# Kleene Algebras and Semimodules for Energy Problems

Zoltán Ésik Uli Fahrenberg Axel Legay Karin Quaas

Univ. Szeged, Hungary / IRISA/Inria Rennes, France / Univ. Leipzig, Germany

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# Motivation

Lower bound problems for energy automata, examples:

- Given finite automaton with integer weights on transitions: does there exist an infinite run in which the accumulated weight never drops below 0?
  - decidable in P
  - Bouyer-F.-Larsen-Markey-Srba: FORMATS'08
- Given timed automaton with integer weights on edges and integer rates in locations: decide the same problem
  - decidable for 1 clock; high complexity
  - by reduction to finite automata with special weight update functions on transitions
  - Bouyer-F.-Larsen-Markey: HSCC'10
- Proof principle: if there's an infinite run, then there's a "lasso"

Goal: Generalize. What's the natural setting?

What is the minimum amount of battery required for the satellite to always be able to send and receive messages?

# Energy Automata

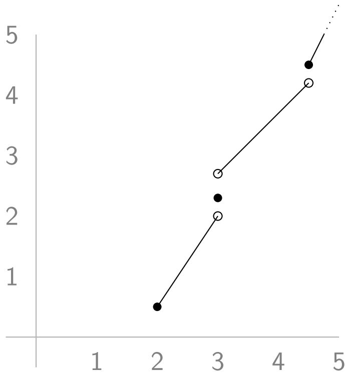
Energy function:

- right-continuous autofunction $f$ on $\{\bot\} \cup \mathbb{R}_{\geq 0} \cup \{\infty\}$
- $\bot$ means "undefined"
- $f(\bot) = \bot$, $f(\infty) = \infty$
- total order: $\bot < x < \infty$
- for $x_1 \leq x_2$: $f(x_2) - f(x_1) \geq x_2 - x_1$
  - "derivative $f' \geq 1$"
- so $f(x) = \bot$ implies $f(x') = \bot$ for all $x' \leq x$: $f$ is defined on a left-closed interval

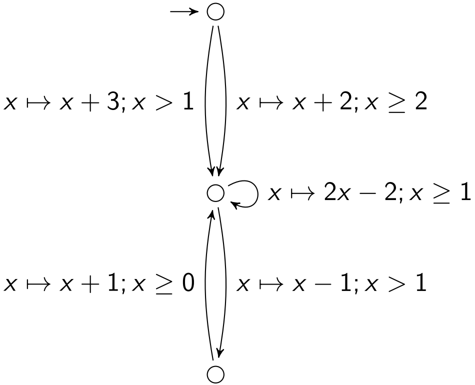
Energy automaton:

- finite automaton with transitions labeled with energy functions
- transitions "transform energy" input $\mapsto$ output
- $f(x) = \bot$ for an $f$-labeled transition: transition is not enabled for input $x$

# Energy Automata, Examples




a simple energy function



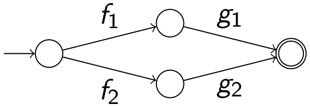

a simple energy automaton

# Energy Function Semiring

Interest: reachability and Büchi acceptance

- Given a set $F$ of accept states and $x_0 \in \mathbb{R}_{\geq 0}$: does there exist a run with initial energy $x_0$ which reaches $F$? does there exist one which visits $F$ infinitely often?

Operations on energy functions: max and $\circ$




becomes $\max(g_1 \circ f_1, g_2 \circ f_2)$

The set $\mathcal{E}$ of energy functions with operations max and $\circ$ is a semiring, with $\mathbf{0} = \lambda x.\bot$, $\mathbf{1} = \lambda x.x$

- without "$f' \geq 1$" condition, only "near-semiring"
- idempotent, positively ordered, complete

# Loops for Reachability

Star: $f^* = \sup_{n\geq 0} f^n$

- for loops which can be taken an arbitrary number of times
- $f^*(x) = \begin{cases} x & \text{if } f(x) \leq x \\ \infty & \text{if } f(x) > x \end{cases}$

**Theorem:** Always, $gf^*h = \sup_{n\geq 0} gf^n h$

- i.e. $\mathcal{E}$ is a star-continuous Kleene algebra

**Corollary:** Let $M$ be the (transposed) transition matrix of an energy automaton

- i.e. $M_{ji}$ is the transition label from state $s_i$ to state $s_j$.

