) of the theorem, assume that $\mathcal{A} \not\preceq_\sigma \mathcal{B}$. Assume that $\sigma = 0$ (the other case being symmetric). We need to prove that for some 0-TAMTL^*_P formula φ, $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$. Consider the turn-based 0-simulation game G_0 between the antagonist and the protagonist at the beginning of Section 3. By the results of Subsection 3.1 we can assume that the timestamps chosen by the antagonist are in the finite set $Times(s_\mathcal{A}, s_\mathcal{B})$, where $(s_\mathcal{A}, s_\mathcal{B})$ is the main current position of the game. It follows that G_0 is finitely-branching. Since $\mathcal{A} \not\preceq_0 \mathcal{B}$, the antagonist has a winning strategy f starting from $(s_0^\mathcal{A}, s_0^\mathcal{B})$, where $s_0^\mathcal{A}$ (resp., $s_0^\mathcal{B}$) is the initial state of $\mathcal{A}$ (resp., $\mathcal{B}$) whose clock-values are *rational*. Hence, the strategy-tree T_f of f from $(s_0^\mathcal{A}, s_0^\mathcal{B})$ is *finite*, and (by def. of *Times*) the timestamps of the moves along the edges of T_f are *rational*. We claim that for each node x_p of T_f labeled by a main position $p = (s_\mathcal{A}, s_\mathcal{B}) \in S_\mathcal{A} \times S_\mathcal{B}$, there is a 0-$\mathsf{TAMTL}^*_P$ formula φ_p (whose unique temporal modality is $\bigcirc$) such that $(\mathcal{A}, s_\mathcal{A}) \models \varphi_p$ and $(\mathcal{B}, s_\mathcal{B}) \not\models \varphi_p$. Hence, the result follows. The proof is by induction on the height of the subtree of T_f rooted at node x_p. By construction, x_p has exactly one child, say x'_p, and the edge from x_p to x'_p corresponds to a move $m^0_\mathcal{A} = (t, a, \delta_\mathcal{A})$ for player 0 in $Mov_\mathcal{A}(0, s_\mathcal{A})$ with $t \in Times(s_\mathcal{A}, s_\mathcal{B}) \subseteq \mathbb{Q}_{\geq 0}$ chosen by the antagonist in Step 1 on page 6. Moreover, the edges from x'_p to its children $y_1, \ldots, y_n$, if any, correspond to *all and only* the matching moves $m^0_\mathcal{B} = (t, a, \delta_\mathcal{B}) \in Mov_\mathcal{B}(0, s_\mathcal{B})$ of $m^0_\mathcal{A}$ for player 0 in $\mathcal{B}$ from $s_\mathcal{B}$. If $n = 0$ (base case), there is no such a matching move. In this case, the 0-TAMTL^*_P formula φ_p satisfying the claim is $\varphi_p = \langle\langle 0\rangle\rangle(\bigvee_{b \in Act_1^\perp} \bigvee_{\kappa \in \{0,1\}} \langle (a, = t), (b, \geq 0), \kappa\rangle)$.

Now, assume that $n \geq 1$. By construction, for each $1 \leq i \leq n$, y_i has a unique child y'_i and the edge from y_i to y'_i is associated with some move $(t', b, \delta'_\mathcal{B}) \in Mov_\mathcal{B}(1, s_\mathcal{B})$ (depending on i) with $t' \in Times(s_\mathcal{A}, s_\mathcal{B}) \subseteq \mathbb{Q}_{\geq 0}$ chosen by the antagonist in Step 3 on page 6. Moreover, the edges from y'_i to its children $z_{i,1}, \ldots, z_{i,m_i}$ represent *all and only* the possible matching moves $(t', b, \delta'_\mathcal{A}) \in Mov_\mathcal{A}(1, s_\mathcal{A})$ (for player 1 in $\mathcal{A}$ from $s_\mathcal{A}$) of the move

$(t', b, \delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1, s_{\mathcal{B}})$. Assume that for each $1 \leq i \leq n$, $m_i \geq 1$, i.e. y'_i is not a leaf (the other case being simpler). By construction, for each $1 \leq l \leq m_i$, $z_{i,l}$ has a unique child $z'_{i,l}$, which is labeled by a main position in $S_{\mathcal{A}} \times S_{\mathcal{B}}$, and the edge from $z_{i,l}$ to $z'_{i,l}$ corresponds to a choice $\kappa = 0, 1$ of the antagonist in Step 5 on page 6. We distinguish two cases:

- $\exists 1 \leq i \leq n.\ \forall 1 \leq l \leq m_i$: the edge from $z_{i,l}$ to $z'_{i,l}$ is associated with the choice $\kappa = 1$;
- $\forall 1 \leq i \leq n.\ \exists 1 \leq l \leq m_i$: the edge from $z_{i,l}$ to $z'_{i,l}$ is associated with the choice $\kappa = 0$.

Here, we focus on the first case. Let $m^1_{\mathcal{B}} = (t', b, \delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1, s_{\mathcal{B}})$ be the move associated with the edge from y_i to y'_i, where $t' \in \mathbb{Q}_{\geq 0}$, and $w_{\mathcal{B}} = Next_{\mathcal{B}}(s_{\mathcal{B}}, m^1_{\mathcal{B}})$. By construction, $t' \leq t$ and the nodes $z'_{i,1}, \ldots, z'_{i,m_i}$ are labeled by positions $(w^1_{\mathcal{A}}, w_{\mathcal{B}}), \ldots, (w^{m_i}_{\mathcal{A}}, w_{\mathcal{B}})$, respectively, where $w^1_{\mathcal{A}}, \ldots, w^{m_i}_{\mathcal{A}}$ are the states of $\mathcal{A}$ obtained from $s_{\mathcal{A}}$ applying all and only the matching moves $(t', b, \delta'_{\mathcal{A}}) \in Mov_{\mathcal{A}}(1, s_{\mathcal{A}})$ of $m^1_{\mathcal{B}}$. By ind. hyp. for each $1 \leq l \leq m_i$, there is a 0-TAMTL$^*_P$ formula ϕ_l s.t. $(\mathcal{A}, w^l_{\mathcal{A}}) \models \phi_l$ and $(\mathcal{B}, w_{\mathcal{B}}) \not\models \phi_l$. Let φ_p be the 0-TAMTL$^*_P$ formula given by

$$\langle\langle 0 \rangle\rangle \{ (\bigvee_{c \in Act^{\perp}_1} \bigvee_{\kappa \in \{0,1\}} \langle (a, = t), (c, \geq 0), \kappa \rangle) \wedge (\langle (a, = t), (b, = t'), 1 \rangle \rightarrow \bigcirc (\phi_1 \vee \ldots \vee \phi_{m_i})) \}$$

Evidently, $(\mathcal{A}, s_{\mathcal{A}}) \models \varphi_p$. Moreover, $(\mathcal{B}, s_{\mathcal{B}}) \not\models \varphi_p$, since for every strategy of player 0 in $\mathcal{B}$ which initially selects from $s_{\mathcal{B}}$ a move of the form (t, a, δ), there is an outcome from $s_{\mathcal{B}}$ of the form $\pi = s_{\mathcal{B}}, \langle (t, a, \delta), (t', b, \delta_{\mathcal{B}}), 1 \rangle, w_{\mathcal{B}}, \ldots$, where by hypothesis $w_{\mathcal{B}} \not\models (\phi_1 \vee \ldots \vee \phi_{m_i})$. A full proof of the converse implication of the theorem can be found in Appendix E. ∎

4 Model checking TG against TAMTL

Fix a TG $\mathcal{A}_{in}$ over (Act_0, Act_1) and a TAMTL formula φ. Let $X_{\mathcal{A}_{in}}$ be the set of clocks of $\mathcal{A}_{in}$. By [3], w.l.o.g. we can assume that the constants occurring in φ (i.e., the constants in the multi-action constraints of φ and the finite bounds in the intervals of the constrained temporal operators of φ) are natural numbers. Moreover, we can assume that $\mathcal{A}_{in}$ uses a clock x_{div}, which is reset whenever the constraint $x_{div} \geq 1$ holds. Let x_φ be a clock not in $X_{\mathcal{A}_{in}}$, and $\mathcal{A}$ be the TG obtained from $\mathcal{A}_{in}$ by simply adding the special clock x_φ (note that x_φ is never used by $\mathcal{A}$). Let K_{max} be the largest constant occurring in $\mathcal{A}$ and φ. We denote by $Reg_{\mathcal{A}_{in}}$ (resp., $Reg_{\mathcal{A}}$) the finite set of equivalences classes of the *region equivalence* on the set $S_{\mathcal{A}_{in}}$ (resp., $S_{\mathcal{A}}$) of states of $\mathcal{A}_{in}$ (resp., $\mathcal{A}$) w.r.t. the constant K_{max} [3], which is defined similarly to the set $Reg_{\mathcal{A} \| \mathcal{B}}$ in Subsection 3.1. We show that checking whether $\mathcal{A}_{in} \models \varphi$ (model-checking problem) can be reduced to solving finite-state games w.r.t. regular objectives. For this, we associate to $\mathcal{A}$ two finite-state games which abstract away from precise time information.

Let $R \in Reg_{\mathcal{A}}$. An abstract time-successor of R is a region $R' \in Reg_{\mathcal{A}}$ such that there is $(q, v) \in R$ so that $(q, v + t) \in R'$ for some $t \in \mathbb{R}_{\geq 0}$, and we write $R \leq R'$. By [3], the previous condition is independent on what representative is chosen in R. Moreover, if R' and R'' are two abstract time-successors of R, then either $R' \leq R''$ or $R'' \leq R'$. For a region R, the set of *abstract moves* available to player σ in R, written $Mov^{abs}_{\mathcal{A}}(\sigma, R)$, is the set of triples $(R', a, \delta) \in Reg_{\mathcal{A}} \times Act^{\perp}_{\sigma} \times \Delta$, such that $R \leq R'$, $\delta = (q, a, g, Y, q')$, q is the location associated with R', and g holds in R'. Given $m = (R', a, \delta) \in Mov^{abs}_{\mathcal{A}}(\sigma, R)$ with $\delta = (q, a, g, Y, q')$, we denote by $Next^{abs}_{\mathcal{A}}(R, m)$ the unique region R'' such that there is $(q, v) \in R'$ so that $(q, v[Y := 0]) \in R''$. By [3], the previous condition is independent on what representative is chosen in R'.

Let $\sigma \in \{0, 1\}$. The finite-state *turn-based* two-player game $\mathcal{A}^{abs}_{\sigma} = \langle P_\sigma = P^{\sigma}_{\sigma} \cup P^{1-\sigma}_{\sigma}, E_\sigma \rangle$ is defined as follows: $P^{\sigma}_{\sigma} = Reg_{\mathcal{A}} \times \{0, 1\}$ is the set of states (or positions) for player σ,

$P_\sigma^{1-\sigma} = \{\langle R, m, l\rangle \mid R \in Reg_{\mathcal{A}},\, m \in Mov^{abs}_{\mathcal{A}}(\sigma, R)$, and $l \in \{0,1\}\}$ is the set of positions for player $1-\sigma$, and $E_\sigma \subseteq (P_\sigma^\sigma \times P_\sigma^{1-\sigma}) \cup (P_\sigma^{1-\sigma} \times P_\sigma^\sigma)$ consists of following edges:

- $(R, l) \twoheadrightarrow (R, m, l)$ for all $R \in Reg_{\mathcal{A}}$, $m \in Mov^{abs}_{\mathcal{A}}(\sigma, R)$, and $l \in \{0,1\}$;
- $(R, \langle R_1, a, \delta_1\rangle, l) \twoheadrightarrow (R', l')$ iff $\exists\langle R_2, b, \delta_2\rangle \in Mov^{abs}_{\mathcal{A}}(1-\sigma, R)$ s.t. *either* $l' = \sigma$, $R_1 \le R_2$, and $R' = Next^{abs}_{\mathcal{A}}(R, \langle R_1, a, \delta_1\rangle)$, *or* $l' = 1-\sigma$, $R_2 \le R_1$, and $R' = Next^{abs}_{\mathcal{A}}(R, \langle R_2, b, \delta_2\rangle)$.

Note that E_σ is total in its first argument. A strategy for player σ in $\mathcal{A}^{abs}_\sigma$ is a function $f : P_\sigma^* \cdot P_\sigma^\sigma \to P_\sigma^{1-\sigma}$ such that for each $\pi = \pi', p \in P_\sigma^* \cdot P_\sigma^\sigma$, $p \twoheadrightarrow f(\pi)$ is an edge of $\mathcal{A}^{abs}_\sigma$. For each $p \in P_\sigma$, the set $Outcomes_{\mathcal{A}^{abs}_\sigma}(\sigma, p, f)$ of *infinite outcomes of f from p* is defined in the usual way. For a finite set of propositions $Prop$, a labeling function $L : P_\sigma \to 2^{Prop}$, a standard LTL formula ξ over $Prop$, and position $p \in P_\sigma$, we say that the strategy f is *winning* in p w.r.t. L and the objective ξ if for each outcome $p_0, p_1, \ldots \in Outcomes_{\mathcal{A}^{abs}_\sigma}(\sigma, p, f)$, $L(p_0), L(p_1), \ldots$ satisfies ξ. The following two lemmata (whose proofs are in Appendix) show the connection between the strategies of player σ in $\mathcal{A}$ and the strategies of player σ in $\mathcal{A}^{abs}_\sigma$.

LEMMA 13. *Let $\sigma \in \{0,1\}$, f be a strategy of player σ in $\mathcal{A}$, $R_0 \in Reg_{\mathcal{A}}$, and $s_0 \in R_0$. Then, there is a strategy f_{abs} of player σ in $\mathcal{A}^{abs}_\sigma$ such that for each path $\pi_{abs} = (R_0, 0), p_0, (R_1, \underline{\sigma_1}), p_1, (R_2, \underline{\sigma_2}) \ldots \in Outcomes_{\mathcal{A}^{abs}_\sigma}(\sigma, (R_0, 0), f_{abs})$, there exists a run $\pi \in Outcomes_{\mathcal{A}}(\sigma, s_0, f)$ of the form $\pi = s_0, \langle m^0_1, m^1_1, \underline{\sigma_1}\rangle, s_1, \langle m^0_2, m^1_2, \underline{\sigma_2}\rangle, \ldots$ so that for each $h \ge 1$, $s_h \in R_h$.*

LEMMA 14. *Let $\sigma \in \{0,1\}$ and f_{abs} be a strategy of player σ in $\mathcal{A}^{abs}_\sigma$, and $R_0 \in Reg_{\mathcal{A}}$. Then, there is a strategy f of player σ in $\mathcal{A}$ s.t. for each $\pi = s_0, \langle m^0_1, m^1_1, \underline{\sigma_1}\rangle, s_1, \langle m^0_2, m^1_2, \underline{\sigma_2}\rangle, \ldots \in Outcomes_{\mathcal{A}}(\sigma, s_0, f)$ with $s_0 \in R_0$, there is $\pi_{abs} \in Outcomes_{\mathcal{A}^{abs}_\sigma}(\sigma, (R_0, 0), f_{abs})$ of the form $\pi_{abs} = (R_0, 0), p_0, (R_1, \underline{\sigma_1}), p_1, (R_2, \underline{\sigma_2}) \ldots$ so that for each $h \ge 1$, $s_h \in R_h$.*

THEOREM 15. *The set of states s_{in} of $\mathcal{A}_{in}$ such that $(\mathcal{A}_{in}, s_{in}) \models \varphi$ is a union of regions in $Reg_{\mathcal{A}_{in}}$, and its (region) representation can be computed in exponential time. Hence, model checking TG against TAMTL is in* EXPTIME.

