Quantitative Analysis: Examples, Applications, Generalities

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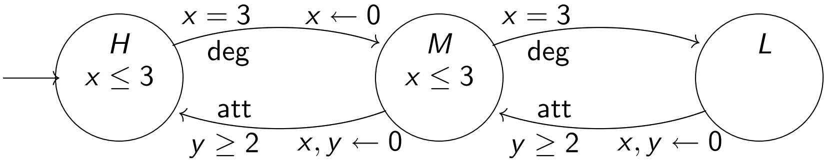
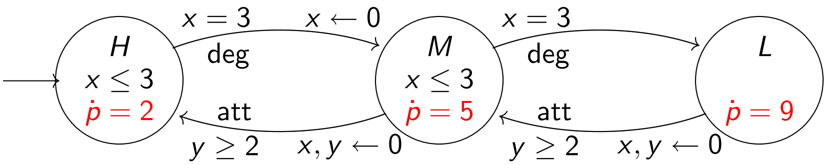
Priced timed automata



- Timed automaton:

  Finite automaton
  + finite set of clocks $C = \{x, y, \dots\}$
  + location invariants
  + edge constraints
  + edge resets

Priced timed automata



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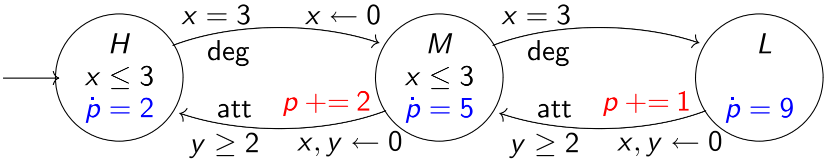
- Priced timed automaton:
  - \+ price rates in locations (cost per time unit)

Priced timed automata



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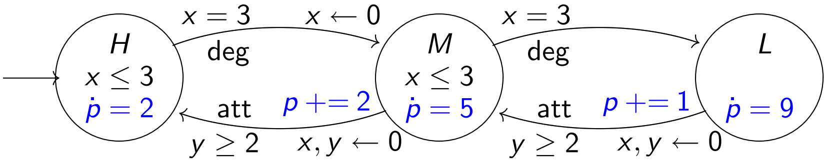
  + location invariants
  + edge constraints
  + edge resets

- Priced timed automaton:

  + price rates in locations (cost per time unit)
  + price updates on transitions

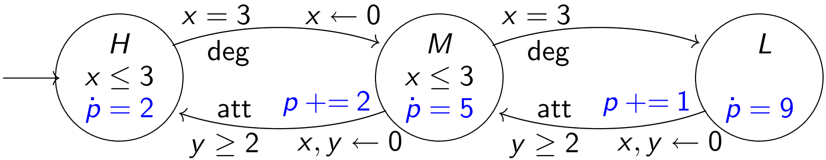
Cheapest infinite runs



Problem: Find cheapest infinite run

- (Motivation: Production plant with High, Medium and Low productivity mode)

Cheapest infinite runs



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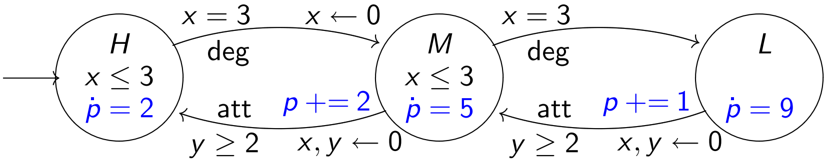
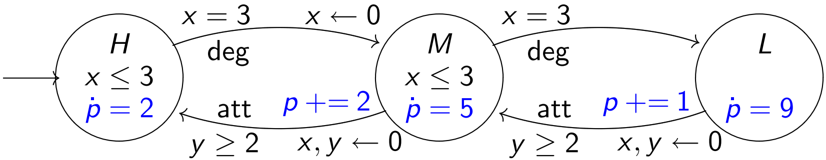
Cheapest infinite runs



Problem: Find cheapest infinite run

- (Motivation: Production plant with High, Medium and Low productivity mode)
- What is the price of an infinite run?
  - Well, ∞ !
- Use mean-payoff
- Or, discounting

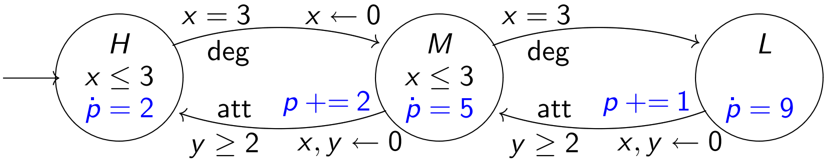
Cheapest infinite runs



An example (finite) run:

loc.	H	d	H	s	M	d	M	s	L	d	L	s	M	d	M	s	H
x	0		3		0		3		3		4		0		2		0
y	0		3		3		6		6		7		0		2		0
t	0		3		3		6		6		7		7		9		9

