Kleene Algebras and Semimodules for Energy Problems

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What is the minimum amount of battery required for the satellite to always be able to send and receive messages?

2007

Infinite Runs in Weighted Timed Automata with Energy Constraints

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MT-LAB Meeting, September 2009

(With slides by Nicolas Markey)

One-slide summary

Goal:

Find infinite schedules
in priced timed automata
which satisfy constraints on total cost

- When should I plan to re-charge my laptop battery if I want to be sure to be able to watch YouTube videos during all my travel?
- How should I re-fill my oil tank so that it never runs out of oil and never runs over?

Results: mixed...
For some problems schedules computable in P, for some uncomputable.

Slogan:

Hybridization of timed automata

Contents

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Energy Constraints

Energy is not only consumed, but can be regained.

$\rightsquigarrow$ "prices" can be negative;

$\rightsquigarrow$ the aim is to continuously satisfy cost constraints

$\rightsquigarrow$ in this paper, we focus on infinite runs.

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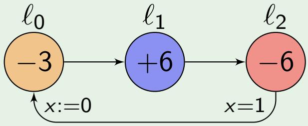

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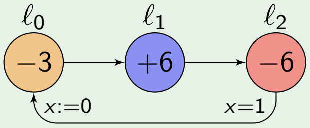

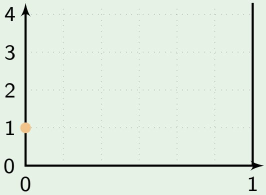

lower-bound problem

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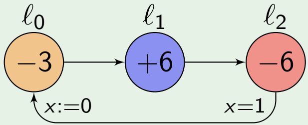

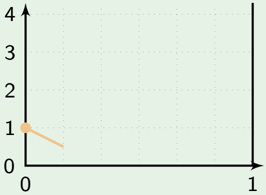

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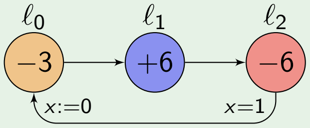

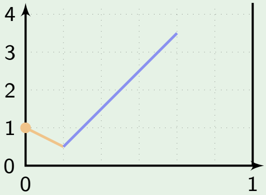

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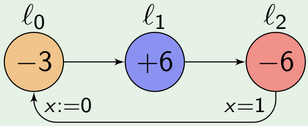

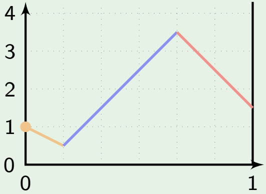

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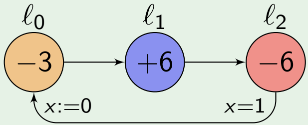

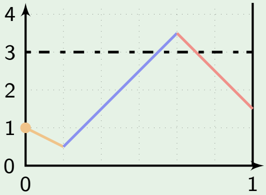

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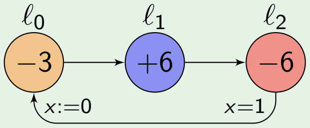

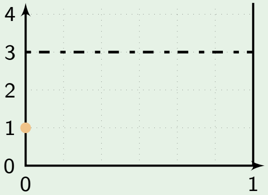

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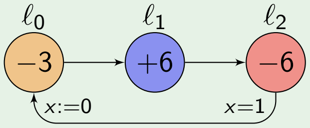

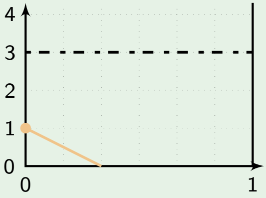

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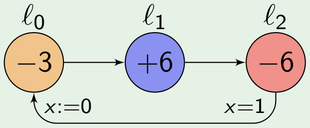

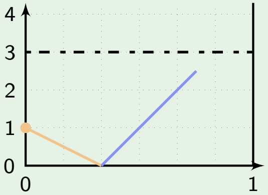

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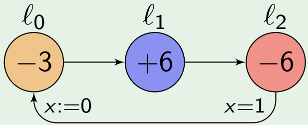

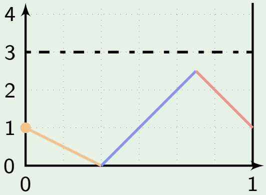

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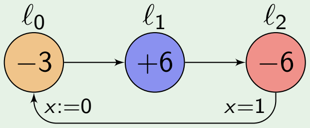

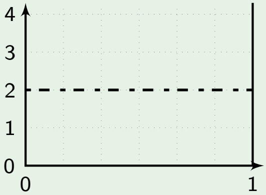

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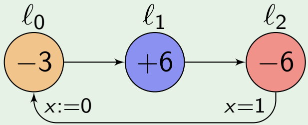