PROOF. We prove by induction on the structure of the formulas that the result holds for each state subformula ϕ of φ. Here, we illustrate the case in which $\phi = \langle\langle\sigma\rangle\rangle_{re}(\phi_1\, \mathcal{U}_I\, \phi_2)$ for some $\sigma \in \{0,1\}$ (the other cases are similar or simpler, and are detailed in Appendix G). For $s \in S_{\mathcal{A}}$, we denote by $Proj(s)$ the associated state in $S_{\mathcal{A}_{in}}$. Let $S_{\mathcal{A}}[x_\varphi := 0]$ be the set of states in $S_{\mathcal{A}}$ such that the value of clock x_φ is 0. Note that for each $s \in S_{\mathcal{A}}$, $(\mathcal{A}, s) \models \phi$ iff $(\mathcal{A}_{in}, Proj(s)) \models \phi$. By ind. hyp. it follows that for each $i = 1, 2$, the set of states $s \in S_{\mathcal{A}}$ such that $(\mathcal{A}, s) \models \phi_i$ is a union of regions in $Reg_{\mathcal{A}}$ whose representation can be computed in exponential time. Evidently, it suffices to show that the last condition continues to hold for the set of states s in $S_{\mathcal{A}}[x_\varphi := 0]$ such that $(\mathcal{A}, s) \models \langle\langle\sigma\rangle\rangle_{re}(\phi_1\, \mathcal{U}_I\, \phi_2)$. Note that by the previous observations, for each $s_0 \in S_{\mathcal{A}}[x_\varphi := 0]$, $(\mathcal{A}, s_0) \models \langle\langle\sigma\rangle\rangle_{re}(\phi_1\, \mathcal{U}_I\, \phi_2)$ iff there is a strategy f of player σ in $\mathcal{A}$ such that for each $\pi = s_0, \langle m^0_1, m^1_1, \sigma_1\rangle, s_1, \langle m^0_2, m^1_2, \sigma_2\rangle, \ldots \in Outcomes_{\mathcal{A}}(\sigma, s_0, f)$, the associate sequence $Reg(s_0), \sigma_1, Reg(s_1), \sigma_2, \ldots$, where $Reg(s_j)$ is the region of s_j, satisfies the following: either (1) for infinitely many $j \ge 0$, $Reg(s_j)$ satisfies the constraint $x_{div} \ge 1$, and there is $k > 0$ such that $Reg(s_k)$ satisfies ϕ_2 and the constraint $x_\varphi \in I$, and $Reg(s_h)$ satisfies ϕ_1 for each $0 < h < k$, or (2) there is $k \ge 0$ such that for each $j \ge k$, $\sigma_j \ne \sigma$ and $Reg(s_j)$ satisfies $x_{div} < 1$. Let $L : P_\sigma \to \{p_{\phi_2}, p_{\phi_1}, (x_{div} \ge 1), (x_\varphi \in I), 0, 1\}$ be the labeling of $\mathcal{A}^{abs}_\sigma$ defined in the obvious way. Then, by Lemmata 13 and 14, for all regions $R_0 \in Reg_{\mathcal{A}}$ satisfying $x_\varphi = 0$ and $s_0 \in R_0$, it holds that $(\mathcal{A}, s_0) \models \langle\langle\sigma\rangle\rangle_{re}(\phi_1\, \mathcal{U}_I\, \phi_2)$ iff there is a winning strategy f_{abs} of player σ in $\mathcal{A}_\sigma^{abs}$ in position $(R_0, 0)$ w.r.t. the labeling L and the LTL objective: $[\Box\Diamond(x_{div} \geq 1) \wedge (p_{\phi_1} \mathcal{U}(p_{\phi_2} \wedge (x_\varphi \in I)))] \vee [\Diamond\Box(\neg(x_{div} \geq 1) \wedge (1-\sigma))]$

Since LTL finite-state games for a fixed LTL formula can be solved in polynomial time [19] and since the size of $\mathcal{A}_\sigma^{abs}$ is exponential in the size of $\mathcal{A}_{in}$, the result follows. ∎

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Appendix

A Reasonable strategies

Fix a TG $\mathcal{A}$ over (Act_0, Act_1) and $\sigma \in \{0,1\}$. Recall that $Blameless_\sigma$ is the set of infinite runs $\pi = s_0, \langle m_1^0, m_1^1, \sigma_1 \rangle, s_1, \langle m_2^0, m_2^1, \sigma_2 \rangle, s_2, \ldots$ of $\mathcal{A}$ such that there is $k \geq 1$ so that for all $j \geq k$, $\sigma_j = 1 - \sigma$. Moreover, a strategy f_σ for player σ is reasonable in a state s iff for all runs $\pi \in Outcomes_{\mathcal{A}}(\sigma, s, f_\sigma)$, either π is divergent or $\pi \in Blameless_\sigma$. Let $\widetilde{Blameless}_\sigma$ be the set of infinite runs $\pi = s_0, \langle m_1^0, m_1^1, \sigma_1 \rangle, s_1, \langle m_2^0, m_2^1, \sigma_2 \rangle, s_2, \ldots$ with $m_j^l = (t_j^l, a_j^l, \delta_j^l)$ (for all $j \geq 1$ and $l \in \{0,1\}$) such that there is $k \geq 1$ so that for all $j \geq k$, either $t_j^\sigma > t_j^{1-\sigma}$ or $Next_{\mathcal{A}}(s_{j-1}, m_j^\sigma) \neq s_j$. Note that $\widetilde{Blameless}_\sigma \subseteq Blameless_\sigma$, but the vice versa in general does not hold. We say that a strategy f_σ for player σ is *strongly reasonable* in a state s iff for all runs $\pi \in Outcomes_{\mathcal{A}}(\sigma, s, f_\sigma)$, either π is divergent or $\pi \in \widetilde{Blameless}_\sigma$. The notion of reasonable strategy given in [9] corresponds to that of strongly reasonable strategy given here.

PROPOSITION 16. *Let f_σ be a strategy of player σ in $\mathcal{A}$ and s be a state of $\mathcal{A}$. Then, f_σ is reasonable in s iff f_σ is strongly reasonable in s.*

PROOF. Since $\widetilde{Blameless}_\sigma \subseteq Blameless_\sigma$, if f_σ is strongly reasonable in s, then f_σ is reasonable in s as well. Now, assume that f_σ is not strongly reasonable in s. We need to show that f_σ is not reasonable in s. By hypothesis, there must be a non-divergent run $\pi \in Outcomes_{\mathcal{A}}(\sigma, s, f_\sigma)$ of the form $\pi = s_0, \langle m_1^0, m_1^1, \sigma_1 \rangle, s_1, \langle m_2^0, m_2^1, \sigma_2 \rangle, s_2, \ldots$ with $m_j^l = (t_j^l, a_j^l, \delta_j^l)$ for all $j \geq 1$ and $l \in \{0,1\}$ s.t. the set $L = \{j \geq 1 \mid t_j^\sigma \leq t_j^{1-\sigma} \wedge Next_{\mathcal{A}}(s_{j-1}, m_j^\sigma) = s_j\}$ is infinite. Let π' obtained from π by replacing for each $j \in L$, σ_j with σ. Evidently, π' is still a non-divergent run in $Outcomes_{\mathcal{A}}(\sigma, s, f_\sigma)$. Moreover, $\pi' \notin Blameless_\sigma$. Hence, f_σ cannot be reasonable in s. ∎

B Proof of Lemma 5

Let $\mathcal{C}$ be a TG. For a finite run $\pi = s_0, \langle m_1^0, m_1^1, \sigma_1 \rangle, s_1, \langle m_2^0, m_2^1, \sigma_2 \rangle, s_2, \ldots, s_k$ of $\mathcal{C}$, the *length* of π is k. For a strategy f_σ of player $\sigma \in \{0,1\}$, state s in $\mathcal{C}$, and $k \geq 0$, $Outcomes_k^{\mathcal{C}}(\sigma, s, f_\sigma)$ denotes the set of finite runs π of $\mathcal{C}$ starting from s of length k such that π is the prefix of some run in $Outcomes_{\mathcal{C}}(\sigma, s, f_\sigma)$. Now, we prove Lemma 5.

Lemma 5. *Let H be a timed alternating simulation for player $\sigma \in \{0,1\}$ from $\mathcal{A}$ to $\mathcal{B}$. Then, for all $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in H$ and strategy $f_{\mathcal{A}}$ of player σ in $\mathcal{A}$, there exists a strategy $f_{\mathcal{B}}$ of player σ in $\mathcal{B}$ such that for every run $\pi_{\mathcal{B}} \in Outcomes_{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}})$, there exists a run $\pi_{\mathcal{A}} \in Outcomes_{\mathcal{A}}(\sigma, s_{\mathcal{A}}, f_{\mathcal{A}})$ so that $H(\pi_{\mathcal{A}}, \pi_{\mathcal{B}})$ and $trace(\pi_{\mathcal{A}}) = trace(\pi_{\mathcal{B}})$.*

PROOF. Fix $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in H$ and a strategy $f_{\mathcal{A}}$ of player σ in $\mathcal{A}$. We claim the following.

Claim: There exists a strategy $f_{\mathcal{B}}$ of player σ in $\mathcal{B}$ such that there is a sequence $(F_k)_{k \in \mathbb{N}}$ of functions $F_k : Outcomes_k^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}}) \to Outcomes_k^{\mathcal{A}}(\sigma, s_{\mathcal{A}}, f_{\mathcal{A}})$ so that for each $k \geq 0$, the following holds:

1. for each $\pi \in Outcomes_k^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}})$, $trace(F_k(\pi)) = trace(\pi)$ and $H(F_k(\pi), \pi)$. Furthermore, if $k > 0$ and $\pi = \pi', \langle m_0, m_1, l \rangle, s$, then $F_k(\pi) = F_{k-1}(\pi'), \langle m_0', m_1', l \rangle, s'$ for some state s' and moves m_0' and m_1' of $\mathcal{A}$.

First, we show that the lemma follows from the claim, and then we prove the claim. Let $f_{\mathcal{B}}$ be a strategy of player σ in $\mathcal{B}$ satisfying the claim, and $\pi_{\mathcal{B}} \in Outcomes_{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}})$. Moreover, let $(F_k)_{k\in\mathbb{N}}$ be as in the statement of the claim, and for each $k \geq 0$, $\pi_{\mathcal{B},k}$ be the prefix of $\pi_{\mathcal{B}}$ of length k, and $\pi_{\mathcal{A},k} = F_k(\pi_{\mathcal{B},k})$. By the claim above, $\pi_{\mathcal{A},k} \in Outcomes_k^{\mathcal{A}}(\sigma, s_{\mathcal{A}}, f_{\mathcal{A}})$, $trace(\pi_{\mathcal{A},k}) = trace(\pi_{\mathcal{B},k})$, $H(\pi_{\mathcal{A},k}, \pi_{\mathcal{B},k})$, and $\pi_{\mathcal{A},k+1} = \pi_{\mathcal{A},k}, \langle m_0, m_1, l\rangle, s$ for some state s, $l \in \{0,1\}$, and moves m_0, m_1 of $\mathcal{A}$. Hence, evidently, $(\pi_{\mathcal{A},k})_{k\in\mathbb{N}}$ represents an (infinite) run in $Outcomes_{\mathcal{A}}(\sigma, s_{\mathcal{A}}, f_{\mathcal{A}})$ satisfying the lemma. Now, we prove the claim above.

Proof of the claim: the strategy $f_{\mathcal{B}}$ is defined by induction on the length n of the finite runs of $\mathcal{B}$ staring from $s_{\mathcal{B}}$. Let $n \geq 0$. Since $Outcomes_n^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f)$ for any strategy f is independent on the values assumed by f over the finite runs of length equal or greater then n, we can assume that the set $Outcomes_n^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}})$ is already given (note that $Outcomes_0^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}}) = \{s_{\mathcal{B}}\}$) and there is a function $F_n : Outcomes_n^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}}) \rightarrow Outcomes_n^{\mathcal{A}}(\sigma, s_{\mathcal{A}}, f_{\mathcal{A}})$ satisfying Condition 1 in the claim for $k = n$ (note that since the function F_0 is independent on the specific strategy, and since $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in H$, for $k = 0$, Condition 1 in the claim trivially holds). Let $\pi \in Outcomes_n^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}})$. Then, $f_{\mathcal{B}}(\pi)$ is defined as follows. Let us consider the finite run $F_n(\pi) \in Outcomes_n^{\mathcal{A}}(\sigma, s_{\mathcal{A}}, f_{\mathcal{A}})$. We have that $trace(F_n(\pi)) = trace(\pi)$ and $H(F_n(\pi), \pi)$. Let $m_\sigma^{\mathcal{A}} = (t, a, \delta_{\mathcal{A}}) = f_{\mathcal{A}}(F_n(\pi))$. Since $(last(F_n(\pi)), last(\pi)) \in H$ and H is an alternating timed simulation for player σ from $\mathcal{A}$ to $\mathcal{B}$, there must exist a matching move $m_\sigma^{\mathcal{B}} = (t, a, \delta_{\mathcal{B}}) \in Mov_{\mathcal{B}}(\sigma, last(\pi))$ such that for every move $m_{1-\sigma}^{\mathcal{B}} = (t', b, \delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1-\sigma, last(\pi))$, there is a matching move $m_{1-\sigma}^{\mathcal{A}} = (t', b, \delta'_{\mathcal{A}}) \in Mov_{\mathcal{A}}(1-\sigma, last(F_n(\pi)))$ so that for all $i = 0, 1$ with $m_i^{\mathcal{A}} \in JDM(m_0^{\mathcal{A}}, m_1^{\mathcal{A}})$,

$$(Next_{\mathcal{A}}(last(F_n(\pi)), m_i^{\mathcal{A}}), Next_{\mathcal{B}}(last(\pi), m_i^{\mathcal{B}})) \in H$$

