PROPERTIES OF THE PINHOLE CAMERA MODEL PROJECTION

NOTES FOR IMAGE REPRESENTATION-EDITING-PERCEPTION

1. Notation

A point $[X_m, Y_m, Z_m]^T$ in the world coordinate is projected into the pixel $[u_x, u_y]^T$ with the following rule:

$$\begin{bmatrix} u_x \\ u_y \\ 1 \end{bmatrix} \equiv \begin{bmatrix} k_x f & s & k_x x_0 \\ 0 & k_y f & k_y y_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & -\mathbf{Rt} \end{bmatrix} \begin{bmatrix} X_m \\ Y_m \\ Z_m \\ 1 \end{bmatrix} \tag{1}$$

2. A distant object looks smaller

Let us consider two objects of the same height h, at different depth Z_1 and Z_2 (with $Z_1 < Z_2$) with respect to the camera plane. Let us consider that the bottom of these objects is at coordinate $X = 0$ and these objects are aligned with the coordinate $Y = 0$.

Let us consider the simple case (without loss of generality) where $k_x = k_y = f = 1$ and that $s = x_0 = y_0 = 0$. Moreover the extrinsic matrix is the identity (i.e., the world coordinate system is the camera one).

Which object looks bigger in the image?

In order to study which object is bigger in the image, we simply have to study the projections of the object tops, *i.e.,* the projections of the two points $[h, 0, Z_1]^T$ and $[h, 0, Z_2]^T$. We have

$$\begin{bmatrix} u_x^1 \\ u_y^1 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} h \\ 0 \\ Z_1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u_x^2 \\ u_y^2 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} h \\ 0 \\ Z_2 \\ 1 \end{bmatrix} \tag{2}$$

then

$$\begin{cases} u_x^1 = h/Z_1 \\ u_y^1 = 0 \end{cases} \quad \text{and} \quad \begin{cases} u_x^2 = h/Z_2 \\ u_y^2 = 0 \end{cases} \tag{3}$$

since $Z_1 < Z_2$, then $u_x^1 > u_x^2$. This means that a distant object looks smaller.

3. The projection of a line is a line

Let us define a line by three slopes (ax, ay, az) and three origin coordinates (X_0, Y_0, Z_0). The equation of the line in the 3D domain is given by

$$\begin{cases} X = a_x t + X_0 \\ Y = a_y t + Y_0 \\ Z = a_z t + Z_0 \end{cases}$$

We project an arbitrary point $\mathbf{P} = [X, Y, Z]^\top$ on the image plane and we have (with the same intrinsic parameters than in the previous section)

$$\begin{cases} u_x = \frac{a_x t + X_0}{a_z t + Z_0} \\ u_y = \frac{a_y t + Y_0}{a_z t + Z_0} \end{cases}$$

Let us consider the case when $a_z = 0$, we have

$$\begin{cases} u_x = \frac{a_x t + X_0}{Z_0} \\ u_y = \frac{a_y t + Y_0}{Z_0} \end{cases}$$

We observe that 1) the projected point lies one a line and 2) the slopes of this line do not depend on X_0 and Y_0 which means that the projection of two parallel lines are still parallel if $a_z = 0$ (i.e., if the 3D line is parallel to the camera plane).

Let us now consider that $a_z \neq 0$, we take $t' = \frac{1}{a_z t + Z_0}$ and we write:

$$\begin{cases} u_x = t'(\frac{a_x}{a_z}(\frac{1}{t'} - Z_0) + X_0) \\ u_y = t'(\frac{a_y}{a_z}(\frac{1}{t'} - Z_0) + Y_0) \end{cases}$$

which gives

$$\begin{cases} u_x = (X_0 - \frac{a_x}{a_z} Z_0)t' + \frac{a_x}{a_z} \\ u_y = (Y_0 - \frac{a_y}{a_z} Z_0)t' + \frac{a_y}{a_z} \end{cases}$$

The projection is still on a line, but the slope now depends on X_0 and Y_0 which means that the projection of two parallel lines are no longer parallel.