INFORMATION THEORY

Master 1 - Informatique - Univ. Rennes 1 / ENS Rennes

Aline Roumy

January 2020

Outline

1. Non mathematical introduction
2. Mathematical introduction: definitions
3. Typical vectors and the Asymptotic Equipartition Property (AEP)
4. Lossless Source Coding
5. Variable length Source coding - Zero error Compression

About me

Aline Roumy
Researcher at Inria, Rennes
Expertise: compression for video streaming
image/signal processing, information theory, machine learning

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Course schedule (tentative)

Information theory (IT):

a self-sufficient course with a lot of connections to probability

- Lecture 1: introduction, reminder on probability
- Lecture 2-3: Data compression (theoretical limits)
- Lecture 4: Construction of codes that can compress data
- Lecture 5: Beyond classical information theory (universality...)

Course organization:

- slides (file available online)
- summary (file available online+hardcopy)
- proofs (see blackboard): take notes!

On my webpage:
http://people.rennes.inria.fr/Aline.Roumy/roumy_teaching.html

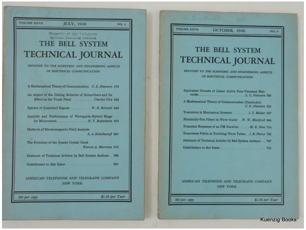
Course grading and documents

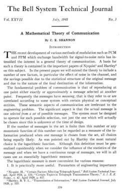
- Homework:
  - exercises (in class and at home)
  - correction in front of the class give bonus points.
- Middle Exam:
  - (in group) written exam,
  - home.
- Final Exam:
  - (individual) written exam
  - questions de cours, et exercices (in French)
  - 2h
- All documents on my webpage:
  http://people.rennes.inria.fr/Aline.Roumy/roumy_teaching.html

Course material

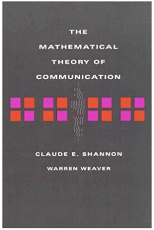
C.E. Shannon, "A mathematical theory of communication", *Bell Sys. Tech. Journal*, 27: 379–423, 623–656, 1948.

seminal paper



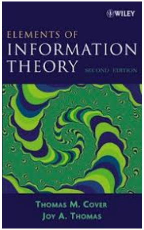

The Bell System Technical Journal

A Mathematical Theory of Communication



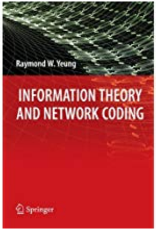
Course material

T.M. Cover and J.A. Thomas. *Elements of Information Theory*. Wiley Series in Telecommunications. Wiley, New York, 2006.
THE reference



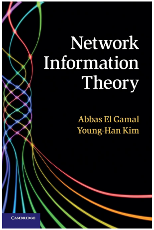
Course material

R. Yeung. *Information Theory and Network Coding*.
Springer 2008.
network coding



Course material

A. E. Gamal and Y-H. Kim. *Network Information Theory*.
Cambridge University Press 2011.
network information theory



Slides:
A. E. Gamal and Y-H. Kim.
Lecture Notes on Network Information Theory. arXiv:1001.3404v5, 2011. **web**

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Lecture 1

Non mathematical introduction

What does “communicating” means?

What it is about? A bit of history...

- Information theory (IT) =
  "The fundamental problem of **communication** is that of reproducing at one point, either exactly or approximately, a message selected at another point."
- IT established by Claude E. Shannon (1916-2001) in 1948.
  - Seminal paper: "A Mathematical Theory of Communication" in the Bell System Technical Journal, 1948.
  - revolutionary and groundbreaking paper

Teaser 1: compression

Hangman game

- **Objective:** play... and explain your strategy

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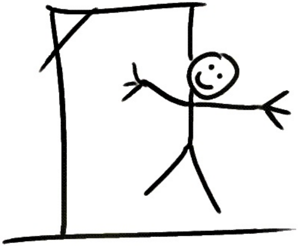

Teaser 1: compression

Hangman game

- **Objective:** play... and explain your strategy

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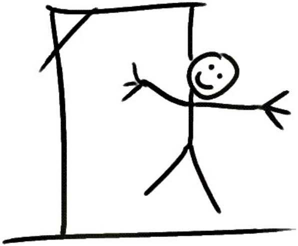


- **2 winning ideas**
  - Letter frequency — **probability**
  - Correlation between successive letters — **dependence**

Teaser 1: compression

Analogy Hangman game-compression

- word
- Answer to a question (yes/no)

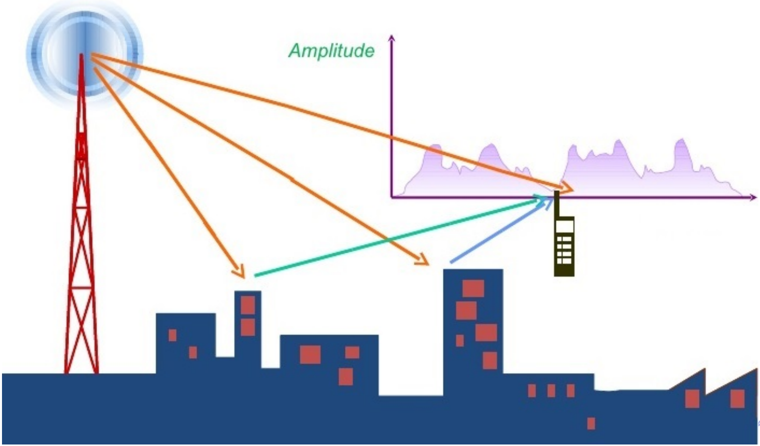
  removes uncertainty in word
- **Goal:** propose a minimum number of letter