We set $f_{\mathcal{B}}(\pi) = m_\sigma^{\mathcal{B}}$. At this point, we can assume that also $Outcomes_{n+1}^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}})$ is already given. It remains to show that there is a function F_{n+1} satisfying Condition 1 in the Claim (for $k = n+1$). The function F_{n+1} is defined as follows. Let $\pi_{n+1} \in Outcomes_{n+1}^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}})$. Hence, $\pi_{n+1} = \pi, \langle m_0, m_1, l\rangle, s$, where $\pi \in Outcomes_n^{\mathcal{B}}(\sigma, s_{\mathcal{B}}, f_{\mathcal{B}})$, $m_0 = (t_0, a_0, \delta_0)$, $m_1 = (t_1, a_1, \delta_1)$, $l \in \{0,1\}$, and $m_l \in JDM(m_0, m_1)$. By construction $f_A(F_n(\pi)) = m'_\sigma$ with $m'_\sigma = (t_\sigma, a_\sigma, \delta'_\sigma)$ for some transition δ'_σ of $\mathcal{A}$, and there are two cases:

- $t_{1-\sigma} \geq t_\sigma$ and $l = \sigma$. By construction $(Next_{\mathcal{A}}(last(F_n(\pi)), m'_\sigma), Next_{\mathcal{B}}(last(\pi), m_\sigma)) \in H$ and there is a move $m'_{1-\sigma} = (t_{1-\sigma}, a_{1-\sigma}, \delta'_{1-\sigma}) \in Mov_{\mathcal{A}}(1-\sigma, last(F_n(\pi)))$ matching the move $m_{1-\sigma}$ in $\mathcal{B}$. Since $Next_{\mathcal{B}}(last(\pi), m_\sigma) = last(\pi_{n+1})$, we obtain that $\pi'_{n+1} = F_n(\pi), \langle m'_0, m'_1, \sigma\rangle, Next_{\mathcal{A}}(last(F_n(\pi)), m'_\sigma)$ is a finite run in $Outcomes_{n+1}^{\mathcal{A}}(\sigma, s_{\mathcal{A}}, f_{\mathcal{A}})$ with $trace(\pi'_{n+1}) = trace(\pi_{n+1})$ and $H(\pi'_{n+1}, \pi_{n+1})$. In this case, we set $F_{n+1}(\pi_{n+1}) = \pi'_{n+1}$.
- $t_{1-\sigma} \leq t_\sigma$ and $l = 1-\sigma$. By construction there is a move $m'_{1-\sigma} = (a_{1-\sigma}, t_{1-\sigma}, \delta'_{1-\sigma}) \in Mov_{\mathcal{A}}(1-\sigma, last(F_n(\pi)))$ matching the move $m_{1-\sigma}$ of $\mathcal{B}$ s.t. $(Next_{\mathcal{A}}(last(F_n(\pi)), m'_{1-\sigma})$, $Next_{\mathcal{B}}(last(\pi), m_{1-\sigma})) \in H$. Since $Next_{\mathcal{B}}(last(\pi), m_{1-\sigma}) = last(\pi_{n+1})$, it follows that $\pi'_{n+1} = F_n(\pi), \langle m'_0, m'_1, 1-\sigma\rangle, Next_{\mathcal{A}}(last(F_n(\pi)), m'_{1-\sigma})$ is in $Outcomes_{n+1}^{\mathcal{A}}(\sigma, s_{\mathcal{A}}, f_{\mathcal{A}})$ with $trace(\pi'_{n+1}) = trace(\pi_{n+1})$ and $H(\pi'_{n+1}, \pi_{n+1})$. We set $F_{n+1}(\pi_{n+1}) = \pi'_{n+1}$.

Evidently, F_{n+1} satisfies Condition 1 in the claim. This concludes the proof of the claim. ∎

C Proof of Lemma 8

Recall that for the two fixed comparable TG $\mathcal{A}$ and $\mathcal{B}$ over (Act_0, Act_1), K_{max} denotes the largest constant occurring in the clock constraints of $\mathcal{A}$ and $\mathcal{B}$, and for $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$,

$$Times(s_{\mathcal{A}}, s_{\mathcal{B}}) = \{0, \ldots, K_{max} + 1 - min(s_{\mathcal{A}}, s_{\mathcal{B}})\} \cup$$
$$\{h - \tau_i \mid i = 1, \ldots, n \text{ and } h = 1, \ldots, K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})\} \cup$$
$$\{h - \tfrac{1}{2}(\tau_i + \tau_{i+1}) \mid i = 0, \ldots, n \text{ and } h = 1, \ldots, K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})\}$$

where $\tau_0 = 0$, $\tau_{n+1} = 1$, $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_n$ denote the fractional parts of the values $t_1, \ldots, t_n$ of the clocks in $(s_{\mathcal{A}}, s_{\mathcal{B}})$ that are not greater than K_{max}, and $min(s_{\mathcal{A}}, s_{\mathcal{B}}) = min\{\lfloor t_1 \rfloor, \ldots, \lfloor t_n \rfloor, K_{max}\}$. In the following, for edge of $\mathcal{A}$ (resp., $\mathcal{B}$), we mean an edge of the associated LTS. Moreover, for distinguishing between edges of $\mathcal{A}$ and $\mathcal{B}$, we use the subscript $\mathcal{A}$ (resp., $\mathcal{B}$) on the source and target states of edges of $\mathcal{A}$ (resp., $\mathcal{B}$). Thus, for example, $s_{\mathcal{A}} \xrightarrow{t,a,\delta} u_{\mathcal{A}}$ denotes an edge of $\mathcal{A}$ and $s_{\mathcal{B}} \xrightarrow{t,a,\delta} u_{\mathcal{B}}$ denotes an edge of $\mathcal{B}$.

Recall that the equivalence relation $\approx_{\mathcal{A}\|\mathcal{B}}$ is defined over $S_{\mathcal{A}} \times S_{\mathcal{B}}$, and $Reg_{\mathcal{A}\|\mathcal{B}}$ denotes the set of its equivalence classes. However, $\approx_{\mathcal{A}\|\mathcal{B}}$ can be obviously extended to all the pairs $((q_{\mathcal{A}}, v_{\mathcal{A}}), (q_{\mathcal{B}}, v_{\mathcal{B}}))$ not in $S_{\mathcal{A}} \times S_{\mathcal{B}}$ such that $q_{\mathcal{A}}$ (resp., $q_{\mathcal{B}}$) is a location of $\mathcal{A}$ (resp., $\mathcal{B}$) and $v_{\mathcal{A}}$ (resp., $v_{\mathcal{B}}$) is a valuation of $\mathcal{A}$ (resp., $\mathcal{B}$), i.e. pairs in which the valuations may not satisfy the invariants of the associated locations. Note that the regions in $Reg_{\mathcal{A}\|\mathcal{B}}$ continue to be equivalence classes of this extension. In order to prove Lemma 8, we need two preliminary results. The first one is given by the following proposition, which corresponds to classical results on timed automata [3], where for $t \geq 0$ and $((q_{\mathcal{A}}, v_{\mathcal{A}}), (q_{\mathcal{B}}, v_{\mathcal{B}})) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$, $((q_{\mathcal{A}}, v_{\mathcal{A}}), (q_{\mathcal{B}}, v_{\mathcal{B}})) + t$ denotes the pair $((q_{\mathcal{A}}, v_{\mathcal{A}} + t), (q_{\mathcal{B}}, v_{\mathcal{B}} + t))$ (which may not be in $S_{\mathcal{A}} \times S_{\mathcal{B}}$).

PROPOSITION 17. *Let* $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t'$. *Then,*

- *if* $s_{\mathcal{A}} \xrightarrow{t,a,\delta} u_{\mathcal{A}}$, *then* $s'_{\mathcal{A}} \xrightarrow{t',a,\delta} u'_{\mathcal{A}}$ *for some* $u'_{\mathcal{A}} \in S_{\mathcal{A}}$;
- *if* $s_{\mathcal{B}} \xrightarrow{t,a,\delta} u_{\mathcal{B}}$, *then* $s'_{\mathcal{B}} \xrightarrow{t',a,\delta} u'_{\mathcal{B}}$ *for some* $u'_{\mathcal{B}} \in S_{\mathcal{B}}$;
- *if* $s_{\mathcal{A}} \xrightarrow{t,a,\delta} u_{\mathcal{A}}$, $s'_{\mathcal{A}} \xrightarrow{t',a,\delta} u'_{\mathcal{A}}$, $s_{\mathcal{B}} \xrightarrow{t,b,\delta'} u_{\mathcal{B}}$, *and* $s_{\mathcal{B}} \xrightarrow{t',b,\delta'} u'_{\mathcal{B}}$, *then* $(u_{\mathcal{A}}, u_{\mathcal{B}}) \approx_{\mathcal{A}\|\mathcal{B}} (u'_{\mathcal{A}}, u'_{\mathcal{B}})$.

The second preliminary result is represented by the following lemma.

LEMMA 18. *Let* $(s_{\mathcal{A}}, s_{\mathcal{B}}) \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}})$. *Then, for each* $t_1 \geq 0$, *there is* $t_2 \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ *such that* $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t_1 \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t_2$ *and: for each* $t \leq t_1$ (*resp.,* $t \geq t_1$), *there is* $t' \leq t_2$ (*resp.,* $t' \geq t_2$) *such that* $t' \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ *and* $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t'$.

PROOF. Let $x_1, \ldots, x_n$ be the clocks in $X_{\mathcal{A}} \cup X_{\mathcal{B}}$ whose values $r_1, \ldots, r_n$ in $(s_{\mathcal{A}}, s_{\mathcal{B}})$ are not greater than K_{max}. Assume w.l.o.g. that the fractional parts $\tau_1, \ldots, \tau_n$ of $r_1, \ldots, r_n$ are in ascending order, i.e. $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_n$. Since $(s_{\mathcal{A}}, s_{\mathcal{B}}) \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}})$, it follows that $x_1, \ldots, x_n$ are all and only the clocks in $X_{\mathcal{A}} \cup X_{\mathcal{B}}$ whose values $r'_1, \ldots, r'_n$ in $(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ are not greater than K_{max}. Moreover, $\lfloor r'_i \rfloor = \lfloor r_i \rfloor$ for each $1 \leq i \leq n$, and the fractional parts $\tau'_1, \ldots, \tau'_n$ of $r'_1, \ldots, r'_n$ satisfy: $\tau'_1 \leq \tau'_2 \leq \ldots \leq \tau'_n$, ($\tau'_i = \tau'_{i+1}$ iff $\tau_i = \tau_{i+1}$) and ($\tau'_i = 0$ iff $\tau_i = 0$) for any i. Moreover, $min(s_{\mathcal{A}}, s_{\mathcal{B}}) = min(s'_{\mathcal{A}}, s'_{\mathcal{B}})$.

Let $Times(s_{\mathcal{A}}, s_{\mathcal{B}}) = \{d_1, \ldots, d_p, d_{p+1}\}$ with $d_1 < d_2 < \ldots < d_{p+1}$. Note that $d_1 = 0$, $d_p = K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})$, and $d_{p+1} = K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}}) + 1$. First, we show the following.

Claim 1: $Times(s'_{\mathcal{A}}, s'_{\mathcal{B}}) = \{d'_1, \ldots, d'_p, d'_{p+1}\}$ with $d'_1 < d'_2 < \ldots < d'_{p+1}$. Moreover, for each $1 \leq i \leq p+1$, $(s_{\mathcal{A}}, s_{\mathcal{B}}) + d_i \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + d'_i$.

Claim 2: for each $t > 0$ such that $d_{i-1} < t < d_i$ with $i \leq p$, there is $d \in \{d_{i-1}, d_i\}$ such that $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s_{\mathcal{A}}, s_{\mathcal{B}}) + d$. Moreover, for each t' such that $d_{i-1} < t' < d_i$, $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s_{\mathcal{A}}, s_{\mathcal{B}}) + t'$.

Claim 1 easily follows from def. of *Times*. Now, we prove Claim 2, Let $t > 0$ such that $d_{i-1} < t < d_i$ with $i \leq p$ (hence, $d_i \leq K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})$). Let $h = \lfloor t \rfloor$. By def. of $Times(s_{\mathcal{A}}, s_{\mathcal{B}})$, one of the following two cases occurs, where $\tau_0 = 0$ and $\tau_{n+1} = 1$:

- there is $0 \leq l < n+1$ such that $d_i = (1 - \tau_l) + h$ and $d_{i-1} = (1 - \frac{1}{2}(\tau_l + \tau_{l+1})) + h$. Note that $\tau_l < \tau_{l+1}$ (otherwise, $d_i = d_{i-1}$).
- there is $0 < l \leq n+1$ such that $d_{i-1} = (1 - \tau_l) + h$ and $d_i = (1 - \frac{1}{2}(\tau_l + \tau_{l-1})) + h$. Note that $\tau_{l-1} < \tau_l$ (otherwise, $d_i = d_{i-1}$).

In the first case, we set $d = d_{i-1}$, and in the second case we set $d = d_i$. It easily follows that for each t' such that $d_{i-1} < t' < d_i$, $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t' \approx_{\mathcal{A}\|\mathcal{B}} (s_{\mathcal{A}}, s_{\mathcal{B}}) + d$. Hence, Claim 2 follows. Now, by using Claims 1 and 2, we prove the lemma. Fix $t_1 \geq 0$. We distinguish two cases:

- $t_1 \leq K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}}) = d_p$. Assume that $t_1 \notin Times(s_{\mathcal{A}}, s_{\mathcal{B}})$ (the other case being simpler). Then, there is $0 < i \leq p$ such that $d_{i-1} < t_1 < d_i$. By Claim 2, for some $d \in \{d_{i-1}, d_i\}$, $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t_1 \approx_{\mathcal{A}\|\mathcal{B}} (s_{\mathcal{A}}, s_{\mathcal{B}}) + d$. If $d = d_{i-1}$, we set $t_2 = d'_{i-1}$: otherwise, we set $t_2 = d'_i$. By Claim 1, we obtain $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t_1 \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t_2$. Now, let $t \leq t_1$ (resp., $t \geq t_1$). We need to show that there is $t' \leq t_2$ (resp., $t' \geq t_2$) with $t' \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ such that $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t'$. We focus on the case $t \leq t_1$ (the other case $t \geq t_1$ being similar). Let $t \leq t_1$. First, assume that $d_{i-1} < t < d_i$. Then, by Claim 2, $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s_{\mathcal{A}}, s_{\mathcal{B}}) + t_1$. Thus, in this case we set $t' = t_2$. Now, assume that $t \leq d_{i-1}$. If $t = d_j$ for some $j \leq i-1$, then we set $t' = d'_j$ (note that $t' \leq t_2$), and the result follows from Claim 1. Otherwise, there is $j \leq i-1$ such that $d_{j-1} < t < d_j$. By Claim 2, there is $r \in \{d_{j-1}, d_j\}$ such that $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s_{\mathcal{A}}, s_{\mathcal{B}}) + r$. Then, we set $t' = d'_{j-1}$ if $r = d_{j-1}$, and $r = d'_j$ otherwise. Note that $t' \leq t_2$. Thus, by Claim 1, the result follows.
- $t_1 > K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})$. We set $t_2 = d'_{p+1} = K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}}) + 1$. Evidently, $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t_1 \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t_2$. Now, let $t \leq t_1$ (resp., $t \geq t_1$). We need to show that there is $t' \leq t_2$ (resp., $t' \geq t_2$) with $t' \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ such that $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t'$. Assume that $t \leq t_1$ (the other case $t \geq t_1$ being similar). If $t \leq d_p$, then we proceed as in the previous case. Otherwise, $t > K_{max} - min(s_{\mathcal{A}}, s_{\mathcal{B}})$. In this case, we set $t' = t_2$, and the result follows.