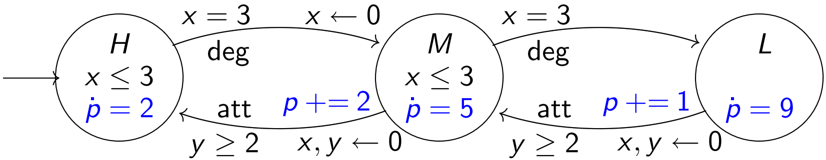
Cheapest infinite runs



An example (finite) run:

loc.	H	d	H	s	M	d	M	s	L	d	L	s	M	d	[illegible]
x	0		3		0		3		3		4		0		[illegible]
y	0		3		3		6		6		7		0		[illegible]
t	0		3		3		6		6		7		7		[illegible]
		$2\int_0^3 \lambda^t dt$		0		$5\int_3^6 \lambda^t dt$		0		$9\int_6^7 \lambda^t dt$		$1\lambda^7$		$5\int_7^9 \lambda^t dt$	

Cheapest infinite runs



An example (finite) run:

loc.	H	d	H	s	M	d	M	s	L	d	L	s	M	d	M	s	H
x	0		3		0		3		3		4		0		2		0
y	0		3		3		6		6		7		0		2		0
t	0		3		3		6		6		7		7		9		9
$\lambda = .9$		5.14		0		9.38		0		4.54		.48		4.31		.77	

- Total discounted price of run: 24.62
- Total discounted price of infinite loop: 40.2

Discounted price

Formally: Let A be a priced timed transition system and $\lambda \in [0,1]$.

- Discounted price of *finite* alternating path $\pi = s_0 \xrightarrow{t_0} s_0' \rightarrow s_1 \rightarrow \cdots \xrightarrow{t_{n-1}} s_{n-1}' \rightarrow s_n$:

$$P(\pi) = \sum_{i=0}^{n-1} \Big(\int_{T_{i-1}}^{T_i} \lambda^t r(s_i^t) dt + \lambda^{T_i} p(s_i' \rightarrow s_{i+1}) \Big)$$

  with $T_i = \sum_{j=0}^{i} t_j$.
- Discounted price of *infinite* alternating path $\pi = s_0 \xrightarrow{t_0} s_0' \rightarrow s_1 \rightarrow \cdots$: limit

$$P(\pi) = \lim_{n \to \infty} P\big(s_0 \rightarrow s_0' \rightarrow \cdots \rightarrow s_{n-1}' \rightarrow s_n\big)$$

  provided that it exists. (!)

Problem and solution

Problem:

- Given: priced timed transition system A, state $s \in A$
- Find: an infinite path from s with lowest discounted price
- (or one that comes arbitrarily close)

Solution:

- For priced timed automata
- with non-negative price rates and updates,
- which are *bounded*
- and *time-divergent*,
- and *rational* λ,
- starting with an *integer valuation*,
- our problem is computable

Proof structure

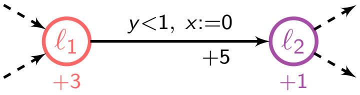
Given priced timed automaton A:

1. Use corner point abstraction to construct *finite weighted graph* $\text{cp}(A)$
2. Find infinite path $\tilde{\pi}$ with lowest discounted price in $\text{cp}(A)$ using linear programming
3. Find cheapest path lying over $\tilde{\pi}$ in A

This needs a soundness and completeness result:

- Given an infinite path $\tilde{\pi}$ in $\text{cp}(A)$ for which $P(\tilde{\pi})$ converges, then for all $\varepsilon > 0$ there exists an infinite path $\pi \in \text{cp}^{-1}(\tilde{\pi})$ for which $\big|P(\pi) - P(\tilde{\pi})\big| < \varepsilon$.
- Given an infinite path π in A, there exists an infinite path $\tilde{\pi} \in \text{cp}(\pi)$ for which $P(\tilde{\pi}) \leq P(\pi)$.