- data (image)
- 1 bit of the bistream that represents the data

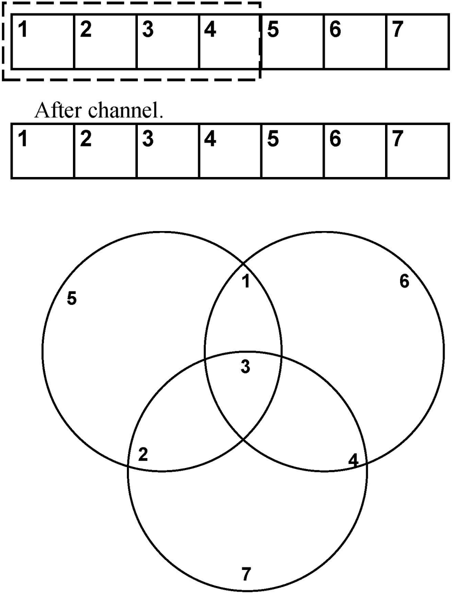
  removes uncertainty in data
- **Goal:** propose a minimum number of bits

Teaser 2: communication over a noisy channel

- Context:
  storing/communicating data on a channel **with errors**
  - scratches on a DVD
  - lost data packets: webpage sent over the internet.
  - lost or modified received signals: wireless links



Teaser 2: communication over a noisy channel



1. choose binary vector (x_1, x_2, x_3, x_4)
2. compute x_5, x_6, x_7 s.t. XOR in each circle is 0
3. add 1 or 2 errors
4. correct errors s.t. rule 2 is satisfied

Quiz 1:

Assume you know how many errors have been introduced.
Can one correct 1 error?
Can one correct 2 errors?

Teaser 2: communication over a noisy channel

Take Home Message (THM):

- To get zero error at the receiver, one can send a **FINITE** number of additional of bits.
- For a finite number of additional of bits, there is a limit on the number of errors that can be corrected.

Summary

- one can **compress** data by using two ideas:
  - Use **non-uniformity** of the probabilities

    this is the source coding theorem (first part of the course)

    very surprising...
  - Use **dependence** between the data

    in middle exam

Summary

- one can **compress** data by using two ideas:
  - Use **non-uniformity** of the probabilities

    this is the source coding theorem (first part of the course)

    very surprising...
  - Use **dependence** between the data

    in middle exam

- one can **send** data over a **noisy** channel and recover the data without any error

  provided the data is **encoded** (send additional data)

  this is the channel coding theorem (second part of the course)

Communicate what?

Definition

Source of information: something that produces **messages**.

Definition

Message: a stream of **symbols** taking their values in an **alphabet**.

Example

Source: camera
Message: picture
Symbol: pixel value: 3 coef. (RGB)
Alphabet=$\{0, \ldots, 255\}^3$

Example

Source: writer
Message: a text
Symbol: letter
Alphabet=$\{a, \ldots, z, !, ., ?, ...\}$

How to model the communication?

- Model for the source:

  communication
  a source of information
  a message of the source
  a symbol of the source
  alphabet of the source

- Model for the communication chain:

How to model the communication?

- Model for the source:

communication		**mathematical model**
a source of information	$\rightarrow$	a random process
a message of the source	$\rightarrow$	a realization of a random **vector**
a symbol of the source	$\rightarrow$	a realization of a random variable
alphabet of the source	$\rightarrow$	alphabet of the random variable

- Model for the communication chain:

Point-to-point Information theory

Shannon proposed and proved three fundamental theorems for point-to-point communication (1 sender / 1 receiver):

1. **Lossless source** coding theorem: For a given source, what is the minimum rate at which the source can be **compressed losslessly**?
   rate = nb bits / source symbol
2. **Lossy source** coding theorem: For a given source and a given distortion D, what is the minimum rate at which the source can be **compressed within distortion** D.
   rate = nb bits / source symbol
3. **Channel coding** theorem: What is the maximum rate at which data can be **transmitted** reliably?
   rate = nb bits / sent symbol over the channel

Application of Information Theory

Information theory is everywhere...

1. **Lossless source** coding theorem:
2. **Lossy source** coding theorem:
3. **Channel coding** theorem:

Quiz 2: On which theorem (1/2/3) rely these applications?

(1) zip compression
(2) jpeg and mpeg compression
(3) sending a jpeg file onto internet
(4) the 15 digit social security number
(5) movie stored on a DVD

Reminder (1)

Definition (Convergence in probability)

Let $(X_n)_{n\geq 1}$ be a sequence of r.v. and X a r.v. both defined over $\mathbb{R}$. $(X_n)_{n\geq 1}$ converges in probability to the r.v. X if

$$\forall \epsilon > 0, \lim_{n\to+\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0.$$

Notation:

$$X_n \xrightarrow{p} X$$

Quiz 3: Which of the following statements are true?

(1) X_n and X are random

(2) X_n is random and X is deterministic (constant)

Reminder (2)

Theorem (Weak Law of Large Numbers (WLLN))

Let $(X_n)_{n\geq 1}$ be a sequence of r.v. over $\mathbb{R}$.
If $(X_n)_{n\geq 1}$ is i.i.d., $\mathcal{L}^2$ (i.e. $\mathbb{E}[X_n^2] < \infty$) then

$$\frac{X_1 + \ldots + X_n}{n} \xrightarrow{p} \mathbb{E}[X_1]$$

Quiz 4: Which of the following statements are true?