This concludes the proof of the lemma. ∎

Now, we can prove Lemma 8.

Lemma 8. *Let $\Gamma \subseteq Reg_{\mathcal{A}\|\mathcal{B}}$ be a set of regions and $R \in \Gamma$ such that R is good in Γ w.r.t. player $\sigma \in \{0,1\}$. Then, the following holds for each $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in R$:*

- *for each move $m^{\mathcal{A}}_{\sigma} = (t, a, \delta_{\mathcal{A}}) \in Mov_{\mathcal{A}}(\sigma, s_{\mathcal{A}})$, there is a matching move $m^{\mathcal{B}}_{\sigma} = (t, a, \delta_{\mathcal{B}}) \in Mov_{\mathcal{B}}(\sigma, s_{\mathcal{B}})$ such that for every move $m^{\mathcal{B}}_{1-\sigma} = (t', b, \delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1-\sigma, s_{\mathcal{B}})$, there is a matching move $m^{\mathcal{A}}_{1-\sigma} = (t', b, \delta'_{\mathcal{A}}) \in Mov_{\mathcal{A}}(1-\sigma, s_{\mathcal{A}})$ so that for all $i = 0,1$ with $m^{\mathcal{A}}_i \in JDM(m^{\mathcal{A}}_0, m^{\mathcal{A}}_1)$, $(Next_{\mathcal{A}}(s_{\mathcal{A}}, m^{\mathcal{A}}_i), Next_{\mathcal{B}}(s_{\mathcal{B}}, m^{\mathcal{B}}_i)) \in R_i$ for some $R_i \in \Gamma$.*

PROOF. Let $R \in \Gamma$ such that R is good in Γ w.r.t. player σ. Evidently, it suffices to show the following:

Claim: for each $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in R$ and edge $s_{\mathcal{A}} \xrightarrow{t_1, a, \delta_{\mathcal{A}}} u_{\mathcal{A}}$ of $\mathcal{A}$ from $s_{\mathcal{A}}$ with $a \in Act^{\perp}_{\sigma}$, there is a matching edge $s_{\mathcal{B}} \xrightarrow{t_1, a, \delta_{\mathcal{B}}} u_{\mathcal{B}}$ in $\mathcal{B}$ from $s_{\mathcal{B}}$ for some $u_{\mathcal{B}} \in S_{\mathcal{B}}$ and transition $\delta_{\mathcal{B}}$ of $\mathcal{B}$ so that:

1. for each $t \geq t_1$ and $b \in Act^{\perp}_{1-\sigma}$, if $s_{\mathcal{B}} \xrightarrow{t, b, \delta} w_{\mathcal{B}}$, then $(u_{\mathcal{A}}, u_{\mathcal{B}}) \in R'$ with $R' \in \Gamma$;

2. for each $t \geq 0$ and $b \in Act^{\perp}_{1-\sigma}$, if $s_{\mathcal{B}} \xrightarrow{t,b,\delta} w_{\mathcal{B}}$, then $s_{\mathcal{A}} \xrightarrow{t,b,\delta'} w_{\mathcal{A}}$ such that if $t \leq t_1$, then $(w_{\mathcal{A}}, w_{\mathcal{B}}) \in R''$ with $R'' \in \Gamma$.

Fix $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in R$ and an edge $s_{\mathcal{A}} \xrightarrow{t_1,a,\delta_{\mathcal{A}}} u_{\mathcal{A}}$ of $\mathcal{A}$ from $s_{\mathcal{A}}$ with $a \in Act^{\perp}_{\sigma}$. Since R is good in Γ w.r.t. player σ, there is $(s'_{\mathcal{A}}, s'_{\mathcal{B}}) \in R$ (hence, $(s_{\mathcal{A}}, s_{\mathcal{B}}) \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}})$) such that $(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ satisfies Condition 1 in Definition 7 of goodness. By Lemma 18, there is $t_2 \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ such that:

(A) $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t_1 \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t_2$ and: for each $t \leq t_1$ (resp., $t \geq t_1$), there is $t' \leq t_2$ (resp., $t' \geq t_2$) with $t' \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ such that $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t'$.

Since $s_{\mathcal{A}} \xrightarrow{t_1,a,\delta_{\mathcal{A}}} u_{\mathcal{A}}$ and $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t_1 \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t_2$ (Property A), by Proposition 17, $s'_{\mathcal{A}} \xrightarrow{t_2,a,\delta_{\mathcal{A}}} u'_{\mathcal{A}}$ for some $\mathcal{A}$-state $u'_{\mathcal{A}}$. Since $t_2 \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ and $(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ satisfies Condition 1 in Definition 7 of goodness, there must be a matching edge $s'_{\mathcal{B}} \xrightarrow{t_2,a,\delta_{\mathcal{B}}} u'_{\mathcal{B}}$ in $\mathcal{B}$ from $s'_{\mathcal{B}}$ so that the following holds:

(B) for each $t \geq t_2$ with $t \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ and $b \in Act^{\perp}_{1-\sigma}$, if $s'_{\mathcal{B}} \xrightarrow{t,b,\delta} w'_{\mathcal{B}}$, then $(u'_{\mathcal{A}}, u'_{\mathcal{B}}) \in R'$ for some $R' \in \Gamma$;

(C) for each $t \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ and $b \in Act^{\perp}_{1-\sigma}$, if $s'_{\mathcal{B}} \xrightarrow{t,b,\delta} w'_{\mathcal{B}}$, then $s'_{\mathcal{A}} \xrightarrow{t,b,\delta'} w'_{\mathcal{A}}$ such that if $t \leq t_2$, then $(w'_{\mathcal{A}}, w'_{\mathcal{B}}) \in R''$ for some $R'' \in \Gamma$.

Since $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t_1 \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t_2$ and $s'_{\mathcal{B}} \xrightarrow{t_2,a,\delta_{\mathcal{B}}} u'_{\mathcal{B}}$, by Proposition 17, there is a $\mathcal{B}$-state $u_{\mathcal{B}}$ such that $s_{\mathcal{B}} \xrightarrow{t_1,a,\delta_{\mathcal{B}}} u_{\mathcal{B}}$. Now, we show that for the edge $s_{\mathcal{B}} \xrightarrow{t_1,a,\delta_{\mathcal{B}}} u_{\mathcal{B}}$ (matching the edge $s_{\mathcal{A}} \xrightarrow{t_1,a,\delta_{\mathcal{A}}} u_{\mathcal{A}}$), Conditions 1 and 2 in the claim are satisfied:

- **Condition 1 of the claim:** assume that $s_{\mathcal{B}} \xrightarrow{t,b,\delta} w_{\mathcal{B}}$ for some $t \geq t_1$ and $b \in Act^{\perp}_{1-\sigma}$. We need to show that $(u_{\mathcal{A}}, u_{\mathcal{B}}) \in R'$ for some $R' \in \Gamma$. By Property A, there is $t' \geq t_2$ such that $t' \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ and $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t'$. By Proposition 17 it follows that $s'_{\mathcal{B}} \xrightarrow{t',b,\delta} w'_{\mathcal{B}}$ for some state $w'_{\mathcal{B}} \in S_{\mathcal{B}}$. Since $t' \geq t_2$ and $t' \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$, by Property B, $(u'_{\mathcal{A}}, u'_{\mathcal{B}}) \in R'$ for some $R' \in \Gamma$. Since $s_{\mathcal{A}} \xrightarrow{t_1,a,\delta_{\mathcal{A}}} u_{\mathcal{A}}$, $s'_{\mathcal{A}} \xrightarrow{t_2,a,\delta_{\mathcal{A}}} u'_{\mathcal{A}}$, $s'_{\mathcal{B}} \xrightarrow{t_2,a,\delta_{\mathcal{B}}} u'_{\mathcal{B}}$, $s_{\mathcal{B}} \xrightarrow{t_1,a,\delta_{\mathcal{B}}} u_{\mathcal{B}}$, and $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t_1 \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t_2$, by Proposition 17, it follows that $(u_{\mathcal{A}}, u_{\mathcal{B}}) \approx_{\mathcal{A}\|\mathcal{B}} (u'_{\mathcal{A}}, u'_{\mathcal{B}})$, hence $(u_{\mathcal{A}}, u_{\mathcal{B}}) \in R'$ and the result holds.
- **Condition 2 of the claim:** assume that $s_{\mathcal{B}} \xrightarrow{t,b,\delta} w_{\mathcal{B}}$ with $b \in Act^{\perp}_{1-\sigma}$ and arbitrary $t \geq 0$, $w_{\mathcal{B}} \in S_{\mathcal{B}}$, and transition δ of $\mathcal{B}$. We need to show that there is a matching edge $s_{\mathcal{A}} \xrightarrow{t,b,\delta'} w_{\mathcal{A}}$ in $\mathcal{A}$ from $s_{\mathcal{A}}$ such that $(w_{\mathcal{A}}, w_{\mathcal{B}}) \in R''$ for some $R'' \in \Gamma$ if $t \leq t_1$. By Property A, there is $t' \in Times(s'_{\mathcal{A}}, s'_{\mathcal{B}})$ such that $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t'$ and $t' \leq t_2$ if $t \leq t_1$. By Proposition 17 it follows that $s'_{\mathcal{B}} \xrightarrow{t',b,\delta} w'_{\mathcal{B}}$ for some state $w'_{\mathcal{B}} \in S_{\mathcal{B}}$. Thus, by Property C, there is a matching edge $s'_{\mathcal{A}} \xrightarrow{t',b,\delta'} w'_{\mathcal{A}}$ in $\mathcal{A}$ from $s'_{\mathcal{A}}$ such that $(w'_{\mathcal{A}}, w'_{\mathcal{B}}) \in R''$ with $R'' \in \Gamma$ if $t' \leq t_2$. Since $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t'$, by Proposition 17, $s_{\mathcal{A}} \xrightarrow{t,b,\delta'} w_{\mathcal{A}}$ for some state $w_{\mathcal{A}}$. Now, we show that $(w_{\mathcal{A}}, w_{\mathcal{B}}) \in R''$ if $t \leq t_1$, hence the result follows. Assume that $t \leq t_1$. Then, $t' \leq t_2$ and $(w'_{\mathcal{A}}, w'_{\mathcal{B}}) \in R''$. Since $s_{\mathcal{B}} \xrightarrow{t,b,\delta} w_{\mathcal{B}}$, $s'_{\mathcal{B}} \xrightarrow{t',b,\delta} w'_{\mathcal{B}}$, $s'_{\mathcal{A}} \xrightarrow{t',b,\delta'} w'_{\mathcal{A}}$, $s_{\mathcal{A}} \xrightarrow{t,b,\delta'} w_{\mathcal{A}}$, and $(s_{\mathcal{A}}, s_{\mathcal{B}}) + t \approx_{\mathcal{A}\|\mathcal{B}} (s'_{\mathcal{A}}, s'_{\mathcal{B}}) + t'$, by Proposition 17, we obtain that $(w_{\mathcal{A}}, w_{\mathcal{B}}) \approx_{\mathcal{A}\|\mathcal{B}} (w'_{\mathcal{A}}, w'_{\mathcal{B}})$, hence $(w_{\mathcal{A}}, w_{\mathcal{B}}) \in R''$ and the result holds.

∎

D Lower bound for timed alternating simulation

For $i = 1, 2$, let $\mathcal{T}_i$ be a TT over Act with set of states S_i. A *timed simulation from $\mathcal{T}_1$ to $\mathcal{T}_2$* is a relation $H \subseteq S_1 \times S_2$ such that for each $(s_1, s_2) \in H$, the following holds: for each edge $s_1 \xrightarrow{t,a,\delta_1} s'_1$ in $[\![\mathcal{T}_1]\!]$ from s_1, there is a matching edge $s_2 \xrightarrow{t,a,\delta_2} s'_2$ in $[\![\mathcal{T}_2]\!]$ from s_2 so that $(s'_1, s'_2) \in H$. For $(s_1, s_2) \in S_1 \times S_2$, we say that *$s_1$ in $\mathcal{T}_1$ is timed simulated by s_2 in $\mathcal{T}_2$* iff there is a timed simulation H from $\mathcal{T}_1$ to $\mathcal{T}_2$ such that $(s_1, s_2) \in H$. The following known result has been proved in [14].

PROPOSITION 19.*[14] For $i = 1, 2$, let $\mathcal{T}_i$ be a* TT *on Act* (possibly, without invariants)[‡] *and s_i be a state of $\mathcal{T}_i$. Then, checking whether s_1 in $\mathcal{T}_1$ is timed simulated by s_2 in $\mathcal{T}_2$ is* EXPTIME*-hard.*

Now, we can prove the desired result.

THEOREM 20. *Given two comparable TG $\mathcal{A}$ and $\mathcal{B}$ and player $\sigma \in \{0, 1\}$, the problem of checking whether $\mathcal{A} \preceq_\sigma \mathcal{B}$ is* EXPTIME*-hard.*

PROOF. By Proposition 19, it suffices to show that checking timed simulation between TT without invariants can be reduced in polynomial time to checking timed alternating simulation between TG. Fix $\sigma \in \{0, 1\}$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two TT over Act without invariants, s_1 be a state of $\mathcal{T}_1$ and s_2 be a state of $\mathcal{T}_2$. We construct in linear time two TG $\mathcal{B}_1$ and $\mathcal{B}_2$ such that s_1 in $\mathcal{T}_1$ is *timed simulated* by s_2 in $\mathcal{T}_2$ iff $\mathcal{B}_1 \preceq_\sigma \mathcal{B}_2$. Hence, the result follows.

Assume that $\sigma = 0$ (the other case is similar). Then, $\mathcal{B}_1 = \langle \mathcal{T}'_1, s_1, Act, \{b\} \rangle$ and $\mathcal{B}_2 = \langle \mathcal{T}'_2, s_2, Act, \{b\} \rangle$ such that $b \notin Act$ is the unique non-null action of player 1, and for each $i = 1, 2$, denoted by q_T a new location whose invariant is *true*, $\mathcal{T}'_i$ is obtained from $\mathcal{T}_i$ by adding for each its location q, the transition $(q, b, true, \emptyset, q_T)$. Now, we prove correctness of the construction.

