From Linear to Branching Distances and Back (via Games)

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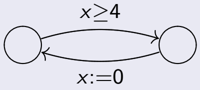
MT-Lab:8

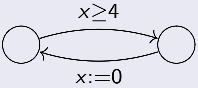
Quantitative Analysis

Quantitative *Models*



Quantitative Quantitative Analysis

Quantitative *Models*

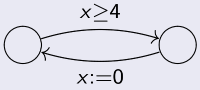


Quantitative *Logics*

$$\mathrm{Pr}_{\le .1}(\Diamond \mathit{error})$$

Quantitative Quantitative Quantitative Analysis

Quantitative *Models*



Quantitative *Logics*

$$\Pr_{\le .1}(\Diamond \mathit{error})$$

Quantitative *Verification*

$$[\![\varphi]\!](s) = 3.14$$
$$d(s,t) = 42$$

Boolean world	**"Quantification"**
Trace equivalence $\equiv$	Linear distance d_L
Bisimilarity $\sim$	Branching distance d_B
$s \sim t$ implies $s \equiv t$	$d_L(s,t) \le d_B(s,t)$

Weighted Automata and Traces

Definition

A weighted automaton: states S, transitions $T \subseteq S \times \mathbb{K} \times S$

- $\mathbb{K}$: Set of weights. Maybe some extra structure. (Lattice? Semiring?)
- Standard example: $\mathbb{K} = L \times \mathbb{R}$. Discrete labels L, real weights $\mathbb{R}$.

Definition

A trace is an infinite sequence of weights; an element of $\mathbb{K}^\omega$.

- Notation: For $s \in S$ in a weighted automaton (S, T), $\text{Tr}(s)$ is the set of traces from s.

Trace distance

Assume given a hemimetric $d_T : \mathbb{K}^\omega \times \mathbb{K}^\omega \to [0, \infty]$.

Examples of Trace Distances

- Let $\mathbb{K} = L \times \mathbb{R}$ and $\lambda \in [0, 1]$ some discounting factor.
- Notation: Trace $\sigma = \big((\sigma_0^\ell, \sigma_0^w), (\sigma_1^\ell, \sigma_1^w), \dots\big)$.

Definition: Point-wise trace distance

$$d_T^\bullet(\sigma, \tau) = \begin{cases} \sup_i \lambda^i |\sigma_i^w - \tau_i^w| & \text{if } \sigma_i^\ell = \tau_i^\ell \text{ for all } i \\ \infty & \text{otherwise} \end{cases}$$

Definition: Accumulating trace distance

$$d_T^+(\sigma, \tau) = \begin{cases} \sum_i \lambda^i |\sigma_i^w - \tau_i^w| & \text{if } \sigma_i^\ell = \tau_i^\ell \text{ for all } i \\ \infty & \text{otherwise} \end{cases}$$

Definition: Maximum-lead trace distance

$$d_T^\pm(\sigma, \tau) = \begin{cases} \sup_i \left|\sum_{j=0}^i \lambda^j \sigma_j^w - \sum_{j=0}^i \lambda^j \tau_j^w\right| & \text{if } \sigma_i^\ell = \tau_i^\ell \text{ for all } i \\ \infty & \text{otherwise} \end{cases}$$

Linear Distance

- (Recall: We assume given a hemimetric $d_T : \mathbb{K}^\omega \times \mathbb{K}^\omega \to [0, \infty]$ on traces.)
- Let $(S, T \subseteq S \times \mathbb{K} \times S)$ be a weighted automaton.
- Linear distance between states $s, t \in S$: use Hausdorff construction:

Definition: Linear distance

$$d_L(s, t) = \sup_{\sigma \in \mathrm{Tr}(s)} \inf_{\tau \in \mathrm{Tr}(t)} d_T(\sigma, \tau)$$

Linear vs. Branching Distance

Definition: Linear distance

$$d_L(s,t) = \sup_{\sigma \in \mathrm{Tr}(s)} \inf_{\tau \in \mathrm{Tr}(t)} d_T(\sigma, \tau)$$

- This is a game!
- Player 1 chooses the worst trace $\sigma \in \mathrm{Tr}(s)$.
- Player 2 matches it with the best trace $\tau \in \mathrm{Tr}(t)$.
- $d_L(s,t) =$ value of the "half-blind weighted bisimulation game": Player 2 has perfect information, Player 1 is blind.

Definition: Branching distance

$d_B(s,t) =$ value of the same game, but with perfect information

- Hence "$d_B(s,t) = \sup_{s \xrightarrow{\sigma_0} s_1} \inf_{t \xrightarrow{\tau_0} t_1} \sup_{s_1 \xrightarrow{\sigma_1} s_2} \inf_{t_1 \xrightarrow{\tau_1} t_2} \cdots \; d_T(\sigma, \tau)$".

Properties

Theorem

For all $s, t \in S$, $d_L(s,t) \leq d_B(s,t)$.

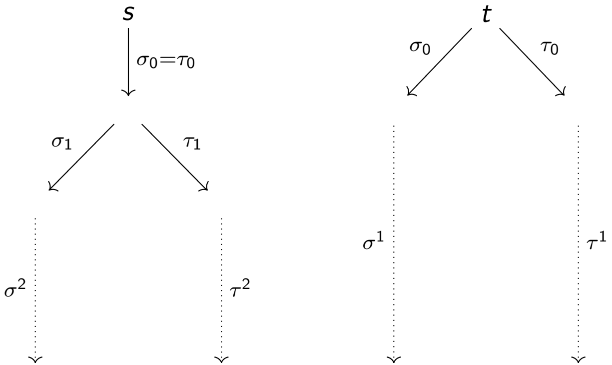
Theorem

There exists a weighted automaton on which d_L *and* d_B *are topologically inequivalent.*

- Unless $\sigma_0 = \tau_0$ implies $d_T(\sigma,\tau) = 0$ for all traces σ, τ.

Proof

Let $\sigma, \tau \in \mathbb{K}^\omega$ such that $\sigma_0 = \tau_0$, $d_T(\sigma, \tau) > 0$, and $d_T(\tau, \sigma) > 0$.



We have $\mathrm{Tr}(s) = \mathrm{Tr}(t)$, hence $d_L(s,t) = 0$. On the other hand, $d_B(s,t) = \min\big(d_T(\sigma, \tau), d_T(\tau, \sigma)\big) > 0$. That's it.

Special Cases

- Back to trace distance examples:

$$d_T^\bullet(\sigma,\tau) = \sup_i \lambda^i |\sigma_i^w - \tau_i^w| = \max\big(|\sigma_0^w - \tau_0^w|, \lambda d_T^\bullet(\sigma^1,\tau^1)\big)$$

Similarly:

$$d_T^+(\sigma,\tau) = |\sigma_0^w - \tau_0^w| + \lambda d_T^+(\sigma^1,\tau^1)$$

Proposition

If $d_T(\sigma,\tau) = f\big(\sigma_0,\tau_0,d_T(\sigma^1,\tau^1)\big)$ for some function $f : \mathbb{K}\times\mathbb{K}\times[0,\infty]\to[0,\infty]$ and all $\sigma,\tau\in\mathbb{K}^\infty$, then $d_B(s,t) = \sup_{s\xrightarrow{x}s'} \inf_{t\xrightarrow{y}t'} f\big(x,y,d_B(s',t')\big)$ for all $s,t\in S$.

- (Needs f to be increasing in the third coordinate.)
- Applies to $d_T^\bullet$ and d_T^+, but not to $d_T^\pm$.
- Works also with $\leq$ instead of $=$.

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