(1) for any nonzero margin, with a sufficiently large sample there will be a very high probability that the average of the observations will be close to the expected value; that is, within the margin.
(2) LHS and RHS are random
(3) averaging kills randomness
(4) the statistical mean ((a.k.a. true mean) converges to the empirical mean (a.k.a. sample mean)

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Lecture 2

Mathematical introduction

Definitions: Entropy and Mutual Information

Some Notation

Specific to information theory are denoted in red

- Upper case letters $X, Y, \ldots$ refer to random process or random variable
- Calligraphic letters $\mathcal{X}, \mathcal{Y}, \ldots$ refer to alphabets
- $|\mathcal{A}|$ is the cardinality of the set $\mathcal{A}$
- $X^n = (X_1, X_2, \ldots, X_n)$ is an n-sequence of random variables or a random vector

  $X_i^j = (X_i, X_{i+1}, \ldots, X_j)$
- Lower case $x, y, \ldots$ and $x^n, y^n, \ldots$ mean scalars/vectors realization
- $X \sim p(x)$ means that the r.v. X has probability mass function (pmf) $\mathbb{P}(X = x) = p(x)$
- $X^n \sim p(x^n)$ means that the discrete random vector X^n has joint pmf $p(x^n)$
- $p(y^n | x^n)$ is the conditional pmf of Y^n given $X^n = x^n$.

Lecture 1: Entropy (1)

Definition (Entropy)

the **entropy** of a **discrete** random variable $X \sim p(x)$:

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x)$$

$H(X)$ in **bits/source sample** is the **average length of the shortest description** of the r.v. X. *(Shown later)*

Notation: $\log := \log_2$

Convention: $0 \log 0 := 0$

Properties

E1 $H(X)$ only depends on the pmf $p(x)$ and not x.

E2 $H(X) = -\mathbb{E}_X \log p(X)$

Entropy (2)

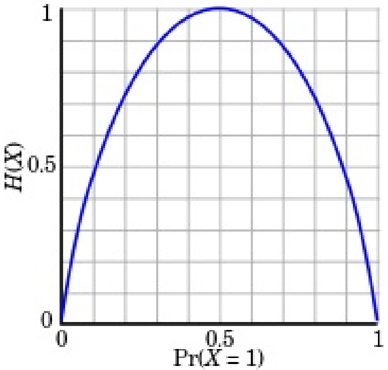
E3 $H(X) \geq 0$ with equality iff X is constant.

E4 $H(X) \leq \log |\mathcal{X}|$. The uniform distribution maximizes entropy.

Example

Binary entropy function: Let $0 \leq p \leq 1$

$$h_b(p) = -p \log p - (1-p) \log (1-p)$$



$H(X)$ for a binary rv.

$H(X)$ measures the amount of uncertainty on the rv X.

Entropy (3)

E4 **(con't)** Alternative proof with the positivity of the **Kullback-Leibler (KL) divergence**.

Definition (Kullback-Leibler (KL) divergence)

Let $p(x)$ and $q(x)$ be 2 pmfs defined on the same set $\mathcal{X}$.
The **KL divergence** between p and q is:

$$D(p||q) = \sum_{x \in \mathcal{X}} p(x) \log \frac{p(x)}{q(x)}$$

Convention: $c \log c/0 = \infty$ for $c > 0$.

Quiz 5: Which of the following statements are true?

(1) $D(p||q) = D(q||p)$.
(2) If $\text{Support}(q) \subset \text{Support}(p)$ then $D(p||q) = \infty$.

Entropy (4)

KL1 **Positivity of KL** [Cover Th. 2.6.3]: $D(p||q) \geq 0$
with equality iff $\forall x, p(x) = q(x)$.

This is a consequence of Jensen's inequality [Cover Th. 2.6.2]:
If f is a convex function and Y is a random variable with numerical values, then

$$\mathbb{E}[f(Y)] \geq f(\mathbb{E}[Y])$$

with equality when $f(.)$ is not strictly convex, or when $f(.)$ is strictly convex and Y follows a degenerate distribution (i.e. is a constant).

KL2 Let $X \sim p(x)$ and $q(x) = \frac{1}{|\mathcal{X}|}$, then $D(p||q) = -H(X) + \log|\mathcal{X}|$

Reminder (independence)

Definition (independence)

The random variables X and Y are independent, denoted by $X \perp\!\!\!\perp Y$, if

$$\forall (x, y) \in \mathcal{X} \times \mathcal{Y}, \quad p(x, y) = p(x)p(y).$$

Definition (Mutual independence – mutuellement indépendant)

For $n \geq 3$, the random variables $X_1, X_2, \ldots, X_n$ are mutually independent if

$$\forall (x_1, \ldots, x_n) \in \mathcal{X}_1 \times \ldots \times \mathcal{X}_n, \quad p(x_1, \ldots, x_n) = p(x_1)p(x_2) \ldots p(x_n).$$

Definition (Pairwise independence – indépendance 2 à 2)

For $n \geq 3$, the random variables X_i, X_j are pairwise independent if $\forall (i, j)$ s.t. $1 \leq i < j \leq n$, X_i and X_j are independent.

Quiz 6

Quiz 6: Which of the following statements are/is true?

(1) mutual independence implies pairwise independence.

(2) pairwise independence implies mutual independence

Reminder (conditional independence)

Definition (conditional independence)

Let X, Y, Z be r.v.
X is independent of Z given Y, denoted by $X \perp\!\!\!\perp Z | Y$, if

$$\forall (x, y, z) \quad p(x, z|y) = \begin{cases} p(x|y)p(z|y) & \text{if } p(y) > 0 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently

$$\forall (x, y, z) \quad p(x, y, z) = \begin{cases} \frac{p(x,y)p(y,z)}{p(y)} = p(x, y)p(z|y) & \text{if } p(y) > 0 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently

$$\forall (x, y, z) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \quad p(x, y, z)p(y) = p(x, y)p(y, z),$$

Definition (Markov chain)

Let $X_1, X_2, ..., X_n$, $n \geq 3$ be r.v.
$X_1 \to X_2 \to ... \to X_n$ forms a Markov chain if $\forall(x_1, ..., x_n)$

$$p(x_1, x_2, ..., x_n) = \begin{cases} p(x_1, x_2)p(x_3|x_2)...p(x_n|x_{n-1}) & \text{if } p(x_2),...,p(x_{n-1})>0 \\ 0 & \text{otherwise} \end{cases}$$

or equivalently $\forall(x_1, ..., x_n)$

$$p(x_1, x_2, ..., x_n)p(x_2)p(x_3)...p(x_{n-1}) = p(x_1, x_2)p(x_2, x_3)...p(x_{n-1}, x_n)$$

Quiz 7: Which of the following statements are true?