Claim: s_1 in $\mathcal{T}_1$ is timed simulated by s_2 in $\mathcal{T}_2$ iff $\mathcal{B}_1 \preceq_0 \mathcal{B}_2$.

Proof of the Claim For each $i = 1, 2$, let S_i be the set of states of $\mathcal{T}_i$. For a state $s_i = (q_i, v_i)$ of $\mathcal{T}_i$, we denote by $s_i + t$ the pair $(q_i, v_i + t)$ (note that since $\mathcal{T}_i$ has no invariants, $s_i + t$ is still a state of $\mathcal{T}_i$). We prove the direct implication ($\Rightarrow$) of the claim (the converse implication being simpler). Assume that s_1 in $\mathcal{T}_1$ is timed simulated by s_2 in $\mathcal{T}_2$. Thus, there is a timed simulation $H \subseteq S_1 \times S_2$ from $\mathcal{T}_1$ to $\mathcal{T}_2$ such that $(s_1, s_2) \in H$. Let H' be obtained from H by adding all the pairs $((q_T, v_1), (q_T, v_2))$ such that v_1 (resp., v_2) is a clock valuation over the set of $\mathcal{T}_i$-clocks. Note that by construction, for $i = 1, 2$, (q_T, v_i) is a state of $\mathcal{B}_i$. We show that H' is a timed alternating simulation for player 0 from $\mathcal{B}_1$ to $\mathcal{B}_2$. Hence, the result follows. First, we observe that since $\mathcal{T}_1$ and $\mathcal{T}_2$ have no invariants, if $(s'_1, s'_2) \in H$, then $(s'_1 + t, s'_2 + t) \in H$ for each $t \geq 0$. Let $(s'_1, s'_2) \in H'$ and $s'_1 \xrightarrow{t,a,\delta_1} s''_1$ be an edge of $[\![\mathcal{T}'_1]\!]$ with $a \in Act \cup \{\perp\}$. Assume that $a \in Act$ (the other case being similar). By construction, $s'_1 \xrightarrow{t,a,\delta_1} s''_1$ is also an edge of $[\![\mathcal{T}_1]\!]$ and $(s'_1, s'_2) \in H$. We need to show that there is a matching edge $s'_2 \xrightarrow{t,a,\delta_2} s''_2$ in $[\![\mathcal{T}'_2]\!]$ such that

1. for each edge in $[\![\mathcal{T}'_2]\!]$ of the form $s'_2 \xrightarrow{t',c,\widehat{\delta}_2} \widehat{s}_2$ with $c \in \{b, \perp\}$, there is a matching edge $s'_1 \xrightarrow{t',c,\widehat{\delta}_1} \widehat{s}_1$ from s'_1 in $[\![\mathcal{T}'_1]\!]$ so that if $t' \leq t$, then $(\widehat{s}_1, \widehat{s}_2) \in H'$, and if $t' \geq t$, then

‡ in the sense that the invariant assigns to each location the *true* clock constraint

$(s_1'', s_2'') \in H'$.

Since $(s_1', s_2') \in H$, $s_1' \xrightarrow{t,a,\delta_1} s_1''$ is an edge of $[\![\mathcal{T}_1]\!]$, and H is a timed simulation from $\mathcal{T}_1$ to $\mathcal{T}_2$, there must be a matching edge $s_2' \xrightarrow{t,a,\delta_2} s_2''$ in $[\![\mathcal{T}_2]\!]$ (that is also an edge of $[\![\mathcal{T}_2']\!]$) such that $(s_1'', s_2'') \in H \subseteq H'$. By construction, for each $t' \geq 0$, $i = 1, 2$, and $c \in \{b, \bot\}$, $s_i' \xrightarrow{t',c,\delta} s_i' + t'$ is an edge of $[\![\mathcal{T}_i']\!]$ for some transition δ, and if $s_i' \xrightarrow{t',c,\delta'} \widehat{s}_i$ is an edge of $[\![\mathcal{T}_i']\!]$, then $\widehat{s}_i = s_i' + t'$. Since $(s_1' + t', s_2' + t') \in H \subseteq H'$ for each $t' \geq 0$, it follows that for the matching edge $s_2' \xrightarrow{t,a,\delta_2} s_2''$ of $s_1' \xrightarrow{t,a,\delta_1} s_1''$, Condition 1 above holds. Hence, the result follows. ∎

E Full proof of the converse implication in Theorem 12

In order to prove the converse implication in Theorem 12, we need additional results. Fix two comparable TG $\mathcal{A}$ and $\mathcal{B}$ over (Act_0, Act_1) with initial states $s^0_{\mathcal{A}}$ and $s^0_{\mathcal{B}}$, respectively. Let $S_{\mathcal{A}}$ (resp., $S_{\mathcal{B}}$) be the set of states of $\mathcal{A}$ (resp., $\mathcal{B}$).

Lemma 21. *For each player $\sigma \in \{0,1\}$, $\mathcal{A} \preceq_\sigma \mathcal{B}$ iff there is a relation $H \subseteq S_{\mathcal{A}} \times S_{\mathcal{B}}$ such that $(s^0_{\mathcal{A}}, s^0_{\mathcal{B}}) \in H$ and for each $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in H$, the following holds:*

- *for every move $m^{\mathcal{A}}_\sigma = (t, a, \delta_{\mathcal{A}}) \in Mov_{\mathcal{A}}(\sigma, s_{\mathcal{A}})$ with $\underline{t \in Times(s_{\mathcal{A}}, s_{\mathcal{B}})}$, there is a matching move $m^{\mathcal{B}}_\sigma = (t, a, \delta_{\mathcal{B}}) \in Mov_{\mathcal{B}}(\sigma, s_{\mathcal{B}})$ such that for every move $m^{\mathcal{B}}_{1-\sigma} = (t', b, \delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1-\sigma, s_{\mathcal{B}})$ with $\underline{t' \in Times(s_{\mathcal{A}}, s_{\mathcal{B}})}$, there is a matching move $m^{\mathcal{A}}_{1-\sigma} = (t', b, \delta'_{\mathcal{A}}) \in Mov_{\mathcal{A}}(1-\sigma, s_{\mathcal{A}})$ so that for all $i = 0, 1$ with $m^{\mathcal{A}}_i \in JDM(m^{\mathcal{A}}_0, m^{\mathcal{A}}_1)$,*

$$(Next_{\mathcal{A}}(s_{\mathcal{A}}, m^{\mathcal{A}}_i), Next_{\mathcal{B}}(s_{\mathcal{B}}, m^{\mathcal{B}}_i)) \in H$$

Proof. Fix $\sigma \in \{0,1\}$. The direct implication is obvious. Now, let us consider the converse implication. Assume that there is $H \subseteq S_{\mathcal{A}} \times S_{\mathcal{B}}$ satisfying the statement of the lemma. We need to show that $\mathcal{A} \preceq_\sigma \mathcal{B}$. Let $\Upsilon_H \subseteq Reg_{\mathcal{A}\|\mathcal{B}}$ be the set of regions R such that $R \cap H \neq \emptyset$. By hypothesis and Definition 7 of goodness in Subsection 3.1, it follows that for each $R \in \Upsilon_H$, R is good in Υ_H w.r.t. player σ. Hence, Υ_H is a fixpoint of the operator Ω_σ defined in Subsection 3.1. By Corollary 10, it follows that $\widetilde{H} = \bigcup_{R \in \Upsilon_H} R$ is an alternating timed simulation for player σ from $\mathcal{A}$ to $\mathcal{B}$. Since $\widetilde{H} \supseteq H$ and $(s^0_{\mathcal{A}}, s^0_{\mathcal{B}}) \in H$, we are done. ∎

Let $\sigma \in \{0,1\}$ and consider the following turn-based two-player game, which corresponds to the game between the antagonist and the protagonist at the beginning of Section 3 with the restriction that if $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$ is the current main position of the game, then the timestamps chosen by the antagonist are in the finite set $Times(s_{\mathcal{A}}, s_{\mathcal{B}})$. The game proceeds in rounds as follows. Each round consists of five steps as follows. Assume that the current main position is $(s_{\mathcal{A}}, s_{\mathcal{B}})$. Then:

1. The antagonist chooses $t \in Times(s_{\mathcal{A}}, s_{\mathcal{B}})$ and a move $m^{\mathcal{A}}_\sigma = (t, a, \delta_{\mathcal{A}}) \in Mov_{\mathcal{A}}(\sigma, s_{\mathcal{A}})$ of player σ in $\mathcal{A}$ from $s_{\mathcal{A}}$ with timestamp t, and moves to position $p_1 = \langle s_{\mathcal{A}}, s_{\mathcal{B}}, m^{\mathcal{A}}_\sigma \rangle$.
2. The protagonist from p_1 chooses a matching move $m^{\mathcal{B}}_\sigma = (t, a, \delta_{\mathcal{B}}) \in Mov_{\mathcal{B}}(\sigma, s_{\mathcal{B}})$ of player σ in $\mathcal{B}$ from $s_{\mathcal{B}}$ and moves to position $p_2 = \langle s_{\mathcal{A}}, s_{\mathcal{B}}, m^{\mathcal{A}}_\sigma, m^{\mathcal{B}}_\sigma \rangle$.
3. The antagonist from position p_2 chooses $t' \in Times(s_{\mathcal{A}}, s_{\mathcal{B}})$ and a move $m^{\mathcal{B}}_{1-\sigma} = (t', b, \delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1-\sigma, s_{\mathcal{B}})$ of player $1-\sigma$ in $\mathcal{B}$ from $s_{\mathcal{B}}$ with timestamp t', and moves to position $p_3 = \langle s_{\mathcal{A}}, s_{\mathcal{B}}, m^{\mathcal{A}}_\sigma, m^{\mathcal{B}}_\sigma, m^{\mathcal{B}}_{1-\sigma} \rangle$.

4. The protagonist from p_3 chooses a matching move $m^{\mathcal{A}}_{1-\sigma} = (t', b, \delta'_{\mathcal{A}}) \in Mov_{\mathcal{A}}(1-\sigma, s_{\mathcal{A}})$ of player $1-\sigma$ in $\mathcal{A}$ from $s_{\mathcal{A}}$ and moves to $p_4 = \langle s_{\mathcal{A}}, s_{\mathcal{B}}, m^{\mathcal{A}}_{\sigma}, m^{\mathcal{B}}_{\sigma}, m^{\mathcal{B}}_{1-\sigma}, m^{\mathcal{A}}_{1-\sigma}\rangle$.
5. The antagonist from position p_4 chooses $i = 0,1$ such that $m^{\mathcal{A}}_i \in JDM(m^{\mathcal{A}}_0, m^{\mathcal{A}}_1)$, and moves to position $(Next_{\mathcal{A}}(s_{\mathcal{A}}, m^{\mathcal{A}}_i), Next_{\mathcal{B}}(s_{\mathcal{B}}, m^{\mathcal{B}}_i))$.

If the game proceeds ad infinitum, then the protagonist wins. Otherwise, the game reaches a position from which the protagonist cannot choose in steps 2 or 4 above a matching move, and the antagonist wins. By Lemma 21, it easily follows that $\mathcal{A} \preceq_\sigma \mathcal{B}$ iff the protagonist has a winning strategy starting at the initial position $(s^{\mathcal{A}}_0, s^{\mathcal{B}}_0)$. Formally the game between the antagonist and the protagonist, which is a zero sum turn-based two-player game, is defined as follows. The underlying game-graph $G_\sigma = \langle P = P_{ant} \cup P_{pro}, \twoheadrightarrow\rangle$ is defined as follows. The set P of states (here, called positions) is partitioned into the set P_{ant} of positions available to the antagonist and the set P_{pro} of positions available to the protagonist, where $P_{ant} = P^0_{ant} \cup P^2_{ant} \cup P^4_{ant}$, $P_{pro} = P^1_{pro} \cup P^3_{prot}$, and the following holds:

$$
\begin{aligned}
P^0_{ant} = \;& S_{\mathcal{A}} \times S_{\mathcal{B}} \\
P^2_{ant} = \;& \{\langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}}), (t,a,\delta_{\mathcal{B}})\rangle \mid s_{\mathcal{A}} \in S_{\mathcal{A}},\ s_{\mathcal{B}} \in S_{\mathcal{B}},\ t \in Times(s_{\mathcal{A}}, s_{\mathcal{B}}), \\
& (t,a,\delta_{\mathcal{A}}) \in Mov_{\mathcal{A}}(\sigma, s_{\mathcal{A}}), \text{ and } (t,a,\delta_{\mathcal{B}}) \in Mov_{\mathcal{B}}(\sigma, s_{\mathcal{B}})\} \\
P^4_{ant} = \;& \{\langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}}), (t,a,\delta_{\mathcal{B}}), (t',b,\delta'_{\mathcal{B}}), (t',b,\delta'_{\mathcal{A}})\rangle \mid s_{\mathcal{A}} \in S_{\mathcal{A}},\ s_{\mathcal{B}} \in S_{\mathcal{B}}, \\
& t, t' \in Times(s_{\mathcal{A}}, s_{\mathcal{B}}),\ (t,a,\delta_{\mathcal{A}}) \in Mov_{\mathcal{A}}(\sigma, s_{\mathcal{A}}),\ (t,a,\delta_{\mathcal{B}}) \in Mov_{\mathcal{B}}(\sigma, s_{\mathcal{B}}) \\
& (t',b,\delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1-\sigma, s_{\mathcal{B}}), \text{ and } (t',b,\delta'_{\mathcal{A}}) \in Mov_{\mathcal{A}}(1-\sigma, s_{\mathcal{A}})\} \\
P^1_{prot} = \;& \{\langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}})\rangle \mid s_{\mathcal{A}} \in S_{\mathcal{A}},\ s_{\mathcal{B}} \in S_{\mathcal{B}},\ t \in Times(s_{\mathcal{A}}, s_{\mathcal{B}}), \text{ and } (t,a,\delta_{\mathcal{A}}) \in Mov_{\mathcal{A}}(\sigma, s_{\mathcal{A}})\} \\
P^3_{pro} = \;& \{\langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}}), (t,a,\delta_{\mathcal{B}}), (t',b,\delta'_{\mathcal{B}})\rangle \mid s_{\mathcal{A}} \in S_{\mathcal{A}},\ s_{\mathcal{B}} \in S_{\mathcal{B}},\ t, t' \in Times(s_{\mathcal{A}}, s_{\mathcal{B}}), \\
& (t,a,\delta_{\mathcal{A}}) \in Mov_{\mathcal{A}}(\sigma, s_{\mathcal{A}}), (t,a,\delta_{\mathcal{B}}) \in Mov_{\mathcal{B}}(\sigma, s_{\mathcal{B}}), \text{ and } (t',b,\delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1-\sigma, s_{\mathcal{B}})\}
\end{aligned}
$$