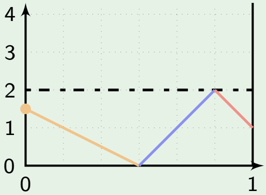

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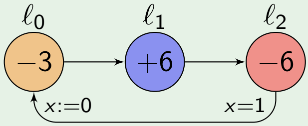

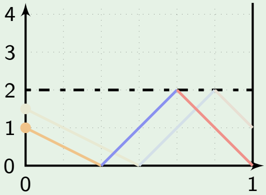

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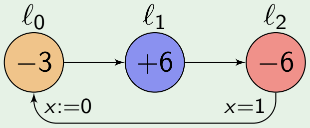

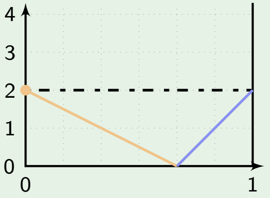

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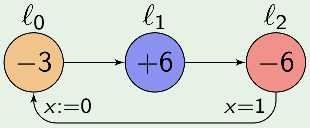

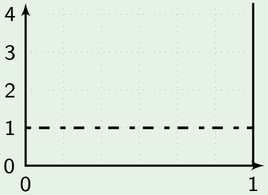

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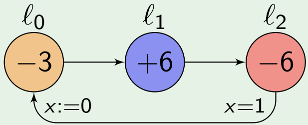

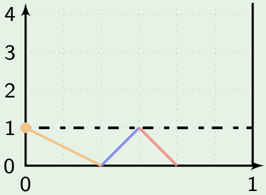

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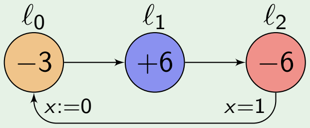
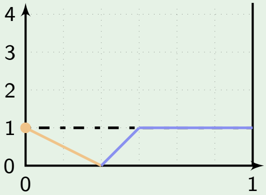


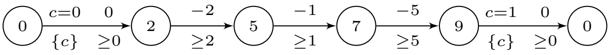


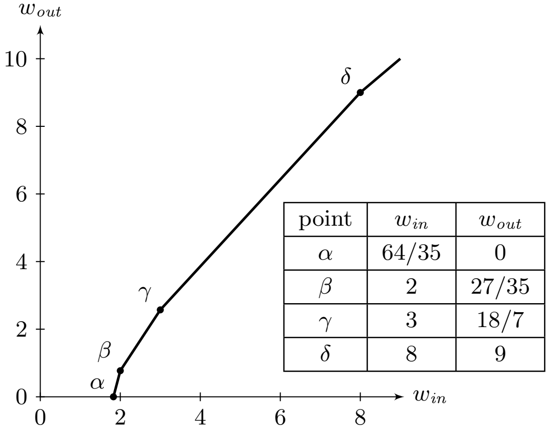
lower-weak-upper-bound problem

2009

after (ω_n, ω_n).

Example 1. *We consider the following example, which is already in normal form. The corresponding function* f_π *then looks as depicted on Figure 5:*





point	w_{in}	w_{out}
α	64/35	0
β	2	27/35
γ	3	18/7
δ	8	9

Figure 5: Function f_π for example with linear observer

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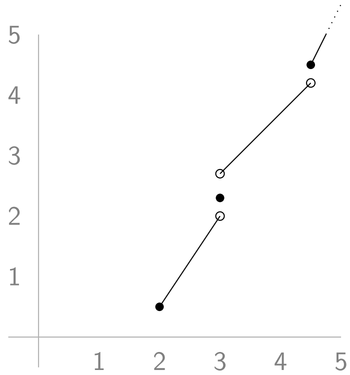
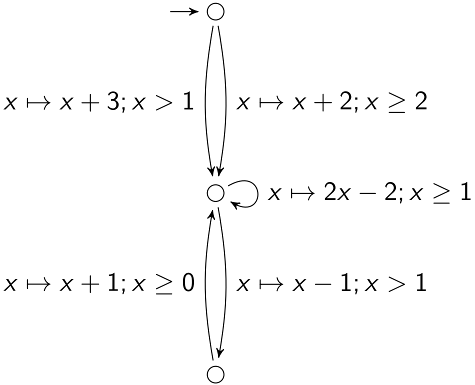
2011

Energy Automata, Examples



a simple energy function



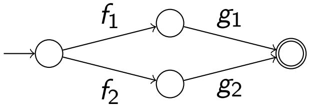
a simple energy automaton

Energy Function Semiring

Interest: reachability and Büchi acceptance

- Given a set F of accept states and $x_0 \in \mathbb{R}_{\geq 0}$: does there exist a run with initial energy x_0 which reaches F? does there exist one which visits F infinitely often?