(1) $X \perp\!\!\!\perp Z|Y$ is equivalent to $X \to Z \to Y$
(2) $X \perp\!\!\!\perp Z|Y$ is equivalent to $X \to Y \to Z$
(3) $X_1 \to X_2 \to ... \to X_n \Rightarrow X_n \to ... \to X_2 \to X_1$

Joint and conditional entropy

Definition (Conditional entropy)

For discrete random variables $(X, Y) \sim p(x, y)$,
the **Conditional entropy** **for a given y** is:

$$H(X \mid Y = y) \quad = \quad -\sum_{x \in \mathcal{X}} p(x|y) \log p(x|y)$$

the **Conditional entropy** is:

$$H(X \mid Y) = \sum_{y \in \mathcal{Y}} p(y) H(X \mid Y = y) = -\mathbb{E}_{XY} \log p(X \mid Y)$$

$$= -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x|y) = -\sum_{y \in \mathcal{Y}} p(y) \sum_{x \in \mathcal{X}} p(x|y) \log p(x|y)$$

$H(X|Y)$ in **bits/source sample** is the **average length of the shortest description** of the r.v. X when Y is known.

Joint entropy

Definition (Joint entropy)

For discrete random variables $(X, Y) \sim p(x, y)$, the **Joint entropy** is:

$$H(X, Y) = -\mathbb{E}_{XY} \log p(X, Y) = -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x, y)$$

$H(X, Y)$ in **bits/source sample** is the **average length of the shortest description** of ???.

Properties

JCE1 **trick** $H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$

JCE2 $H(X,Y) \leq H(X) + H(Y)$ with equality iff X and Y are independent (denoted $X \perp\!\!\!\perp Y$).

JCE3 **Conditioning reduces entropy**
$H(X|Y) \leq H(X)$ with equality iff $X \perp\!\!\!\perp Y$

JCE4 **Chain rule** for entropy (formule des conditionnements successifs)
Let X^n be a discrete random vector

$$\begin{aligned} H(X^n) &= H(X_1) + H(X_2|X_1) + \ldots + H(X_n|X_{n-1},\ldots,X_1) \\ &= \sum_{i=1}^{n} H(X_i|X_{i-1},\ldots,X_1) \\ &= \sum_{i=1}^{n} H\left(X_i|X^{i-1}\right) \leq \sum_{i=1}^{n} H(X_i) \end{aligned}$$

with notation $H(X_1|X^0) = H(X_1)$.

JCE5 $H(X|Y) \geq 0$ with equality iff $X = f(Y)$ a.s.

JCE6 $H(X|X) = 0$ and $H(X, X) = H(X)$

JCE7 **Data processing inequality.** Let X be a discrete random variable and $g(X)$ be a function of X, then

$$H(g(X)) \leq H(X)$$

with equality iff $g(x)$ is injective on the support of $p(x)$.

JCE8 **Fano's inequality:** link between entropy and error prob. Let $(X, Y) \sim p(x, y)$ and $P_e = \mathbb{P}\{X \neq Y\}$, then

$$H(X|Y) \leq h_b(P_e) + P_e \log(|\mathcal{X}| - 1) \leq 1 + P_e \log(|\mathcal{X}| - 1)$$

JCE9 $H(X|Z) \geq H(X|Y, Z)$ with equality iff X and Y are independent given Z (denoted $X \perp\!\!\!\perp Y|Z$).

JCE10 $H(X, Y|Z) \leq H(X|Z) + H(Y|Z)$ with equality iff $X \perp\!\!\!\perp Y|Z$.

Venn diagram

	is represented by	
X (a r.v.)	$\rightarrow$	set (set of realizations)
$H(X)$	$\rightarrow$	area of the set
$H(X, Y)$	$\rightarrow$	area of the union of sets

Exercise

1. Draw a Venn Diagram for 2 r.v. X and Y.
   Show $H(X), H(Y), H(X, Y)$ and $H(Y|X)$.
2. Show the case $X \perp\!\!\!\perp Y$
3. Draw a Venn Diagram for 3 r.v. X, Y and Z and show the decomposition $H(X, Y, Z) = H(X) + H(Y|X) + H(Z|X, Y)$.
4. Show the case $X \perp\!\!\!\perp Y|Z$

Mutual Information

Definition (Mutual Information)

For discrete random variables $(X, Y) \sim p(x, y)$, the **Mutual Information** is:

$$\begin{aligned} I(X;Y) &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log\frac{p(x,y)}{p(x)p(y)} \\ &= H(X) - H(X|Y) = H(Y) - H(Y|X) \\ &= H(X) + H(Y) - H(X,Y) \end{aligned}$$

Exercise Show $I(X;Y)$ on the Venn Diagram representing X and Y.