Finally, the set $\twoheadrightarrow \subseteq (P_{pro} \times P_{ant}) \cup (P_{ant} \times P_{pro}) \cup (P^4_{ant} \times P^0_{ant})$ contains all and only the (labeled) edges of the following form:

$$
\begin{aligned}
&(s_{\mathcal{A}}, s_{\mathcal{B}}) && \xrightarrow{(t,a,\delta_{\mathcal{A}})} \langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}})\rangle \\
&\langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}})\rangle && \xrightarrow{(t,a,\delta_{\mathcal{B}})} \langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}}), (t,a,\delta_{\mathcal{B}})\rangle \\
&\langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}}), (t,a,\delta_{\mathcal{B}})\rangle && \xrightarrow{(t',b,\delta'_{\mathcal{B}})} \langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}}), (t,a,\delta_{\mathcal{B}}), (t',b,\delta'_{\mathcal{B}})\rangle \\
&\langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}}), (t,a,\delta_{\mathcal{B}}), (t',b,\delta'_{\mathcal{B}})\rangle && \xrightarrow{(t',b,\delta'_{\mathcal{A}})} \langle s_{\mathcal{A}}, s_{\mathcal{B}}, (t,a,\delta_{\mathcal{A}}), (t,a,\delta_{\mathcal{B}}), (t',b,\delta'_{\mathcal{B}}), (t',b,\delta'_{\mathcal{A}})\rangle
\end{aligned}
$$

and for each position $p = \langle s_{\mathcal{A}}, s_{\mathcal{B}}, m^{\mathcal{A}}_{\sigma}, m^{\mathcal{B}}_{\sigma}, m^{\mathcal{B}}_{1-\sigma}, m^{\mathcal{A}}_{1-\sigma}\rangle \in P^4_{ant}$ and for each $i = 0,1$ such that $m^{\mathcal{A}}_i \in JDM(m^{\mathcal{A}}_0, m^{\mathcal{A}}_1)$, the following edge

$$p \overset{i}{\twoheadrightarrow} (Next_{\mathcal{A}}(s_{\mathcal{A}}, m^{\mathcal{A}}_i), Next(s_{\mathcal{B}}, m^{\mathcal{B}}_i))$$

Note that since $Times(s_{\mathcal{A}}, s_{\mathcal{B}})$ is finite for each pair $(s_{\mathcal{A}}, s_{\mathcal{B}}) \in S_{\mathcal{A}} \times S_{\mathcal{B}}$, the number of successors of each position in G_σ is finite. A strategy f_{pro} of the protagonist is a mapping $f_{pro} : P^* \cdot P_{pro} \to P \cup \{\vdash\}$ ($\vdash$ is for undefined) assigning to each sequence $\pi = \pi', p \in P^* \cdot P_{pro}$ leading to a position $p \in P_{pro}$ an element in $P \cup \{\vdash\}$ such that (1) if $f_{pro}(\pi) \neq \vdash$, then $f_{pro}(\pi)$ is a successor of p in G_σ, and (2) if $f_{pro}(\pi) = \vdash$, then p has no successors in G_σ. For a position p, the set of outcomes of f_{pro} starting from p is the set of maximal paths

$\pi = p_1, p_2, \ldots$ in G_σ such that $p_1 = p$ and for each $k < |\pi|$, if $p_k \in P_{pro}$, then $p_{k+1} = f_{pro}(p_1, \ldots, p_k)$. The strategy f_{pro} is winning for position $p \in P$ iff each outcome of f_{pro} from p is *infinite*. A strategy f_{ant} of the antagonist is a mapping $f_{ant} : P^* \cdot P_{ant} \to P$ assigning to each sequence $\pi = \pi', p \in P^* \cdot P_{ant}$ leading to a position $p \in P_{ant}$ a position in P such that $f_{ant}(\pi)$ is a successor of p in G_σ. Note that the set of successors of each position in P_{ant} is not empty since for each pair $(s_\mathcal{A}, s_\mathcal{B}) \in S_\mathcal{A} \times S_\mathcal{B}$, $0 \in Times(s_\mathcal{A}, s_\mathcal{B})$. For a position p, the set of outcomes of f_{ant} starting from position p is defined similarly to the set of outcomes for a strategy of the protagonist. The strategy f_{ant} is winning from position p iff each outcome of f_{ant} from p is *finite*. Note that the considered game can be trivially converted into an "equivalent" infinite finitely-branching turn-based two-player safety game. Since (perfect-information) turn-based two-player safety games are determined [19], by Lemma 21, we obtain the following result.

LEMMA 22. *If $\mathcal{A} \not\preceq_\sigma \mathcal{B}$, then there is a winning strategy of the* antagonist *from position $(s_0^\mathcal{A}, s_0^\mathcal{B})$ in the game G_σ.*

Let f_{ant} be a strategy of the antagonist in the game G_σ and p_0 a position in the game. The *strategy-tree* $T(f_{ant}, p_0)$ of f_{ant} starting from p_0 is the finitely-branching tree with nodes labeled by positions in G_σ and labeled edges, inductively defined as follows:

- the root is labeled by the initial position p_0;
- let x_p be a node labeled by a position $p \in P_{pro}$: then, for each successor p' of p in G_σ, there is exactly one edge in the tree from x_p to a child $x_{p'}$ labeled by p'. The label of this edge coincides with the label of the unique edge of G_σ from p to p';
- let x_p be a node labeled by a position $p \in P_{ant}$: then, x_p has a unique child y_p labeled by $f_{ant}(\pi)$, where $\pi = \pi', p$ is the sequence of positions labeling the nodes of the partial path from the root to x_p. The label of the edge from x_p to y_p coincides with the label of some edge in G_σ from p to $f_{ant}(\pi)$. Note that if $p \notin P_{ant}^4$, then there is a unique edge in G_σ from p to $f_{ant}(\pi)$. If instead $p \in P_{ant}^4$, by construction there may be two edges in G_σ from p to $f_{ant}(\pi)$, one labeled by 0 and the other one labeled by 1.

Evidently, the maximal paths of $T(f_{ant}, p_0)$ from the root correspond to the outcomes of f_{ant} from position p_0. Note that for each $(s_\mathcal{A}, s_\mathcal{B}) \in S_\mathcal{A} \times S_\mathcal{B}$, if the clock-values in $(s_\mathcal{A}, s_\mathcal{B})$ are rational, then $Times(s_\mathcal{A}, s_\mathcal{B}) \subseteq \mathbb{Q}_{\geq 0}$. Since the clock-values in the initial states of $\mathcal{A}$ and $\mathcal{B}$ are rational, we obtain the following result.

LEMMA 23. *Let f_{ant} be a strategy of the antagonist in the game G_σ. Then, for the strategy-tree $T(f_{ant}, (s_0^\mathcal{A}, s_0^\mathcal{B}))$, all the timestamps associated with the edge-labels are in $\mathbb{Q}_{\geq 0}$.*

Now, we can prove the converse implication of Theorem 12.

THEOREM 24. *For each player $\sigma \in \{0,1\}$, if $\mathcal{A} \not\preceq_\sigma \mathcal{B}$, then there is a σ-TAMTL$_P^*$ formula φ such that $\mathcal{A} \models \varphi$ and $\mathcal{B} \not\models \varphi$.*

PROOF. Let $\sigma \in \{0,1\}$ and assume that $\mathcal{A} \not\preceq_\sigma \mathcal{B}$. Assume that $\sigma = 0$ (the other case being symmetric). By Lemma 22, there is a winning strategy f_{ant} of the antagonist in the game G_0 starting from position $p_0 = (s_0^\mathcal{A}, s_0^\mathcal{B})$. It follows that each path in the strategy-tree $T(f_{ant}, p_0)$ is finite. Since $T(f_{ant}, p_0)$ is finitely-branching, $T(f_{ant}, p_0)$ is also a finite tree. We claim that for each node x_p of $T(f_{ant}, p_0)$ labeled by a position p of the form $p = (s_\mathcal{A}, s_\mathcal{B}) \in S_\mathcal{A} \times S_\mathcal{B}$ (note that the root satisfies this condition), there is a 0-TAMTL$_P^*$ formula φ_p such that $(\mathcal{A}, s_\mathcal{A}) \models \varphi_p$ and $(\mathcal{B}, s_\mathcal{B}) \not\models \varphi_p$. Hence, the result follows. The proof is by induction on the height h of the subtree of $T(f_{ant}, p_0)$ rooted at node x_p (since $T(f_{ant}, p_0)$ is a finite tree, h is well-defined). Since x_p is labeled by a position $p = (s_{\mathcal{A}}, s_{\mathcal{B}})$ in P_{ant}, x_p has exactly one child, say x'_p, and the label of the edge from x_p to x'_p is a move $m^0_{\mathcal{A}}$ for player 0 in $Mov_{\mathcal{A}}(0, s_{\mathcal{A}})$ of the form $m^0_{\mathcal{A}} = (t, a, \delta_{\mathcal{A}})$. Moreover, by Lemma 23, $t \in \mathbb{Q}_{\geq 0}$.

Base Step ($h = 1$): in this case x'_p is a leaf. By construction, there is no matching move for player 0 in $Mov_{\mathcal{B}}(0, s_{\mathcal{B}})$ of the form $(t, a, \delta_{\mathcal{B}})$. Let φ_p be the 0-TAMTL$^*_P$ formula given by

$$\varphi_p = \langle\langle 0 \rangle\rangle \Big(\bigvee_{b \in Act_1^{\perp}} \bigvee_{\kappa \in \{0,1\}} \langle (a, = t), (b, \geq 0), \kappa \rangle \Big)$$

Evidently, $(\mathcal{A}, s_{\mathcal{A}}) \models \varphi_p$ and $(\mathcal{B}, s_{\mathcal{B}}) \not\models \varphi_p$. Thus, for the base case, the result holds.

Induction Step ($h > 1$): let $y_1, \ldots, y_n$ be the children of x'_p (with $n \geq 1$). By construction, the labels of the edges from x'_p corresponds to *all and only* the matching moves $m^0_{\mathcal{B}} = (t, a, \delta_{\mathcal{B}}) \in Mov_{\mathcal{B}}(0, s_{\mathcal{B}})$ of $m^0_{\mathcal{A}} = (t, a, \delta_{\mathcal{A}})$ for player 0 in $\mathcal{B}$ from $s_{\mathcal{B}}$. Moreover, for each $1 \leq i \leq n$, let y'_i be the unique child of y_i (since y_i is labeled by a position in P_{ant}, this condition is satisfied).

First, assume that for some $1 \leq i \leq n$, y'_i is a leaf. By construction the edge from y_i to y'_i is labeled by a move $(t', b, \delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1, s_{\mathcal{B}})$ of player 1 in $\mathcal{B}$ from $s_{\mathcal{B}}$, and there is no matching move $(t', b, \delta'_{\mathcal{A}})$ in $Mov_{\mathcal{A}}(1, s_{\mathcal{A}})$. By Lemma 23, $t' \in \mathbb{Q}_{\geq 0}$. Let φ_p be the 0-TAMTL$^*_P$ formula defined as follows:

$$\varphi_p = \langle\langle 0 \rangle\rangle \Big\{ \Big(\bigvee_{c \in Act_1^{\perp}} \bigvee_{\kappa \in \{0,1\}} \langle (a, = t), (c, \geq 0), \kappa \rangle \Big) \wedge \neg \Big(\bigvee_{\kappa \in \{0,1\}} \langle (a, = t), (b, = t'), \kappa \rangle \Big) \Big\}$$

Evidently, $(\mathcal{A}, s_{\mathcal{A}}) \models \varphi_p$ and $(\mathcal{B}, s_{\mathcal{B}}) \not\models \varphi_p$. Thus, in this case the result holds.

Now, assume that for each $1 \leq i \leq n$, y'_i is not a leaf. For each $1 \leq i \leq n$, let $z_{i,1}, \ldots, z_{i,m_i}$ (with $m_i \geq 1$) be the children of y'_i, which are labeled by positions in P^4_{ant}, and for each $1 \leq l \leq m_i$, let $z'_{i,l}$ be the unique child of $z_{i,l}$, which is labeled by a position in $S_{\mathcal{A}} \times S_{\mathcal{B}}$. By construction, for each $1 \leq i \leq n$, the labels of the edges from y'_i to the children $z_{i,1}, \ldots, z_{i,m_i}$ represent *all and only* the possible matching moves $(t', b, \delta'_{\mathcal{A}}) \in Mov_{\mathcal{A}}(1, s_{\mathcal{A}})$ (for player 1 in $\mathcal{A}$ from $s_{\mathcal{A}}$) of the move $(t', b, \delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1, s_{\mathcal{B}})$ (depending on i) labeling the edge from y_i to y'_i. Moreover, for all $1 \leq i \leq n$ and $1 \leq l \leq m_i$, the edge from $z_{i,l}$ to its unique child $z'_{i,l}$ is labeled by either 0 or 1. We distinguish two cases:

- there is $1 \leq i \leq n$ such that for each $1 \leq l \leq m_i$, the edge from $z_{i,l}$ to its unique child $z'_{i,l}$ is labeled by 1. Let $m^1_{\mathcal{B}} = (t', b, \delta'_{\mathcal{B}}) \in Mov_{\mathcal{B}}(1, s_{\mathcal{B}})$ be the move labeling the edge from y_i to y'_i, and let $w_{\mathcal{B}} = Next_{\mathcal{B}}(s_{\mathcal{B}}, m^1_{\mathcal{B}})$. By construction, $t' \leq t$ and the nodes $z'_{i,1}, \ldots, z'_{i,m_i}$ are labeled by positions $(w^1_{\mathcal{A}}, w_{\mathcal{B}}), \ldots, (w^{m_i}_{\mathcal{A}}, w_{\mathcal{B}})$, respectively, where $w^1_{\mathcal{A}}, \ldots, w^{m_i}_{\mathcal{A}}$ are the states of $\mathcal{A}$ obtained from $s_{\mathcal{A}}$ applying all and only the matching moves $(t', b, \delta'_{\mathcal{A}}) \in Mov_{\mathcal{A}}(1, s_{\mathcal{A}})$ of $m^1_{\mathcal{B}}$. Moreover, by Lemma 23, $t' \in \mathbb{Q}_{\geq 0}$, and by ind. hyp. for each $1 \leq l \leq m_i$, there is a 0-TAMTL$^*_P$ formula ϕ_l such that $(\mathcal{A}, w^l_{\mathcal{A}}) \models \phi_l$ and $(\mathcal{B}, w_{\mathcal{B}}) \not\models \phi_l$. Let φ_p be the 0-TAMTL$^*_P$ formula defined as:

$$\varphi_p = \langle\langle 0 \rangle\rangle \Big\{ \Big(\bigvee_{c \in Act_1^{\perp}} \bigvee_{\kappa \in \{0,1\}} \langle (a, = t), (c, \geq 0), \kappa \rangle \Big) \wedge \\ \Big(\langle (a, = t), (b, = t'), 1 \rangle \rightarrow \bigcirc (\phi_1 \vee \ldots \vee \phi_{m_i}) \Big) \Big\}$$

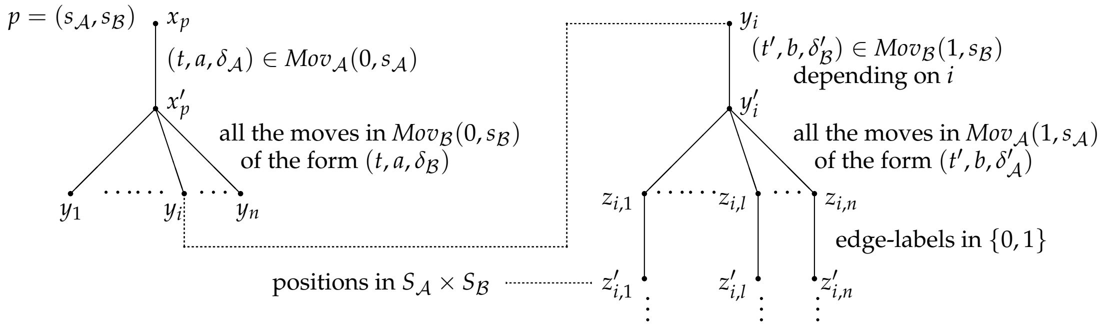

Figure 1: Structure of the subtree of $T(f_{ant}, p_0)$ rooted at an $(S_\mathcal{A} \times S_\mathcal{B})$-node x_p.

