Linear and Branching Distances for Weighted Automata

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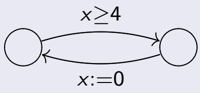
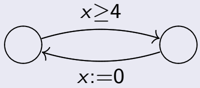
Quantitative Analysis

Quantitative *Models*



Quantitative Quantitative Analysis

Quantitative *Models*

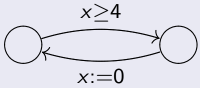


Quantitative *Logics*

$$\Pr_{\leq .1}(\Diamond \mathit{error})$$

Quantitative Quantitative Quantitative Analysis

Quantitative *Models*



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Quantitative *Verification*

$$[\![\varphi]\!](s) = 3.14$$

Quantitative Quantitative Quantitative Analysis

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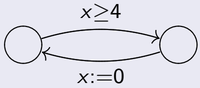

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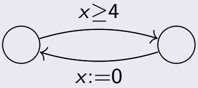
Quantitative *Verification*

$[\![\varphi]\!](s) = 3.14$

Boolean world	**“Quantification”**
Trace equivalence $\equiv$	Linear distance d_L
Bisimilarity $\sim$	Branching distance d_B
$s \sim t$ implies $s \equiv t$	$d_L(s,t) \leq d_B(s,t)$
$s \models \varphi$ or $s \not\models \varphi$	$[\![\varphi]\!](s)$ is a quantity
$s \sim t$ iff $\forall \varphi : s \models \varphi \Leftrightarrow t \models \varphi$	$d_B(s,t) = \sup_\varphi d\big([\![\varphi]\!](s), [\![\varphi]\!](t)\big)$

Quantitative Quantitative Quantitative Analysis

Quantitative *Models*



Quantitative *Logics*

$\mathrm{Pr}_{\leq .1}(\Diamond \mathit{error})$

Quantitative *Verification*

$[\![\varphi]\!](s) = 3.14$

- CT, UF, KGL: Quantitative analysis of weighted transition systems. JLAP, to appear.
- UF, KGL, CT: A quantitative characterization of weighted Kripke structures in temporal logic. CAI, to appear.
- UF, CT, KGL: Linear and branching distances for weighted automata. To be written.

Weighted automata and traces

Definition

A weighted automaton: states S, transitions $T \subseteq S \times \mathbb{R} \times S$

(Yes, we can deal with more general weights than $\mathbb{R}$. Also: labels.)

Definition

A trace is an infinite sequence of weights.

Definition: Trace distances (values in $\mathbb{R} \cup \{\infty\}$)

point-wise	**accumulating**
$d_L^\bullet(\sigma, \tau) = \sup_i \lambda^i \lvert\sigma_i - \tau_i\rvert$	$d_L^+(\sigma, \tau) = \sum_i \lambda^i \lvert\sigma_i - \tau_i\rvert$

$\lambda \in [0, 1]$ is a fixed discounting factor.

(Yes, there are other interesting trace distances.)

Linear distance

Linear distance between states: use Hausdorff distance:

Definition

$$d_L^\cdot(s,t) = \sup \begin{cases} \sup\limits_{\sigma\in \mathrm{Tr}(s)} \inf\limits_{\tau\in \mathrm{Tr}(t)} d_L^\cdot(\sigma,\tau) \\ \sup\limits_{\tau\in \mathrm{Tr}(t)} \inf\limits_{\sigma\in \mathrm{Tr}(s)} d_L^\cdot(\sigma,\tau) \end{cases}$$

Lemma

$$d_L^\bullet(s,t) \le \sup \begin{cases} \sup\limits_{s\xrightarrow{x}s'} \inf\limits_{t\xrightarrow{y}t'} \max\big(|x-y|, \lambda d_L^\bullet(s',t')\big) \\ \sup\limits_{t\xrightarrow{y}t'} \inf\limits_{s\xrightarrow{x}s'} \max\big(|x-y|, \lambda d_L^\bullet(s',t')\big) \end{cases}$$

and similarly for d_L^+

Branching distances

Definition: Branching distances are minimal fixed points

$$d_B^\bullet(s,t) = \sup \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} \max\left(|x-y|, \lambda d_B^\bullet(s',t')\right) \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} \max\left(|x-y|, \lambda d_B^\bullet(s',t')\right) \end{cases}$$

$$d_B^+(s,t) = \sup \begin{cases} \sup\limits_{s \xrightarrow{x} s'} \inf\limits_{t \xrightarrow{y} t'} |x-y| + \lambda d_B^+(s',t') \\ \sup\limits_{t \xrightarrow{y} t'} \inf\limits_{s \xrightarrow{x} s'} |x-y| + \lambda d_B^+(s',t') \end{cases}$$

Theorem

$d_L^{\cdot}(s,t) \leq d_B^{\cdot}(s,t)$

Metric properties

- $d_L^\bullet$ and $d_B^\bullet$ are topologically inequivalent
- Likewise, d_L^+ and d_B^+ are topologically inequivalent
- For $\lambda = 1$,
  - $d_L^\bullet$ and d_L^+ are topologically inequivalent
  - and so are $d_B^\bullet$ and d_B^+
- For $\lambda < 1$,
  - $d_L^\bullet$ and d_L^+ are Lipschitz equivalent
  - and so are $d_B^\bullet$ and d_B^+

Logical characterization

For both point-wise and accumulating branching distance, there is an adequate logical characterization using weighted CTL (with two different semantics).

Where to go from here?

- Other interesting distances: *e.g.* maximum-lead distance

$$d_L^{\pm}(\sigma, \tau) = \sup_i \lambda^i \Big| \sum_{j=0}^{i} \sigma_j - \sum_{j=0}^{i} \tau_j \Big|$$

  Want corresponding branching distance ✓
- General picture: Linear distances are easy to define, branching distances are easy to compute
- Want general framework for linear distances on $\mathbb{K}$-weighted automata (for a semiring $\mathbb{K}$) and general recipy for how to go from linear to branching distances.
- (We're almost there.)