Mutual Information: properties

MI1 $I(X;Y)$ is a function of $p(x,y)$

MI2 $I(X;Y)$ is symmetric: $I(X;Y) = I(Y;X)$

MI3 $I(X;X) = H(X)$

MI4 $I(X;Y) = D(p(x,y) || p(x)p(y))$

MI5 $I(X;Y) \geq 0$
with equality iff $X \perp\!\!\!\perp Y$

MI6 $I(X;Y) \leq \min(H(X), H(Y))$
with equality iff $X = f(Y)$ **a.s. or** $Y = f(X)$ **a.s.**

Conditional Mutual Information

Definition (Conditional Mutual Information)

For discrete random variables $(X, Y, Z) \sim p(x, y, z)$, the **Conditional Mutual Information** is:

$$
\begin{aligned}
I(X; Y|Z) &= \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} \sum_{z\in\mathcal{Z}} p(x, y, z) \log \frac{p(x, y|z)}{p(x|z)p(y|z)} \\
&= H(X|Z) - H(X|Y, Z) \\
&= H(Y|Z) - H(Y|X, Z)
\end{aligned}
$$

Exercise Show $I(X; Y|Z)$ and $I(X; Z)$ on the Venn Diagram representing X, Y, Z.

CMI1 $I(X; Y|Z) \geq 0$ with equality iff $X \perp\!\!\!\perp Y|Z$

Exercise Compare $I(X; Y, Z)$ with $I(X; Y|Z) + I(X; Z)$ on the Venn Diagram representing X, Y, Z.

CMI2 Chain rule

$$I\left(X^n;Y\right)=\sum_{i=1}^{n} I\left(X_i;Y\left|X^{i-1}\right.\right)$$

CMI3 If $X\to Y\to Z$ form a Markov chain, then $I(X;Z|Y)=0$

CMI4 Corollary: If $X\to Y\to Z$, then $I(X;Y)\geq I(X;Y|Z)$

CMI5 Corollary: **Data processing inequality:**
If $X\to Y\to Z$ form a Markov chain, then $I(X;Y)\geq I(X;Z)$

Exercise Draw the Venn Diagram of the Markov chain $X\to Y\to Z$

CMI6 There is **no order relation** between $I(X;Y)$ and $I(X;Y|Z)$
Faux amis: Recall $H(X|Z)\leq H(X)$

Hint: show an example s.t. $I(X;Y)>I(X;Y|Z)$ and an example s.t. $I(X;Y)<I(X;Y|Z)$

Exercise Show the area that represents $I(X;Y)-I(X;Y|Z)$ on the Venn Diagram...

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Lecture 3

Typical vectors and
Asymptotic **Equipartition** Property (AEP)

Re-reminder

Definition (Convergence in probability)

Let $(X_n)_{n\geq 1}$ be a sequence of r.v. and X a r.v. both defined over $\mathbb{R}^d$.
$(X_n)_{n\geq 1}$ converges in probability to the r.v. X if

$$\forall \epsilon > 0, \lim_{n\to+\infty} \mathbb{P}(|X_n - X| > \epsilon) = 0.$$

Notation:

$$X_n \xrightarrow{p} X$$

Theorem (Weak Law of Large Numbers (WLLN))

Let $(X_n)_{n\geq 1}$ be a vector of r.v. over $\mathbb{R}$.
If $(X_n)_{n\geq 1}$ is i.i.d., $\mathcal{L}^2$ (i.e. $\mathbb{E}[X_n^2] < \infty$) then

$$\frac{X_1 + \ldots + X_n}{n} \xrightarrow{p} \mathbb{E}[X_1]$$

Theorem (Asymptotic Equipartition Property (AEP))

Let $X_1, X_2, \ldots$ *be i.i.d.* $\sim p(x)$ **finite** *random process (source), let us denote* $p(x^n) = \prod_{i=1}^{n} p(x_i)$, *then*

$$-\frac{1}{n} \log p(X^n) \to H(X) \qquad \textit{in probability}$$

Definition (Typical set)

Let $\epsilon > 0, n > 0$ and $X \sim p(x)$, the set $A_\epsilon^{(n)}(X)$ of ϵ-**typical** vectors x^n, where $p\left(x^n\right) = \prod_{i=1}^n p\left(x_i\right)$ is defined as

$$A_\epsilon^{(n)}(X) = \left\{ x^n : \left| -\frac{1}{n} \log p\left(x^n\right) - H(X) \right| \leq \epsilon \right\}$$

Properties

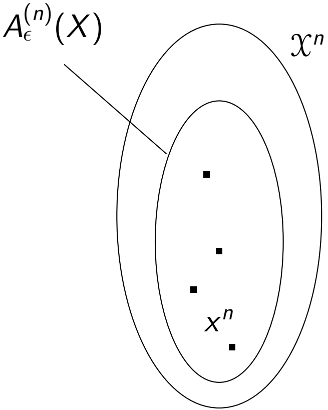
AEP1 $\forall(\epsilon, n)$, all these statements are equivalent:

$$\begin{aligned} x^n \in A_\epsilon^{(n)} \quad &\Leftrightarrow \quad 2^{-n(H(X)+\epsilon)} \leq p\left(x^n\right) \leq 2^{-n(H(X)-\epsilon)} \\ &\Leftrightarrow \quad p\left(x^n\right) \doteq 2^{-n(H(X)\pm\epsilon)} \end{aligned}$$

Notation: $a_n \doteq 2^{n(b\pm\epsilon)} \Leftrightarrow \left| \frac{1}{n} \log a_n - b \right| \leq \epsilon$ for n sufficiently large.

"uniform distribution on the typical set"

Interpretation of typicality



Example of typical vectors

$\mathbb{P}[X = x] = p(x)$
$x^n = (x_1, \ldots x_i, \ldots x_n)$
$n_x = |\{i : x_i = x\}|$

Let x^n satisfies $\frac{n_x}{n} = p(x)$ then

$$p(x^n) = \prod_i p(x_i) = \prod_{x \in \mathcal{X}} p(x)^{n_x}$$
$$= 2^{\sum_x np(x) \log p(x)} = 2^{-nH(X)}$$

x^n represents well the distribution
So, x^n is ϵ-typical, $\forall \epsilon$.