Evidently, $(\mathcal{A}, s_\mathcal{A}) \models \varphi_p$. Moreover, $(\mathcal{B}, s_\mathcal{B}) \not\models \varphi_p$, since for every strategy of player 0 in $\mathcal{B}$ which selects from the initial state $s_\mathcal{B}$ a move of the form $(t, a, \delta) \in Mov_\mathcal{B}(0, s_\mathcal{B})$, there is an outcome from $s_\mathcal{B}$ of the form $\pi = s_\mathcal{B}, \langle (t, a, \delta), (t', b, \delta_\mathcal{B}), 1 \rangle, w_\mathcal{B}, \ldots$, where by hypothesis $w_\mathcal{B} \not\models (\phi_1 \vee \ldots \vee \phi_{m_i})$. Thus, in this case the result holds.

- for each $1 \leq i \leq n$ there is $1 \leq l_i \leq m_i$ such that the edge from z_{i,l_i} to its unique child z'_{i,l_1} is labeled by 0. Then, by construction the nodes $z'_{1,l_1}, \ldots, z'_{n,l_n}$ are labeled by positions $(u_\mathcal{A}, u^1_\mathcal{B}), \ldots, (u_\mathcal{A}, u^n_\mathcal{B})$, where $u_\mathcal{A} = Next_\mathcal{A}(s_\mathcal{A}, m^0_\mathcal{A})$, $m^0_\mathcal{A} = (t, a, \delta_\mathcal{A}) \in Mov_\mathcal{A}(0, s_\mathcal{A})$ is the move which labels the edge from x_p to x'_p, and $u^1_\mathcal{B}, \ldots, u^n_\mathcal{B}$ are the states of $\mathcal{B}$ obtained from $s_\mathcal{B}$ applying all and only the matching moves $(t, a, \delta_\mathcal{B}) \in Mov_\mathcal{B}(0, s_\mathcal{B})$ of $m^0_\mathcal{A}$. Moreover, both $Mov_\mathcal{B}(1, s_\mathcal{B})$ and $Mov_\mathcal{A}(1, s_\mathcal{A})$ contain moves for player 1 whose timestamp is equal or greater than t. By ind. hyp. for each $1 \leq l \leq n$, there is a 0-TAMTL$^*_p$ formula ϕ_l such that $(\mathcal{A}, u_\mathcal{A}) \models \phi_l$ and $(\mathcal{B}, u^l_\mathcal{B}) \not\models \phi_l$. Let φ_p be the 0-TAMTL$^*_p$ formula defined as follows:

$$\varphi_p := \langle\langle 0 \rangle\rangle \Big\{ \Big(\bigvee_{c \in Act_1^\perp} \bigvee_{\kappa \in \{0,1\}} \langle (a, = t), (c, \geq 0), \kappa \rangle \Big) \wedge \Big(\Big(\bigvee_{c \in Act_1^\perp} \langle (a, = t), (c, \geq 0), 0 \rangle \Big) \rightarrow \bigcirc (\phi_1 \wedge \ldots \wedge \phi_n) \Big) \Big\}$$

Evidently, $(\mathcal{A}, s_\mathcal{A}) \models \varphi_p$ and $(\mathcal{B}, s_\mathcal{B}) \not\models \varphi_p$. Thus, also in this case the result holds.

This concludes the proof of the theorem. ∎

F Proofs of Lemmata 13 and 14

Fix $\sigma \in \{0, 1\}$. The length of a finite path $p_0, p_1, \ldots, p_k$ of $\mathcal{A}^{abs}_\sigma$ is $\lfloor \frac{k}{2} \rfloor$. Recall that for a strategy f_{abs} of player σ in $\mathcal{A}^{abs}_\sigma$ and position $p \in P_\sigma$, $Outcomes_{\mathcal{A}^{abs}_\sigma}(\sigma, p, f)$ denotes the set of infinite paths $p_0, p_1, \ldots$ of $\mathcal{A}^{abs}_\sigma$ such that $p_0 = p$ and for each i with $p_i \in P^\sigma_\sigma$, $f(p_1, \ldots, p_i) = p_{i+1}$. Moreover, for each $k \geq 0$, we denote by $Outcomes^{\mathcal{A}^{abs}_\sigma}_k(\sigma, p, f_{abs})$ the set of finite paths π of $\mathcal{A}^{abs}_\sigma$ of length k and leading to a position in P^σ_σ such that π is the prefix of some (infinite) path in $Outcomes_{\mathcal{A}^{abs}_\sigma}(\sigma, p, f_{abs})$. We also consider for each strategy f of player σ in $\mathcal{A}$, each state s of $\mathcal{A}$, and $k \geq 0$, the set $Outcomes^\mathcal{A}_k(\sigma, s, f)$, which has been defined at the beginning of Section B. For a $\mathcal{A}$-state $s = (q, v)$, we denote by $Reg(s)$ the region (in $Reg_\mathcal{A}$) of s, and by

$s+t$ the pair $(q, v+t)$ (note that $v+t$ may not satisfy the invariant of location q, hence $s+t$ may not be a $\mathcal{A}$-state).
Let $\pi = s_0, \langle m_1^0, m_1^1, \sigma_1\rangle, s_1, \ldots, \langle m_k^0, m_k^1, \sigma_k\rangle, s_k, \ldots$ be a (finite or infinite) run of $\mathcal{A}$ such that for each j and $i = 0, 1$, $m_j^i = (t_j^i, a_j^i, \delta_j^i)$. We associate to π, the (finite or infinite) sequence

$$Reg(\sigma,\pi) = \langle Reg(s_0), \sigma_0\rangle, p_0, \langle Reg(s_1), \sigma_1\rangle, p_1, \ldots, p_{k-1}, \langle Reg(s_k), \sigma_k\rangle, \ldots$$

where $\sigma_0 = 0$, and for each j, $p_j = \langle Reg(s_j), (Reg(s_j + t_j^\sigma), a_j^\sigma, \delta_j^\sigma), \sigma_j\rangle$. By definition of $\mathcal{A}_\sigma^{abs}$ it easily follows that $Reg(\sigma,\pi)$ is a path of $\mathcal{A}_\sigma^{abs}$. Note that if π is finite, then $Reg(\sigma,\pi)$ leads to a position in P_σ^σ.

F.1 Proof of Lemma 13

Lemma 13 directly follows from the following lemma.

LEMMA 25. *Let $\sigma \in \{0,1\}$, f be a strategy of player σ in $\mathcal{A}$, $R_0 \in Reg_{\mathcal{A}}$, and $s_0 \in R_0$. Then, there is a strategy f_{abs} of player σ in $\mathcal{A}_\sigma^{abs}$ such that for each $\pi_{abs} \in Outcomes_{\mathcal{A}_\sigma^{abs}}(\sigma, (R_0, 0), f_{abs})$, there is $\pi \in Outcomes_{\mathcal{A}}(\sigma, s_0, f)$ so that $Reg(\sigma,\pi) = \pi_{abs}$.*

PROOF. Fix $\sigma \in \{0,1\}$, a strategy f of player σ in $\mathcal{A}$, $R_0 \in Reg_{\mathcal{A}}$, and $s_0 \in R_0$. We claim the following.

Claim: There is a strategy f_{abs} of player σ in $\mathcal{A}_\sigma^{abs}$ and a sequence $(F_k)_{k\in\mathbb{N}}$ of functions $F_k : Outcomes_k^{\mathcal{A}_\sigma^{abs}}(\sigma, (R_0, 0), f_{abs}) \rightarrow Outcomes_k^{\mathcal{A}}(\sigma, s_0, f)$ so that for each $k \geq 0$, we have:

1. for each $\pi_{abs} \in Outcomes_k^{\mathcal{A}_\sigma^{abs}}(\sigma, (R_0, 0), f_{abs})$, $Reg(\sigma, F_k(\pi_{abs})) = \pi_{abs}$. Furthermore, if $k > 0$ and $\pi_{abs} = \pi'_{abs}, p_{1-\sigma}, p_\sigma$, then it holds that $F_k(\pi_{abs}) = F_{k-1}(\pi'_{abs}), \langle m_0, m_1, l\rangle, s$ for some $l \in \{0,1\}$, $\mathcal{A}$-state s and moves m_0 and m_1 of $\mathcal{A}$.

First, we show that the lemma follows from the claim, and then we prove the claim. Let f_{abs} be a strategy of player σ in $\mathcal{A}_\sigma^{abs}$ satisfying the claim and $\pi_{abs} \in Outcomes_{\mathcal{A}_\sigma^{abs}}(\sigma, (R_0, 0), f_{abs})$. Moreover, let $(F_k)_{k\in\mathbb{N}}$ be as in the statement of the claim, and for each $k \geq 0$, $\pi_{abs,k}$ be the prefix of π_{abs} of length k leading to a position in P_σ^σ, and $\pi_k = F_k(\pi_{abs,k})$. By the claim above, $\pi_k \in Outcomes_k^{\mathcal{A}}(\sigma, s_0, f)$, $Reg(\sigma, \pi_k) = \pi_{abs,k}$, and $\pi_{k+1} = \pi_k, \langle m_0, m_1, l\rangle, s$ for some $l \in \{0,1\}$, $\mathcal{A}$-state s and moves m_0 and m_1 of $\mathcal{A}$. Hence, evidently, $(\pi_k)_{k\in\mathbb{N}}$ represents an (infinite) run $\pi \in Outcomes_{\mathcal{A}}(\sigma, s_0, f)$ such that $Reg(\sigma,\pi) = \pi_{abs}$, hence the lemma follows. Now, we prove the claim above.

Proof of the claim: The strategy f_{abs} is defined by induction on the length n of the finite paths of $\mathcal{A}_\sigma^{abs}$ starting from $(R_0, 0)$ and leading to a position in P_σ^σ. Let $n \geq 0$. Since $Outcomes_n^{\mathcal{A}_\sigma^{abs}}(\sigma, (R_0, 0), f'_{abs})$ for any strategy f'_{abs} is independent on the values assumed by f'_{abs} over the finite paths of $\mathcal{A}_\sigma^{abs}$ of length equal or greater then n, we can assume that the set $Outcomes_n^{\mathcal{A}_\sigma^{abs}}(\sigma, (R_0, 0), f_{abs})$ is already given (note that $Outcomes_0^{\mathcal{A}_\sigma^{abs}}(\sigma, (R_0, 0), f_{abs}) = \{(R_0, 0)\}$) and there is a function $F_n : Outcomes_n^{\mathcal{A}_\sigma^{abs}}(\sigma, (R_0, 0), f_{abs}) \rightarrow Outcomes_n^{\mathcal{A}}(\sigma, s_0, f)$ satisfying Condition 1 in the claim for $k = n$ (note that since the function F_0 is independent on the specific strategy and $s_0 \in R_0$, for $k = 0$, Condition 1 in the claim trivially holds). Let $\pi_{abs} \in Outcomes_n^{\mathcal{A}_\sigma^{abs}}(\sigma, (R_0, 0), f_{abs})$. Then, $f_{abs}(\pi_{abs})$ is defined as follows. Let us consider the run $F_n(\pi_{abs}) \in Outcomes_n^{\mathcal{A}}(\sigma, s_0, f)$, where $Reg(\sigma, F_n(\pi_{abs})) = \pi_{abs}$. Let $s = last(F_n(\pi_{abs}))$ and $(t, a, \delta_a) = f(F_n(\pi_{abs}))$. Then, $last(\pi_{abs}) = (Reg(s), l)$ for some

$l \in \{0,1\}$. Since $(t, a, \delta_a) \in Mov_{\mathcal{A}}(\sigma, s)$, it follows that $p_{1-\sigma} = \langle Reg(s), (Reg(s+t), a, \delta_a), l\rangle$ is a successor of $last(\pi_{abs})$ in $\mathcal{A}^{abs}_{\sigma}$. Thus, we define $f_{abs}(\pi_{abs}) = p_{1-\sigma}$.