Compute $M^* = \sup_{n\geq 0} M^n$
Then $s_j$ is reachable from $s_i$ with initial energy $x_0$ iff $M^*_{ji}(x_0) \neq \bot$.

# Loops for Infinite Runs

Omega: " $f^\omega = \lim_{n\to\infty} f^n$ "

- for loops which are taken infinitely often
- $f^\omega(x) = \begin{cases} \bot & \text{if } f(x) < x \quad \text{or } x = \bot \\ \top & \text{if } f(x) \geq x \quad \text{and } x \neq \bot \end{cases}$
- important: two-valued; $\mathcal{V}$: energy functions into $\{\bot, \top\}$

**Theorem:** $(\mathcal{E}, \mathcal{V})$ is a Conway semiring-semimodule pair

**Corollary:** Let $M$ be the (transposed) transition matrix of an energy automaton

- i.e. $M_{ji}$ is the transition label from state $s_i$ to state $s_j$.

Compute " $M^\omega = \lim_{n\to\infty} M^n$ "
Then there is an infinite run from $s_i$ with initial energy $x_0$ iff $M_i^\omega(x_0) \neq \bot$.

# Some Technical Details for Reachability

(Applying work by S. Bloom, Z. Ésik, W. Kuich and others)

For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in \mathcal{E}^{k \times k}$ and $d \in \mathcal{E}^{m \times m}$ (and $k + m = n$), let

$$M^* = \begin{bmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^* bd^* \\ (d \vee ca^*b)^* ca^* & (d \vee ca^*b)^* \end{bmatrix} \in \mathcal{E}^{n \times n}$$

**Lemma:** $M^*$ does not depend on $k$ and $m$, and always $NM^*P = \sup_n NMP$.

- can also use (generalized) Floyd-Warshall algorithm to compute $M^*$; generally faster

**Theorem:** For any $\mathcal{E}$-automaton $(S, M)$ with $S = \{1, \dots, n\}$, $F = \{1, \dots, k\}$, $k \le n$, $s_0 \le n$, and $x_0 \in \mathbb{R}_{\ge 0}$, $\text{Reach}(s_0, x_0, F) = \mathbf{tt}$ iff ${}_{\mathsf{t}}F^{\le k} M^* I^{s_0}(x_0) \neq \bot$.

# Some Technical Details for Büchi Acceptance

(Extending work by S. Bloom, Z. Ésik, W. Kuich and others)

For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in \mathcal{E}^{k\times k}$ and $d \in \mathcal{E}^{m\times m}$ (and $k + m = n$), let

$$M^\omega = {}_{\mathrm{t}}\begin{bmatrix} (a \vee bd^*c)^\omega \vee d^\omega c(a \vee bd^*c)^* \\ (d \vee ca^*b)^\omega \vee a^\omega b(d \vee ca^*b)^* \end{bmatrix} \in \mathcal{E}^{1\times n}$$

$$M^{\omega_k} = {}_{\mathrm{t}}\begin{bmatrix} (a \vee bd^*c)^\omega \\ (a \vee bd^*c)^\omega bd^* \end{bmatrix} \in \mathcal{E}^{1\times n}$$

**Theorem:** For any $\mathcal{E}$-automaton $(S, M)$ with $S = \{1, \dots, n\}$, $F = \{1, \dots, k\}$, $k \le n$, $s_0 \le n$, and $x_0 \in \mathbb{R}_{\ge 0}$,
Büchi$(s_0, x_0, F) = \mathtt{tt}$ iff $M^{\omega_k} I^{s_0}(x_0) \ne \bot$.

# Conclusion

- Energy problems can be solved using the theory of semiring-weighted automata and semiring-semimodule pairs
  - for reachability, use star; for Büchi, use omega
- Extensions to multi-dimension or games: semiring techniques do not seem to apply
  - but techniques from well-structured transition systems do
  - for multi-dimensional games, undecidability is quickly reached
- Extension to energy automata with discrete inputs?
  - modeling discrete control problems

What is the minimum amount of battery required, and which control actions do I need to apply, for the satellite to always be able to send and receive messages?