Quiz

- Let $X \sim \mathcal{B}(0.2)$, $\epsilon = 0.1$ and n=10. Which of the following x^n vector is ϵ-typical? $a = (0100000100)$ $b = (1100000000)$ $c = (1111111111)$
- Let $X \sim \mathcal{B}(0.5)$, $\epsilon = 0.1$ and n=10. Which x^n vectors are ϵ-typical?

Properties

AEP2 $\forall \epsilon > 0, \lim_{n \to +\infty} \mathbb{P}\left(\left\{X^n \in A_\epsilon^{(n)}(X)\right\}\right) = 1$

"for a given ϵ, asymptotically a.s. typical"

Theorem (CT Th. 3.1.2)

Given $\epsilon > 0$. Assume that $\forall n, X^n \sim \prod_{i=1}^n p(x_i)$.

Then, for n **sufficiently large**, we have

1. $\mathbb{P}\left(A_\epsilon^{(n)}(X)\right) = \mathbb{P}\left(\left\{X^n \in A_\epsilon^{(n)}(X)\right\}\right) > 1 - \epsilon$
2. $\left|A_\epsilon^{(n)}(X)\right| \leq 2^{n(H(X)+\epsilon)}$
3. $\left|A_\epsilon^{(n)}(X)\right| > (1-\epsilon) 2^{n(H(X)-\epsilon)}$

2 and 3 can be summarized in $\left|A_\epsilon^{(n)}\right| \doteq 2^{n(H(X) \pm 2\epsilon)}$.

Outline

1. Non mathematical introduction
2. Mathematical introduction: definitions
3. Typical vectors and the Asymptotic Equipartition Property (AEP)
4. Lossless Source Coding
5. Variable length Source coding - Zero error Compression

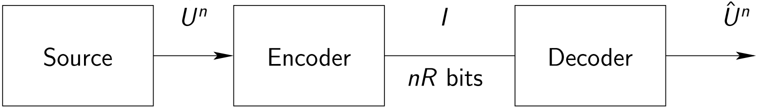
Lecture 4

Lossless Source Coding

↓

data compression

Compression system model:



We assume a **finite** alphabet i.i.d. source $U_1, U_2, \ldots \sim p(u)$.

Definition (Fixed-length Source code (FLC))

Let $R \in \mathbb{R}^+, n \in \mathbb{N}^*$. A $\left(2^{nR}, n\right)$ fixed-length source code consists of:

1. An **encoding function** that assigns to each $u^n \in \mathcal{U}^n$ an index $i \in \{1, 2, \ldots, 2^{nR}\}$, i.e., a codeword of length nR bits:

$$\begin{aligned} \mathcal{U}^n &\rightarrow \mathcal{I} = \{1, 2, ..., 2^{nR}\} \\ u^n &\mapsto i(u^n) \end{aligned}$$

2. A **decoding function** that assigns an estimate $\hat{u}^n(i)$ to each received index i

$$\begin{aligned} \mathcal{I} &\rightarrow \mathcal{U}^n \\ i &\mapsto \hat{u}^n(i) \end{aligned}$$

Definition (Probability of decoding error)

Let $n \in \mathbb{N}^*$. The **probability of decoding error** is $P_e^{(n)} = \mathbb{P}\{\hat{U}^n \neq U^n\}$

R is called the **compression rate**: number of bits per source sample.

Definition (Achievable rate)

Let $R \in \mathbb{R}^+$. A rate R is **achievable** if there exists a sequence of $(2^{nR}, n)$ codes with $P_e^{(n)} \to 0$ as $n \to \infty$

Source Coding Theorem

The source coding problem is to find the infimum of all achievable rates.

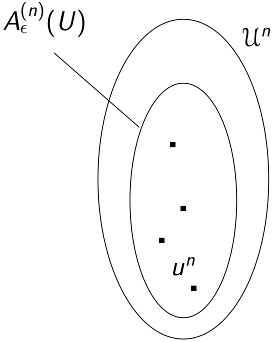
Theorem (Source coding theorem (Shannon'48))

Let $U \sim p(u)$ be a **finite** alphabet i.i.d. source. Let $R \in \mathbb{R}^+$.
[Achievability]. If $R > H(U)$,
then there exists a sequence of $\left(2^{nR}, n\right)$ codes s.t. $P_e^{(n)} \to 0$.

[Converse]. For any sequence of $\left(2^{nR}, n\right)$ codes s.t. $P_e^{(n)} \to 0$,
$R \geq H(U)$

Classical (and equivalent) statement of **[Converse]**:
If there exists a sequence of $\left(2^{nR}, n\right)$ codes s.t. $P_e^{(n)} \to 0$,
then $R \geq H(U)$

Proof of achievability [CT Th. 3.2.1]



Let $U \sim p(u)$ a finite alphabet i.i.d. process.
Let $R \in \mathbb{R}$, $\epsilon > 0$.