Corner point abstraction



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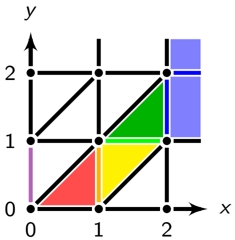

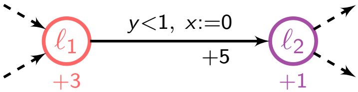

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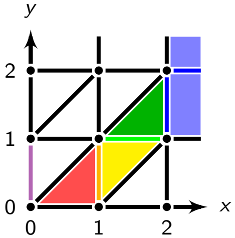

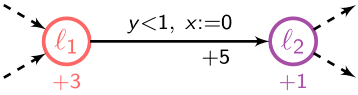

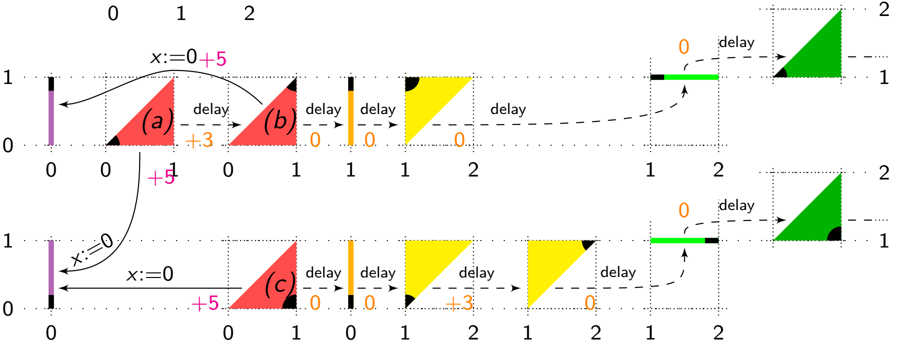

Proof details (skip)

Given an infinite path π in A, there exists an infinite path $\tilde{\pi} \in \text{cp}(\pi)$ for which $P(\tilde{\pi}) \le P(\pi)$.

Write $\pi = (q_0, \nu_0) \to (q_0, \nu_0 + t_0) \xrightarrow[p_0]{} (q_1, \nu_1) \to (q_1, \nu_1 + t_1) \xrightarrow[p_1]{} \cdots$

Then
$$P(\pi) = \sum_{i=0}^{\infty} \Big(\int_{T_{i-1}}^{T_i} \lambda^t r(q_i) dt + \lambda^{T_i} p_i \Big)$$
$$= \sum_{i=0}^{\infty} \Big(\frac{1}{\ln \lambda} r(q_i) \big(\lambda^{T_i} - \lambda^{T_{i-1}} \big) + p_i \lambda^{T_i} \Big)$$

– a function in variables $T_0, T_1, \dots$.

$\Rightarrow$ Optimization problem: Minimize $P(\pi) = f(T_0, T_1, \dots)$ under the constraint that $T_0, T_1, \dots$ lie in a specific *zone* defined by π above.

Proof details, 2.

(skip)

$$P(\pi) = f(T_0, T_1, \dots) = \sum_{i=0}^{\infty} \Big(\frac{1}{\ln \lambda} r(q_i)\big(\lambda^{T_i} - \lambda^{T_{i-1}}\big) + p_i \lambda^{T_i} \Big)$$

Task: Minimize $f(T_0, T_1, \dots)$ under the constraint that $(T_0, T_1, \dots) \in Z$ for a given (bounded, closed) zone Z.

- (can be shown that) f is (weakly) monotonic
- easy to see: monotonic functions over *finite-dimensional* closed zones attain their minimum in a *corner point*

$\Rightarrow$ for *finite paths*, the corner point abstraction can "see" the path with lowest price (because it goes through corners) $\Rightarrow$ done

- Need: Generalization of the above to infinite-dimensional zones
- not easy, because infinite-dimensional zones are not compact
- difficult part: show that infimum is attained somewhere

Proof details, 3. (skip)

Theorem: $Z \subseteq \mathbb{R}^\infty$ bounded and closed (in the supremum metric)

- $f_1, f_2, \dots$ continuous functions $\mathrm{pr}_i\, Z \to \mathbb{R}_{\geq 0}$ (*non-negative* values!)
- $f(x_1, x_2, \dots) = \sum_{i=1}^\infty f_i(x_i) : Z \to [0, \infty]$ converges for some $x \in Z$

$\Rightarrow$ exists $z \in Z$ for which $f(z) = \inf_{y \in Z} f(y)$

Proof: Let $x : \mathbb{N} \to Z$ be a sequence for which $\lim f(x_i) = \inf_{y \in Z} f(y)$

Standard argument: Z is compact $\Rightarrow$ x contains converging subsequence x' $\Rightarrow$ let $z = \lim x'$ $\Rightarrow$ done.

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But Z is not compact.