At this point, we can assume that also $Outcomes^{\mathcal{A}^{abs}_{\sigma}}_{n+1}(\sigma, (R_0, 0), f_{abs})$ is already given. It remains to show that there is a function F_{n+1} satisfying Condition 1 in the Claim (for $k = n+1$). The function F_{n+1} is defined as follows. Let $\pi_{abs,n+1} \in Outcomes^{\mathcal{A}^{abs}_{\sigma}}_{n+1}(\sigma, (R_0, 0), f_{abs})$. Hence, $\pi_{abs,n+1} = \pi_{abs}, p_{1-\sigma}, p_{\sigma}$, where $\pi_{abs} \in Outcomes^{\mathcal{A}^{abs}_{\sigma}}_{n}(\sigma, (R_0, 0), f_{abs})$. By construction $p_{1-\sigma} = \langle Reg(s), (Reg(s+t_{\sigma}), a_{\sigma}, \delta_{\sigma}), l\rangle$ for some $l \in \{0,1\}$, where $(t_{\sigma}, a_{\sigma}, \delta_{\sigma}) \in f(F_n(\pi_{abs}))$ and $s = last(F_n(\pi_{abs}))$. Since $p_{\sigma} \in P^{\sigma}_{\sigma}$ is a successor of $p_{1-\sigma}$ in $\mathcal{A}^{abs}_{\sigma}$, there must be an abstract move $m_{1-\sigma} = (R_{1-\sigma}, a_{1-\sigma}, \delta_{1-\sigma}) \in Mov_{\mathcal{A}^{abs}_{\sigma}}(1-\sigma, Reg(s))$ such that one of the following holds:

- $Reg(s+t_{\sigma}) \leq R_{1-\sigma}$ and $p_{\sigma} = (Next^{abs}_{\mathcal{A}}(Reg(s), m_{\sigma}), \sigma)$, where $m_{\sigma} = (Reg(s+t_{\sigma}), a_{\sigma}, \delta_{\sigma})$. Since $Reg(s+t_{\sigma}) \leq R_{1-\sigma}$ and $(R_{1-\sigma}, a_{1-\sigma}, \delta_{1-\sigma}) \in Mov_{\mathcal{A}^{abs}_{\sigma}}(1-\sigma, Reg(s))$, there must be $t_{1-\sigma} \geq t_{\sigma}$ such that $(t_{1-\sigma}, a_{1-\sigma}, \delta_{1-\sigma}) \in Mov_{\mathcal{A}}(1-\sigma, s)$. It follows that the sequence

$$\pi_{n+1} = F_n(\pi_{abs}), \langle (t_0, a_0, \delta_0), (t_1, a_1, \delta_1), \sigma\rangle, Next_{\mathcal{A}}(s, (t_{\sigma}, a_{\sigma}, \delta_{\sigma}))$$

  is a run in $Outcomes^{\mathcal{A}}_{n+1}(\sigma, s_0, f)$ such that $Reg(\sigma, \pi_{n+1}) = \pi_{abs,n+1}$. In this case, we set $F_{n+1}(\pi_{abs,n+1}) = \pi_{n+1}$.
- $R_{1-\sigma} \leq Reg(s+t_{\sigma})$ and $p_{\sigma} = (Next^{abs}_{\mathcal{A}}(Reg(s), m_{1-\sigma}), 1-\sigma)$. Thus, there must be $t_{1-\sigma} \geq 0$ such that $s + t_{1-\sigma} \in R_{1-\sigma}$ and $(t_{1-\sigma}, a_{1-\sigma}, \delta_{1-\sigma}) \in Mov_{\mathcal{A}}(1-\sigma, s)$. Since $Reg(s) \leq R_{1-\sigma} \leq Reg(s+t_{\sigma})$, we can choose $t_{1-\sigma}$ in such a way $t_{1-\sigma} \leq t_{\sigma}$. It follows that the sequence

$$\pi_{n+1} = F_n(\pi_{abs}), \langle (t_0, a_0, \delta_0), (t_1, a_1, \delta_1), 1-\sigma\rangle, Next_{\mathcal{A}}(s, (t_{1-\sigma}, a_{1-\sigma}, \delta_{1-\sigma}))$$

  is a run in $Outcomes^{\mathcal{A}}_{n+1}(\sigma, s_0, f)$ such that $Reg(\sigma, \pi_{n+1}) = \pi_{abs,n+1}$. In this case, we set $F_{n+1}(\pi_{abs,n+1}) = \pi_{n+1}$.

Evidently, F_{n+1} satisfies Condition 1 in the claim. This concludes the proof of the claim. ∎

F.2 Proof of Lemma 14

Lemma 14 directly follows from the following lemma.

LEMMA 26. *Let $\sigma \in \{0,1\}$ and f_{abs} be a strategy of player σ in $\mathcal{A}^{abs}_{\sigma}$, and $R_0 \in Reg_{\mathcal{A}}$. Then, there is a strategy f of player σ in $\mathcal{A}$ s.t. for each $\pi = s_0, \langle m^0_1, m^1_1, \sigma_1\rangle, s_1, \langle m^0_2, m^1_2, \sigma_2\rangle, \ldots \in Outcomes_{\mathcal{A}}(\sigma, s_0, f)$ with $s_0 \in R_0$, it holds that $Reg(\sigma, \pi) \in Outcomes_{\mathcal{A}^{abs}_{\sigma}}(\sigma, (R_0, 0), f_{abs})$.*

PROOF. Fix $\sigma \in \{0,1\}$, a strategy f_{abs} of player σ in $\mathcal{A}^{abs}_{\sigma}$, and $R_0 \in Reg_{\mathcal{A}}$. The strategy f of player σ in $\mathcal{A}$ is defined as follows. Let $\pi = s_0, \langle m^0_1, m^1_1, \sigma_1\rangle, s_1, \ldots, s_k$ be a finite run of $\mathcal{A}$ with $s_0 \in R_0$, and let $p_k = f_{abs}(Reg(\sigma, \pi))$. Since $Reg(\sigma, \pi)$ leads to position $(Reg(s_k), \sigma_k)$, by definition of $\mathcal{A}^{abs}_{\sigma}$, $p_k = \langle Reg(s_k), (R, a, \delta), \sigma_k\rangle$, where (R, a, δ) is an abstract move in $Mov_{\mathcal{A}^{abs}_{\sigma}}(\sigma, Reg(s_k))$. Hence, there is $t \geq 0$ such that $s_k + t \in R$ and $(t, a, \delta) \in Mov_{\mathcal{A}}(\sigma, s_k)$. We set $f(\pi) = (t, a, \delta)$. By construction it easily follows that for each $\pi = s_0, \langle m^0_1, m^1_1, \sigma_1\rangle, s_1, \langle m^0_2, m^1_2, \sigma_2\rangle, \ldots \in Outcomes_{\mathcal{A}}(\sigma, s_0, f)$ with $s_0 \in R_0$, $Reg(\sigma, \pi) \in Outcomes_{\mathcal{A}^{abs}_{\sigma}}(\sigma, (R_0, 0), f_{abs})$. ∎

G The remaining cases in the proof of Theorem 15

Theorem 15. *The set of states s_{in} of $\mathcal{A}_{in}$ such that $(\mathcal{A}_{in}, s_{in}) \models \varphi$ is a union of regions in $Reg_{\mathcal{A}_{in}}$, and its (region) representation can be computed in exponential time. Hence, model checking TG against TAMTL is in* EXPTIME.

PROOF. The non-trivial cases to consider are those in which the arbitrary state formula ϕ of φ has the form $\phi = \mathcal{O}_\sigma(\phi_1 \,\mathcal{U}_I\, \phi_2)$, or $\phi = \mathcal{O}_\sigma(\phi_1 \,\widetilde{\mathcal{U}}_I\, \phi_2)$, or $\phi = \mathcal{O}_\sigma \psi_0$, where $\sigma \in \{0,1\}$, $\mathcal{O}_\sigma \in \{\langle\langle\sigma\rangle\rangle, \langle\langle\sigma\rangle\rangle_{re}\}$, and ψ_0 is a boolean combination of (timed) multi-action constraints. Here, we assume that $\mathcal{O}_\sigma = \langle\langle\sigma\rangle\rangle_{re}$ (the other cases being simpler). The case $\phi = \langle\langle\sigma\rangle\rangle_{re}(\phi_1 \,\mathcal{U}_I\, \phi_2)$, has been examined at the end of Section 4. For the other two ones, we have the following:

- $\phi = \langle\langle\sigma\rangle\rangle_{re}(\phi_1 \,\widetilde{\mathcal{U}}_I\, \phi_2)$: it suffices to show that the set of states s in $S_{\mathcal{A}}[x_\varphi := 0]$ such that $(\mathcal{A}, s) \models \langle\langle\sigma\rangle\rangle_{re}(\phi_1 \widetilde{\mathcal{U}}_I \phi_2)$ is a union of regions in $Reg_{\mathcal{A}}$ whose representation can be computed in exponential time. Since for each $s \in S_{\mathcal{A}}$, $(\mathcal{A}, s) \models \phi$ iff $(\mathcal{A}_{in}, Proj(s)) \models \phi$, by ind. hyp. for each $i = 1, 2$, this last condition holds for the set of states $s \in S_{\mathcal{A}}$ such that $(\mathcal{A}, s) \models \phi_i$. Note that the path formula $(\phi_1 \,\widetilde{\mathcal{U}}_I\, \phi_2)$ is equivalent to the path formula $(\Box_I\, \phi_2) \vee (\phi_2 \,\mathcal{U}\, (\phi_1 \wedge \phi_2))$. Let $L : P_\sigma \to \{p_{\phi_2}, p_{\phi_1}, (x_{div} \geq 1), (x_\varphi \in I), 0, 1\}$ be the labeling of $\mathcal{A}^{abs}_\sigma$ defined in the obvious way (for example, proposition $(x_\varphi \in I)$ labels a position $p \in P_\sigma$ iff the associated region satisfies the constraint $(x_\varphi \in I)$). Then, by Lemmata 13 and 14, for all regions $R_0 \in Reg_{\mathcal{A}}$ satisfying $x_\varphi = 0$ and $s_0 \in R_0$, it holds that $(\mathcal{A}, s_0) \models \langle\langle\sigma\rangle\rangle_{re}(\phi_1 \widetilde{\mathcal{U}}_I \phi_2)$ iff there is a winning strategy f_{abs} of player σ in $\mathcal{A}^{abs}_\sigma$ in position $(R_0, 0)$ w.r.t. the labeling L and the LTL objective:

$$[\Box\Diamond(x_{div} \geq 1) \wedge \xi] \vee [\Diamond\Box(\neg(x_{div} \geq 1) \wedge (1 - \sigma))]$$

$$\xi := \{\Box((x_\varphi \in I) \to p_{\phi_2})\} \vee \{p_{\phi_2} \,\mathcal{U}\, (p_{\phi_2} \wedge p_{\phi_1})\}$$

  Since LTL finite-state games for a fixed formula can be solved in polynomial and since the size of $\mathcal{A}^{abs}_\sigma$ is exponential in the size of $\mathcal{A}_{in}$, the result follows.
- $\phi = \langle\langle\sigma\rangle\rangle_{re}\, \psi_0$, where ψ_0 is a boolean combination of multi-action constraints. We need additional definitions. An *abstract multi-action* of $\mathcal{A}$ is a triple $\langle(R_0, a_0), (R_1, a_1), l\rangle$, where $R_0, R_1 \in Reg_{\mathcal{A}}$, $l \in \{0,1\}$, $R_l \leq R_0, R_1$, and $a_i \in Act_i^{\perp}$ for each $i = 0, 1$. Given a multi-action constraint $\chi = \langle(a'_0, \sim_0 c_0), (a'_1, \sim_1 c_1), l'\rangle$ with $c_0, c_1 \leq K_{max}$ and in $\mathbb{N}$ (recall that K_{max} is the largest constant occurring in $\mathcal{A}$ and φ), we say that the abstract multi-action $\langle(R_0, a_0), (R_1, a_1), l\rangle$ satisfies χ iff $l' = l$ and for each $i = 0, 1$, $a'_i = a_i$ and R_i satisfies the clock constraint $x_\varphi \sim_i c_i$. The above satisfaction relation can be extended to boolean combinations of multi-action constraints in the obvious way. Moreover, for a position $p_{1-\sigma} = \langle R, (R_1, a, \delta_1), l\rangle \in P^{1-\sigma}_\sigma$ of $\mathcal{A}^{abs}_\sigma$ (recall that $(R_1, a, \delta_1) \in Mov^{abs}_{\mathcal{A}}(\sigma, R)$) and abstract move $(R_2, b, \delta_2) \in Mov^{abs}_{\mathcal{A}}(1 - \sigma, R)$, we say that a successor $p_\sigma = (R', l')$ of $p_{1-\sigma}$ is a (R_2, b, δ_2)-successor of $p_{1-\sigma}$ iff *either* $l' = \sigma$, $R' = Next^{abs}_{\mathcal{A}}(R, (R_1, a, \delta_1))$, and $R_1 \leq R_2$, *or* $l' = 1 - \sigma$, $R' = Next^{abs}_{\mathcal{A}}(R, (R_2, b, \delta_2))$ and $R_2 \leq R_1$. Note that all the successors of $p_{1-\sigma}$ can be obtained in this way. Now, let us consider the formula $\phi = \langle\langle\sigma\rangle\rangle_{re}\, \psi_0$. By the previous case, we can assume that the result holds for the formula $\phi_0 = \langle\langle\sigma\rangle\rangle_{re}\, true$. Hence, the set of states of $S_{\mathcal{A}}$ satisfying ϕ_0 is a union of regions in $Reg_{\mathcal{A}}$, which can be computed in exponential time. Let $L : P_\sigma \to \{p_{\phi_0}\}$ be the labeling of $\mathcal{A}^{abs}_\sigma$ defined in the obvious way.

  Let $R_0 \in Reg_{\mathcal{A}}$ satisfying $x_\varphi = 0$ and $s_0 \in R_0$. Note that $(A, s_0) \models \langle\langle\sigma\rangle\rangle_{re}\, \psi_0$ iff

- there is a move $m_\sigma = (t_\sigma, a_\sigma, \delta_\sigma) \in Mov_{\mathcal{A}}(\sigma, s)$ such that for each move $m_{1-\sigma} = (t_{1-\sigma}, a_{1-\sigma}, \delta_{1-\sigma}) \in Mov_{\mathcal{A}}(1-\sigma, s)$, and for each $l = 0, 1$ with $m_l \in JDM(m_0, m_1)$: the multi-action $\langle (t_0, a_0), (t_1, a_1), l \rangle$ satisfies ψ_0 and $Next_{\mathcal{A}}(s, m_l) \models \langle\langle \sigma \rangle\rangle_{re}\, true$.

Since R_0 satisfies $x_\varphi = 0$ and $s_0 \in R_0$, it easily follows that $(A, s_0) \models \langle\langle \sigma \rangle\rangle_{re}\, \psi_0$ iff

- there is a successor $p_{1-\sigma} = \langle R_0, (R'_\sigma, a_\sigma, \delta_\sigma), 0 \rangle \in P_\sigma^{1-\sigma}$ of position $(R_0, 0)$ in $\mathcal{A}_\sigma^{abs}$ such that for all abstract moves $(R'_{1-\sigma}, a_{1-\sigma}, \delta_{1-\sigma}) \in Mov_{\mathcal{A}}^{abs}(1-\sigma, R_0)$ and $(R'_{1-\sigma}, a_{1-\sigma}, \delta_{1-\sigma})$-successors $p_\sigma = (R', l')$ of $p_{1-\sigma}$, we have: (R', l') is labeled by p_{ϕ_0} and the abstract multi-action $\langle (R'_0, a_0), (R'_1, a_1), l' \rangle$ satisfies ψ_0.

Since the above condition can be checked in exponential time, we are done.

∎