- Assume that $R > H(U) + \epsilon$.
  Then $\left|A_\epsilon^{(n)}\right| \leq 2^{n(H(U)+\epsilon)} < 2^{nR}$.
  Assume that nR is an integer.
- Encoding: Assign a distinct index $i\left(u^n\right)$ to each $u^n \in A_\epsilon^{(n)}$
  Assign the same index (not assigned to any typical vector) to all $u^n \notin A_\epsilon^{(n)}$
- The probability of error
  $P_e^{(n)} = 1 - \mathbb{P}\left(A_\epsilon^{(n)}\right) \to 0$ as $n \to \infty$

Proof of converse [Yeung Sec. 5.2, ElGamal Page 3-34]

- Given a sequence of $\left(2^{nR}, n\right)$ codes with $P_e^{(n)} \to 0$, let I be the random variable corresponding to the index of the $\left(2^{nR}, n\right)$ encoder.
  By Fano's inequality

  $$H\left(U^n \,|I\right) \leq H\left(U^n \,\middle|\, \hat{U}^n\right) \leq nP_e^{(n)} \log |\mathcal{U}| + 1 \triangleq n\epsilon_n$$

  where $\epsilon_n \to 0$ as $n \to \infty$, since $|\mathcal{U}|$ is finite.
- Now consider

  $$\begin{aligned} nR &\geq H(I) \\ &= I\left(U^n; I\right) \\ &= nH(U) - H\left(U^n|I\right) \geq nH(U) - n\epsilon_n \end{aligned}$$

  Thus as $n \to \infty, R \geq H(U)$
- The above source coding theorem also holds for any discrete stationary and ergodic source

Outline

Lecture 5

Variable length Source coding

Zero error Data Compression

A code

Definition (Variable length Source code (VLC))

Let X be a r.v. with finite alphabet $\mathcal{X}$. A **variable-length source code** C for a random variable X is a mapping

$$C : \mathcal{X} \rightarrow \mathcal{A}^*$$

where $\mathcal{X}$ is a set of M **symbols**,
$\mathcal{A}$ is a set of D **letters**, and
$\mathcal{A}^*$ the set of finite length sequences (or strings) of letters from $\mathcal{A}$.
$C(x)$ denotes the **codeword** corresponding to the symbol x.

In the following, we will say Source code for VLC.
Examples 1, 2

The length of a code

Let $L : \mathcal{A}^* \to \mathbb{N}$ denote the **length mapping** of a codeword (sequence of letters).

$L(C(x))$ is the number of letters of $C(x)$, and
$L(C(x)) \log |\mathcal{A}|$ the number of bits.

Definition

The **expected length** $L(C)$ of a source code C for a random variable X with pmf $p(x)$ is given by:

$$L(C) = \mathbb{E}[L(C(X))] = \sum_{x \in \mathcal{X}} L(C(x))p(x)$$

Goal Find a source code C for X with **smallest $L(C)$**.

Encoding a sequence of source symbols

Definition

A **source message**= a sequence of symbols
A **coded sequence**= a sequence of codewords

Definition

The **extension** of a code C is the mapping from finite length sequences of $\mathcal{X}$ (of any length) to finite length strings of $\mathcal{A}$, defined by:

$$\begin{array}{rccll} C: & \mathcal{X}^* & \rightarrow & \mathcal{A}^* & \\ & (x_1, \ldots, x_n) & \mapsto & C(x_1, \ldots, x_n) & = C(x_1)C(x_2)\ldots C(x_n) \end{array}$$

where $C(x_1)C(x_2)\ldots C(x_n)$ indicates the concatenation of the corresponding codewords.

Characteristics of good codes

Definition

A (source) code C is said to be **non-singular** iff C is injective:

$$\forall (x_i, x_j) \in \mathcal{X}^2, x_i \neq x_j \Rightarrow C(x_i) \neq C(x_j)$$

Definition

A code is called **uniquely decodable** iff its extension is non-singular.

Definition

A code is called a **prefix code** (or an **instantaneous code**) if no codeword is a prefix of any other codeword.

prefix code	$\Rightarrow$	**uniquely decodable**
uniquely decodable	$\not\Rightarrow$	**prefix** code

Examples

Kraft inequality

Theorem (prefix code $\Leftrightarrow$ KI [CT Th 5.2.1])

Let C be an **prefix code** *for the source X with $|\mathcal{X}| = M$ over an alphabet $\mathcal{A} = \{a_1, ..., a_D\}$ of size D. Let $l_1, l_2, ..., l_M$ the lengths of the codewords associated to the realizations of X. These codeword lengths must satisfy the* **Kraft inequality**

$$\sum_{i=1}^{M} D^{-l_i} \leq 1 \qquad \text{(KI)}$$

Conversely, *let $l_1, l_2, ..., l_M$ be M lengths that satisfy this inequality (KI), there exists an* **prefix code** *with M symbols, constructed with D letters, and with these word lengths.*

- from the lengths, one can always construct a prefix code
- finding prefix code is equivalent to finding the codeword lengths

uniquely decodable

Theorem (uniquely decodable code $\Leftrightarrow$ KI [CT Th 5.5.1])

The codeword lengths of any **uniquely decodable** *code must satisfy the Kraft inequality.*
Conversely, *given a set of codeword lengths that satisfy this inequality, it is possible to construct a* **uniquely decodable** *code with these codeword lengths.*

uniquely decodable

Theorem (uniquely decodable code ⇔ KI [CT Th 5.5.1])

The codeword lengths of any **uniquely decodable** *code must satisfy the Kraft inequality.*
Conversely, *given a set of codeword lengths that satisfy this inequality, it is possible to construct a* **uniquely decodable** *code with these codeword lengths.*

Good news!!

$$\begin{array}{rcl} \text{prefix code} & \Leftrightarrow & \text{KI} \\ \text{uniquely decodable code (UDC)} & \Leftrightarrow & \text{KI} \end{array}$$

$\Rightarrow$ same set of achievable codeword lengths for UDC and prefix

$\Rightarrow$ restrict the search of good codes to the set of prefix codes.

Optimal source codes

Let X be a r.v. taking M values in $\mathcal{X} = \{\alpha_1, \alpha_2, ..., \alpha_M\}$, with probabilities $p_1, p_2, ..., p_M$.

Each symbol α_i is associated with a codeword W_i i.e. a sequence of l_i letters, where each letter takes value in an alphabet of size D.