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$x = \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9$

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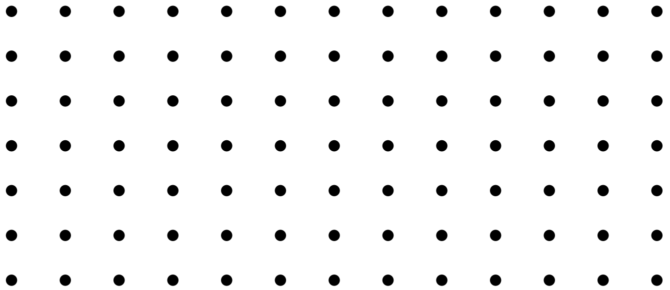

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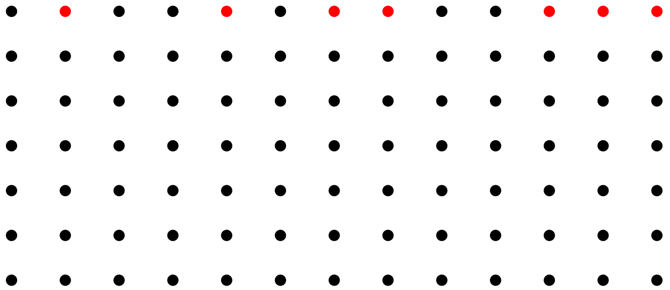

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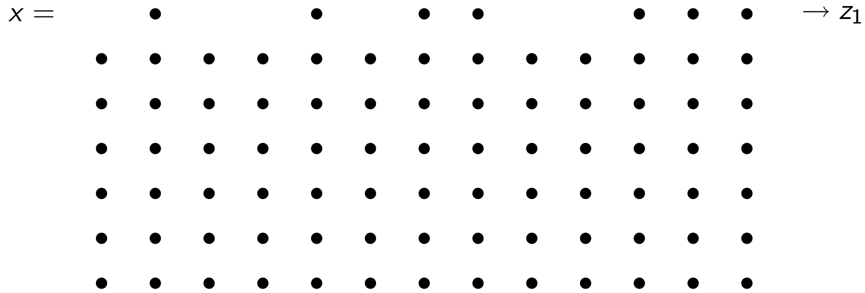

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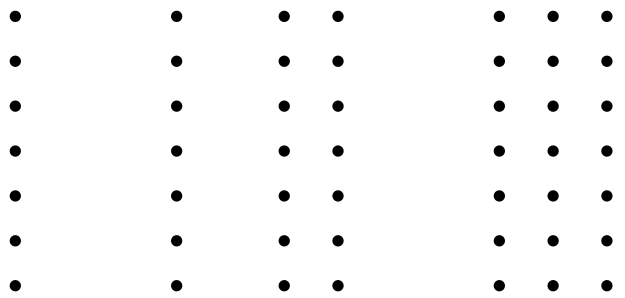

$\to z_1$

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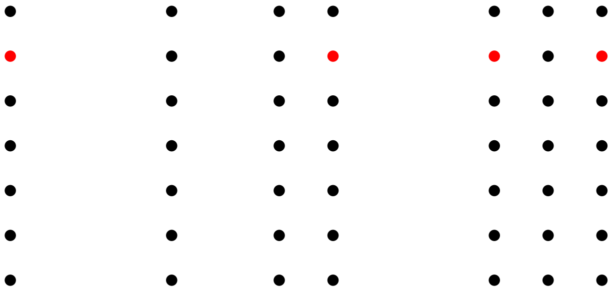

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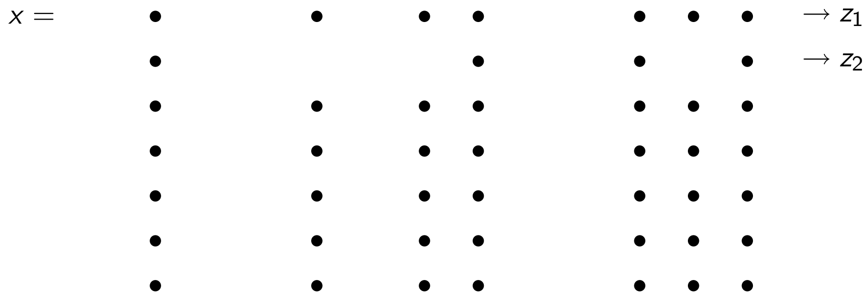

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Corollary: $Z \subseteq \mathbb{R}^\infty$ bounded and closed zone

- $f_1, f_2, \dots$ monotonous continuous functions $\text{pr}_i\, Z \to \mathbb{R}_{\geq 0}$
- $f(x_1, x_2, \dots) = \sum_{i=1}^\infty f_i(x_i) : Z \to [0, \infty]$ converges for some $x \in Z$

$\Rightarrow$ exists corner point z of Z for which $f(z) = \inf_{y \in Z} f(y)$

So corner point abstraction *can* see the path with lowest price $\Rightarrow$ done

Summary

- The discount-optimal infinite path problem is computable for priced timed automata
- (with arbitrarily many clocks)
- (under certain restrictions; only "real" restriction: time divergence)
- Uses generalization of a well-known fact about monotonous functions defined on zones, to infinite zones
- (Other applications?)