Operations on energy functions: max and $\circ$



becomes $\max(g_1 \circ f_1, g_2 \circ f_2)$

The set $\mathcal{E}$ of energy functions with operations max and $\circ$ is a semiring, with $\mathbf{0} = \lambda x.\bot$, $\mathbf{1} = \lambda x.x$

- without "$f' \geq 1$" condition, only "near-semiring"
- idempotent, positively ordered, complete

Loops for Reachability

Star: $f^* = \sup_{n\geq 0} f^n$

- for loops which can be taken an arbitrary number of times
- $f^*(x) = \begin{cases} x & \text{if } f(x) \leq x \\ \infty & \text{if } f(x) > x \end{cases}$

Theorem: Always, $gf^*h = \sup_{n\geq 0} gf^n h$

- i.e. $\mathcal{E}$ is a star-continuous Kleene algebra

Corollary: Let M be the (transposed) transition matrix of an energy automaton

- i.e. M_{ji} is the transition label from state s_i to state s_j.

Compute $M^* = \sup_{n\geq 0} M^n$
Then s_j is reachable from s_i with initial energy x_0 iff $M^*_{ji}(x_0) \neq \perp$.

Loops for Infinite Runs

Omega: " $f^\omega = \lim_{n\to\infty} f^n$ "

- for loops which are taken infinitely often
- $f^\omega(x) = \begin{cases} \bot & \text{if } f(x) < x \quad \text{or } x = \bot \\ \top & \text{if } f(x) \geq x \quad \text{and } x \neq \bot \end{cases}$
- important: two-valued; $\mathcal{V}$: energy functions into $\{\bot, \top\}$

Theorem: $(\mathcal{E}, \mathcal{V})$ is a Conway semiring-semimodule pair

Corollary: Let M be the (transposed) transition matrix of an energy automaton

- i.e. M_{ji} is the transition label from state s_i to state s_j.

Compute " $M^\omega = \lim_{n\to\infty} M^n$ "
Then there is an infinite run from s_i with initial energy x_0 iff $M_i^\omega(x_0) \neq \bot$.

Some Technical Details for Reachability

(Applying work by S. Bloom, Z. Ésik, W. Kuich and others)

For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in \mathcal{E}^{k\times k}$ and $d \in \mathcal{E}^{m\times m}$ (and $k + m = n$), let

$$M^* = \begin{bmatrix} (a \vee bd^*c)^* & (a \vee bd^*c)^* bd^* \\ (d \vee ca^*b)^* ca^* & (d \vee ca^*b)^* \end{bmatrix} \in \mathcal{E}^{n\times n}$$

Lemma: M^* does not depend on k and m, and
always $NM^*P = \sup_n NMP$.

- can also use (generalized) Floyd-Warshall algorithm to compute M^*; generally faster

Theorem: For any $\mathcal{E}$-automaton (S, M) with $S = \{1, \dots, n\}$, $F = \{1, \dots, k\}$, $k \leq n$, $s_0 \leq n$, and $x_0 \in \mathbb{R}_{\geq 0}$, $\text{Reach}(s_0, x_0, F) = \mathbf{tt}$ iff ${}_{\mathrm{t}}F^{\leq k} M^* I^{s_0}(x_0) \neq \bot$.

Some Technical Details for Büchi Acceptance

(Extending work by S. Bloom, Z. Ésik, W. Kuich and others)

For a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $a \in \mathcal{E}^{k \times k}$ and $d \in \mathcal{E}^{m \times m}$ (and $k + m = n$), let

$$M^\omega = {}_{\mathrm{t}}\begin{bmatrix} (a \vee bd^* c)^\omega \vee d^\omega c(a \vee bd^* c)^* \\ (d \vee ca^* b)^\omega \vee a^\omega b(d \vee ca^* b)^* \end{bmatrix} \in \mathcal{E}^{1 \times n}$$

$$M^{\omega_k} = {}_{\mathrm{t}}\begin{bmatrix} (a \vee bd^* c)^\omega \\ (a \vee bd^* c)^\omega bd^* \end{bmatrix} \in \mathcal{E}^{1 \times n}$$

Theorem: For any $\mathcal{E}$-automaton (S, M) with $S = \{1, \dots, n\}$, $F = \{1, \dots, k\}$, $k \leq n$, $s_0 \leq n$, and $x_0 \in \mathbb{R}_{\geq 0}$,
Büchi$(s_0, x_0, F) = \mathbf{tt}$ iff $M^{\omega_k} I^{s_0}(x_0) \neq \bot$.

Conclusion

- Energy problems can be solved using the theory of semiring-weighted automata and semiring-semimodule pairs
  - for reachability, use star; for Büchi, use omega
- Extensions to multi-dimension or games: semiring techniques do not seem to apply
  - but techniques from well-structured transition systems do
  - for multi-dimensional games, undecidability is quickly reached
- Extension to energy automata with discrete inputs?
  - modeling discrete control problems