Goal

Find a uniquely decodable code with minimum expected length.

$$\Longleftrightarrow$$

Find a prefix code with minimum expected length.

$$\Longleftrightarrow$$

Find a set of lengths satisfying KI with minimum expected length.

$$\{l_1^*, l_2^*, ..., l_M^*\} = \arg \min_{\{l_1, l_2, ..., l_M\}} \sum_{i=1}^{M} p_i l_i \qquad \text{(Pb1)}$$

$$\text{s.t. } \forall i, l_i \geq 0 \text{ and } \sum_{i=1}^{M} D^{-l_i} \leq 1$$

Battle plan to solve (Pb1)

1. find a lower bound for $L(C)$,
2. find an upper bound,
3. construct an optimal prefix code.

Lower bound of prefix code

Theorem (Lower bound on the expected length of any prefix code [CT Th. 5.3.1])

The **expected length** $L(C)$ *of* **any prefix** *D-ary code for the r.v. X taking M values in $\mathcal{X} = \{\alpha_1, \alpha_2, \ldots, \alpha_M\}$, with probabilities $p_1, p_2, \ldots, p_M$,*
is greater than or equal to the **entropy** $H(X)/\log(D)$ *i.e.,*

$$L(C) = \sum_{i=1}^{M} p_i l_i \geq \frac{H(X)}{\log D}$$

with equality iff $p_i = D^{-l_i}$, for $i = 1, \ldots, M$, and $\sum_{i=1}^{M} D^{-l_i} = 1$

Lower and upper bound of Shannon code

Definition

A **Shannon** code (defined on an alphabet with D symbols) for each source symbol $\alpha_i \in \mathcal{X} = \{\alpha_i\}_{i=1}^{M}$ of probability $p_i > 0$, assigns codewords of length $L(C(\alpha_i)) = l_i = \lceil -\log_D(p_i) \rceil$.

Theorem (Expected length of a Shannon code [CT Sec. 5.4])

Let X be a r.v. with entropy $H(X)$. The **Shannon code** *for the source X can be turned* **into a prefix code** *and its* **expected length $L(C)$ satisfies**

$$\frac{H(X)}{\log D} \leq L(C) < \frac{H(X)}{\log D} + 1 \quad (1)$$

Lower and upper bound of Shannon code

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Corollary

Let X be a r.v. with entropy $H(X)$. There **exists a prefix code** *with expected length $L(C)$ that satisfies (1).*

Lower and upper bound of optimal code

Definition

A code is **optimal** if it achieves the lowest expected length **among all prefix codes**.

Theorem (Lower and upper bound on the expected length of an optimal code [CT Th 5.4.1])

Let X be a r.v. with entropy $H(X)$. Any **optimal code** C^* *for* X *with codeword lengths* $l_1^*, \dots, l_M^*$ *and* **expected length** $L(C^*) = \sum p_i l_i^*$ **satisfies**

$$\frac{H(X)}{\log D} \leq L(C^*) < \frac{H(X)}{\log D} + 1$$

Quiz Improve the upper bound.

Improved upper bound

Theorem (Lower and upper bound on the expected length of an optimal code for a sequence of symbols[CT Th 5.4.2])

Let X be a r.v. with entropy $H(X)$. Any **optimal code** *C^* for a sequence of s i.i.d. symbols $(X_1, ..., X_s)$ with expected length $L(C^*)$* **per source symbol** *X satisfies*

$$\frac{H(X)}{\log D} \leq L(C^*) < \frac{H(X)}{\log D} + \frac{1}{s}$$

This is the zero-error source coding Theorem.

Same average achievable rate for **vanishing** and **error-free** compression.
This is not true in general for distributed coding of multiple sources.

Construction of optimal codes

Lemma (Necessary conditions on optimal prefix codes[CT Le5.8.1])

Given a **binary** *prefix code C with word lengths $l_1, ..., l_M$ associated with a set of symbols with probabilities $p_1, ..., p_M$.*
Without loss of generality, assume that
(i) $p_1 \geq p_2 \geq ... \geq p_M$,
(ii) a group of symbols with the same probability is arranged in order of increasing codeword length (i.e. if $p_i = p_{i+1} = ... = p_{i+r}$ then $l_i \leq l_{i+1} ... \leq l_{i+r}$).
If **C is optimal** *within the class of* **prefix** *codes,* **C must satisfy**:

1. **higher** *probabilities symbols have* **shorter** *codewords ($p_i > p_k \Rightarrow l_i < l_k$),*
2. *the two least probable symbols have* **equal** *length ($l_M = l_{M-1}$),*
3. *among the codewords of* **length l_M**, *there must be at least two words that* **agree in all digits except the last**.

Huffman code

Let X be a r.v. taking M values in $\mathcal{X} = \{\alpha_1, \alpha_2, ..., \alpha_M\}$, with probabilities $p_1, p_2, ..., p_M$ s.t. $p_1 \geq p_2 \geq ... \geq p_M$.
Each letter α_i is associated with a codeword W_i i.e. a sequence of l_i letters, where each letter takes value in an alphabet of size $D = 2$.

1. **Combine** the last 2 symbols α_{M-1}, α_M into an **equivalent symbol** $\alpha_{M,M-1}$ w.p. $p_M + p_{M-1}$,
2. Suppose we can construct an optimal code C_2 $(W_1, ..., W_{M,M-1})$ for the new set of symbols $\{\alpha_1, \alpha_2, ..., \alpha_{M,M-1}\}$.
   Then, construct the code C_1 for the original set as:

$$
\begin{aligned}
C_1: \quad \alpha_i &\mapsto W_i, \ _{\forall i \in [1, M-2]}, \text{ same codewords as in } C_2 \\
\alpha_{M-1} &\mapsto W_{M,M-1}\, 0 \\
\alpha_M &\mapsto W_{M,M-1}\, 1
\end{aligned}
$$

Theorem (Huffman code is optimal [CT Th. 5.8.1])