- Computable yes, feasible NO
- (The corner point abstraction is HUGE, and the Linear Programming problem to solve is also HUGE)
- For zone-based iterative computations: need some fixed-point property / additivity

Additivity

- For $\rightarrow$ a switch and π a path out of the target of the switch:

$$P(\rightarrow \circ \pi) = p(\rightarrow) + P(\pi)$$

For $\xrightarrow{t}$ a delay and π a path out of the end state of the delay:

$$P(\xrightarrow{t} \circ \pi) = p(\xrightarrow{t}) + \lambda^t P(\pi)$$

- Similar property not satisfied for other formalisms (*e.g.* mean-payoff, or discounting by discrete steps)
- Question: What do we know about the cost function if we require

$$P(\xrightarrow{t} \circ \pi) = p(\xrightarrow{t}) + g(t) P(\pi)$$

- (Natural property also because of additivity of delays !)
- Answer: Everything!

Additivity

Theorem: If P is an accumulated price function in a PTA which satisfies

$$P(s \xrightarrow{t_1} s' \xrightarrow{t_2} s'') = P(s \xrightarrow{t_1} s') + g(t_1)\, P(s' \xrightarrow{t_2} s'')$$

for all states and delays, then

$$g(t) = \lambda^t \quad \text{and} \quad P(s \xrightarrow{t} s') = \alpha(s) \int_0^t \lambda^t dt$$

for some $\lambda \in \mathbb{R}_{\geq 0}$ and $\alpha : S \to \mathbb{R}$.

Reason: The functional equation

$$f(t_1 + t_2) = f(t_1) + g(t_1)\, f(t_2)$$

has essentially only one solution.

Proof: Just do like Cauchy in 1821 ! (or skip)

Proof

(skip)

Proof: We need to solve the functional equation

$$f(x+y) = f(x) + g(x)f(y)$$

and we find inspiration in Cauchy's 1821 textbook *Cours d'analyse*:

Proof (skip)

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- By induction:

$$f(kx) = f((k-1)x) + g((k-1)x)f(x) = f(x)\Big(\sum_{i=0}^{k-1} g(ix)\Big)$$

$$f(kx) = f(x) + g(x)f((k-1)x) = f(x)\Big(\sum_{i=0}^{k-1} (g(x))^i\Big)$$

Proof

(skip)

Proof: We need to solve the functional equation

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- Put $\beta = f(1)$, then $f(n) = \beta \cdot \sum_{i=1}^{n-1} \lambda^i = \beta \frac{1-\lambda^n}{1-\lambda}$ *etc.*

Summary

With time-based discounting, you get

- computability of cheapest infinite runs in non-negatively priced timed automata,

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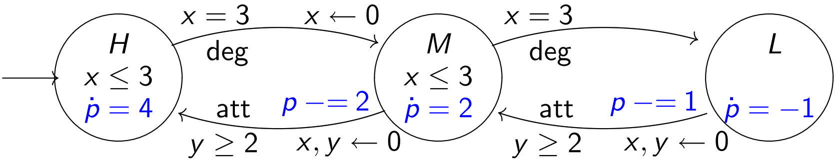
- feasible zone-based algorithms ?

And additivity you can only have for time-based discounting.

- Is additivity necessary and / or sufficient for a zone-based approximation algorithm ?

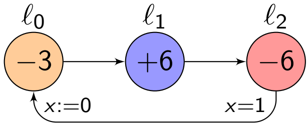
Infinite runs with energy constraints

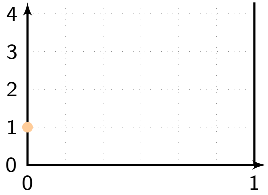


New problem: Given some initial value of p, does there exist an infinite run in which p always is ≥ 0 ?

- (Same motivation as before)
- No discounting this time
- (Can we introduce discounting here ?)

Another example





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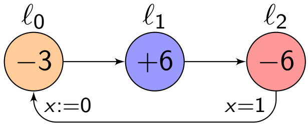
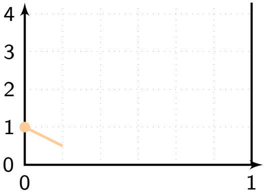
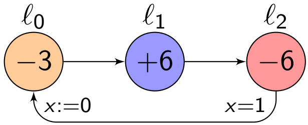
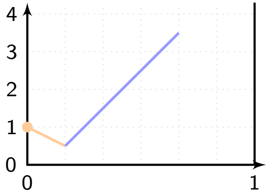




lower-bound problem

Another example





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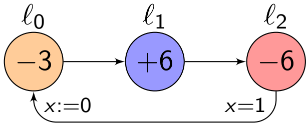

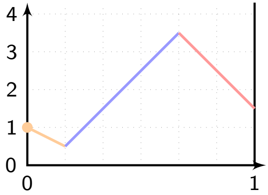

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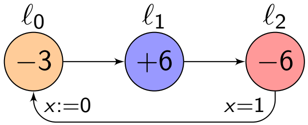

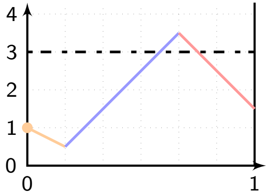

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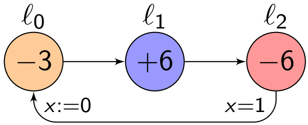

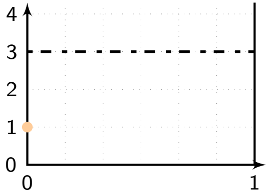

lower-upper-bound problem

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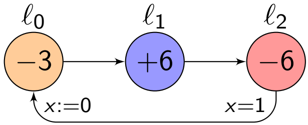

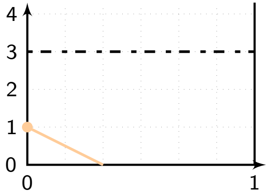

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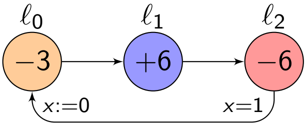

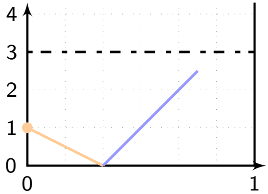

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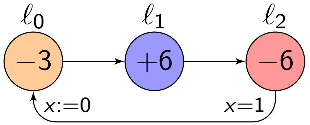

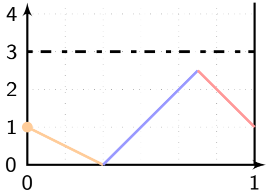

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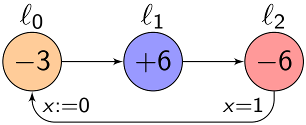

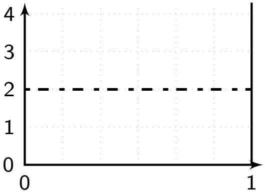

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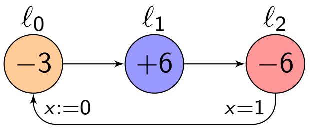

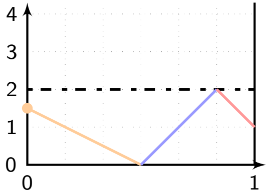

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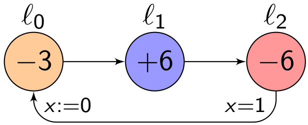

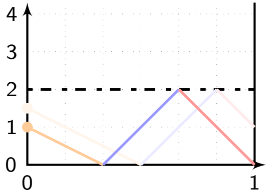

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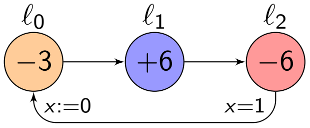

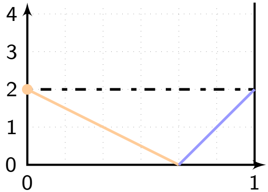

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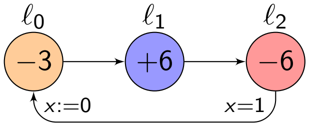

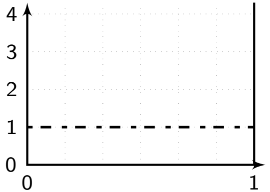

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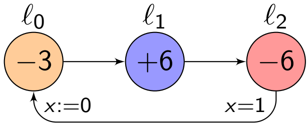

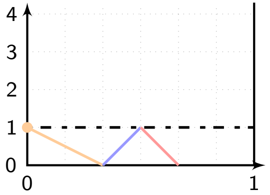

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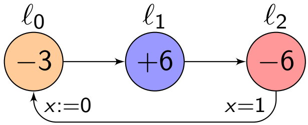

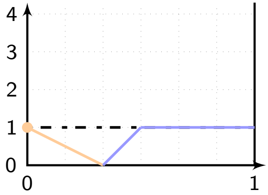

lower-weak-upper-bound problem

Problems and solutions

Problems:

- Given: priced transition system A, state $s \in A$, initial credit p
- Find: an infinite path from s with initial credit p in which the accumulated credit
  - (L) never goes below 0, or
  - (L+W) never goes below 0, under restricted capacity, or
  - (L+U) always stays within interval bounds $[0, u]$
- (or one that comes arbitrarily close)
- Also: Does the above hold for all infinite paths from s ?
- And can we solve games ?

Solutions:

- For finite priced transition systems: almost completely solved
- For 1-clock priced timed automata: some results
- For priced timed automata with 2 clocks: open
- For ≥ 3 clocks: probably all undecidable

Results for the untimed case

	exist. problem	univ. problem	games
L	$\in$ P	$\in$ P	$\in$ UP $\cap$ coUP P-hard
L+W	$\in$ P	$\in$ P	$\in$ NP $\cap$ coNP P-hard
L+U	$\in$ PSPACE NP-hard	$\in$ P	EXPTIME-c.

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- Bellman-Ford algorithm

Results for the untimed case

	exist. problem	**univ. problem**	**games**
L	$\in$ P	$\in$ P	$\in$ UP $\cap$ coUP P-hard
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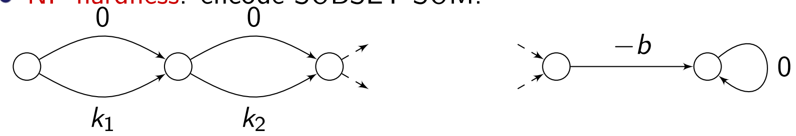
L+U	$\in$ PSPACE NP-hard	$\in$ P	EXPTIME-c.

- PSPACE: guess an infinite path in the configuration graph
- NP-hardness: encode SUBSET-SUM:



Results for the untimed case

	exist. problem	univ. problem	games
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L+W	$\in$ P	$\in$ P	$\in$ NP $\cap$ coNP P-hard
L+U	$\in$ PSPACE NP-hard	$\in$ P	EXPTIME-c.

- EXPTIME: play the game in the configuration graph
- EXPTIME-hardness: encode COUNTDOWN-GAME

Results for the untimed case

	exist. problem	univ. problem	games
L	$\in$ P	$\in$ P	$\in$ UP $\cap$ coUP P-hard
L+W	$\in$ P	$\in$ P	$\in$ NP $\cap$ coNP P-hard
L+U	$\in$ PSPACE NP-hard	$\in$ P	EXPTIME-c.

- Mean-payoff games
- (See also recent work by Raskin-Doyen-Henzinger)

Results for the 1-clock case (without discrete updates !)

	exist. problem	**univ. problem**	**games**
L	$\in$ P	$\in$ P	?
L+W	$\in$ P	$\in$ P	?
L+U	?	?	undecidable

Results for the 1-clock case (without discrete updates !)

	exist. problem	univ. problem	games
L	$\in$ P	$\in$ P	?
L+W	$\in$ P	$\in$ P	?
L+U	?	?	undecidable

- corner-point abstraction

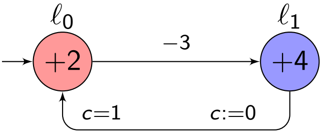
Results for the 1-clock case (without discrete updates !)

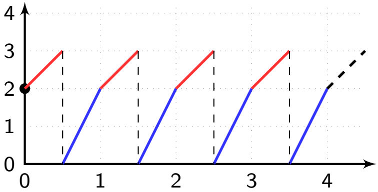
	exist. problem	**univ. problem**	**games**
L	$\in$ P	$\in$ P	?
L+W	$\in$ P	$\in$ P	?
L+U	?	?	undecidable

- two-counter machines

Extensions

- For 1-clock PTA with discrete updates, the corner point abstraction is not complete:





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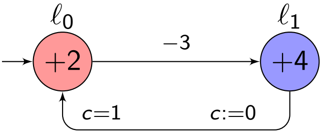
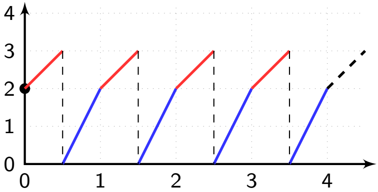
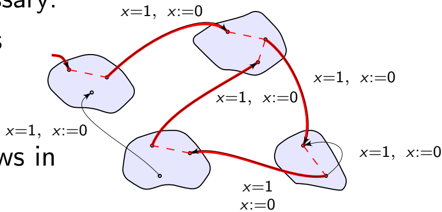




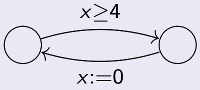
- Completely different approach necessary:
  - Optimize along reset-free paths
  - Compute "energy automaton" abstraction



- Extends also to exponential price laws in locations: $\dot{p} = k\,p$ instead of $\dot{p} = k$
- "Hybridization" ?

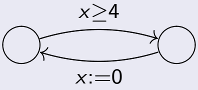
Quantitative Analysis

Quantitative *Models*



Quantitative Quantitative Analysis

Quantitative *Models*

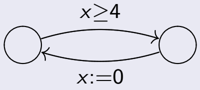


Quantitative *Logics*

$$\Pr_{\leq .1}(\Diamond \mathit{error})$$

Quantitative Quantitative Quantitative Analysis

Quantitative *Models*



Quantitative *Logics*

$\Pr_{\leq .1}(\Diamond \mathit{error})$

Quantitative *Verification*

$[\![\varphi]\!](s) = 3.14$

Boolean world	**"Quantification"**
Trace equivalence $\equiv$	Linear distance d_L
Bisimilarity $\sim$	Branching distance d_B
$s \sim t$ implies $s \equiv t$	$d_L(s,t) \leq d_B(s,t)$
$s \models \varphi$ or $s \not\models \varphi$	$[\![\varphi]\!](s)$ is a quantity
$s \sim t$ iff $\forall \varphi : s \models \varphi \Leftrightarrow t \models \varphi$	$d_B(s,t) = \sup_\varphi d\big([\![\varphi]\!](s), [\![\varphi]\!](t)\big)$

Weighted automata and traces

Definition

A weighted automaton: states S, transitions $T \subseteq S \times \mathbb{R} \times S$

(Yes, we can deal with more general weights than $\mathbb{R}$. Also: labels.)

Definition

A trace is an infinite sequence of weights.

Definition: Trace distances (values in $\mathbb{R} \cup \{\infty\}$)

point-wise	**accumulating**
$d_L^\bullet(\sigma, \tau) = \sup_i \lambda^i \lvert\sigma_i - \tau_i\rvert$	$d_L^+(\sigma, \tau) = \sum_i \lambda^i \lvert\sigma_i - \tau_i\rvert$

$\lambda \in [0, 1]$ is a fixed discounting factor.
(Yes, there are other interesting trace distances.)

Linear distance

Linear distance between states: use Hausdorff distance:

Definition

$$d_L^{\cdot}(s,t) = \sup \begin{cases} \sup\limits_{\sigma\in\mathrm{Tr}(s)} \inf\limits_{\tau\in\mathrm{Tr}(t)} d_L^{\cdot}(\sigma,\tau) \\ \sup\limits_{\tau\in\mathrm{Tr}(t)} \inf\limits_{\sigma\in\mathrm{Tr}(s)} d_L^{\cdot}(\sigma,\tau) \end{cases}$$

Lemma

$$d_L^{\bullet}(s,t) \le \sup \begin{cases} \sup\limits_{s\xrightarrow{x}s'} \inf\limits_{t\xrightarrow{y}t'} \max\left(|x-y|, \lambda d_L^{\bullet}(s',t')\right) \\ \sup\limits_{t\xrightarrow{y}t'} \inf\limits_{s\xrightarrow{x}s'} \max\left(|x-y|, \lambda d_L^{\bullet}(s',t')\right) \end{cases}$$

and similarly for d_L^{+}

Branching distances

Definition: Branching distances are minimal fixed points

$$d_B^{\bullet}(s,t) = \sup \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} \max\left(|x-y|, \lambda d_B^{\bullet}(s',t')\right) \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} \max\left(|x-y|, \lambda d_B^{\bullet}(s',t')\right) \end{cases}$$

$$d_B^{+}(s,t) = \sup \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} |x-y| + \lambda d_B^{+}(s',t') \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} |x-y| + \lambda d_B^{+}(s',t') \end{cases}$$

Theorem

$d_L^{\cdot}(s,t) \leq d_B^{\cdot}(s,t)$

Metric properties

- $d_L^\bullet$ and $d_B^\bullet$ are topologically inequivalent
- Likewise, d_L^+ and d_B^+ are topologically inequivalent
- For $\lambda = 1$,
  - $d_L^\bullet$ and d_L^+ are topologically inequivalent
  - and so are $d_B^\bullet$ and d_B^+
- For $\lambda < 1$,
  - $d_L^\bullet$ and d_L^+ are Lipschitz equivalent
  - and so are $d_B^\bullet$ and d_B^+

Logical characterization

For both point-wise and accumulating branching distance, there is an adequate logical characterization using weighted CTL (with two different semantics).

Where to go from here?

- Other interesting distances: *e.g.* maximum-lead distance (Henzinger-Majumdar-Prabhu)

$$d_L^{\pm}(\sigma, \tau) = \sup_i \lambda^i \Big| \sum_{j=0}^{i} \sigma_j - \sum_{j=0}^{i} \tau_j \Big|$$

  Corresponding branching distance ✓
- General picture: Linear distances are easy to define, branching distances are easy to compute
- General framework for linear distances on $\mathbb{K}$-weighted automata (for a semiring $\mathbb{K}$) and general recipy for how to go from linear to branching distances

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