Price War in Heterogeneous Wireless Networks

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Abstract

Wireless users have the opportunity to choose between heterogeneous access modes, such as 3G, WiFi or WiMAX for instance, which operate with different distance ranges. Due to the increasing commercial interest in access networks, those technologies are often managed by competing providers. The goal of this paper is to study the price war occuring in the case of two providers, with one provider operating in a sub-area of the other. A typical example is that of a WiFi operator against a WiMAX one, WiFi being operated in the smaller area. Using a simple model, we discuss how, for fixed prices, (elastic) demand is split among providers, and then characterize the Nash equilibria for the price war. We derive the conditions on provider capacities and coverage areas under which providers share demand on the common area. A striking additional result is that among the Nash equilibria, the one for which providers set the largest price corresponds to the case when the competitive environment does not bring any loss in terms of social welfare with respect to the socially optimal situation: at equilibrium, the

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overall utility of the system is maximized. The price of stability is one.

Key words: Wireless networks, Pricing, Competition, Game theory

1. Introduction

1.1. Context

Broadband access networks are becoming prominent in nowadays life, with various applications such as Internet access, wired or wireless telephony, television... One of the main trends is the convergence of all those services in a single network. At the same time, personal devices such as laptops or cellular phones are reliant on ubiquitous connectivity: there is now the possibility to access the network by different means in terms of provider and technology. Each user may have the opportunity to choose his access mode depending on the service availability first, and then the feasible quality of service (QoS), pondered by the corresponding access charge. Among the numerous network access technologies, we can mention

- cable modem, fiber optic links and digital subscriber line (xDSL), that require fixed access from houses or offices,
- 3G (for third generation) wireless that may be accessed from most inhabited areas,
- WiFi (for Wireless Fidelity) technology, that has been developed by working group IEEE 802.11 to provide wireless access from local area networks or hotspots [1],

- WiMAX [2, 3] (for Worldwide Interoperability for Microwave Access), that has been more recently standardized by working group IEEE 802.16, in order to reach devices at further distances.

With respect to WiFi, WiMAX is a long-range system, covering many kilometers, while WiFi typically covers tens of meters, but WiMAX and WiFi also provide different Quality of Service (QoS).

Apart from this diversity in access technologies, another trend in networking is the transition from monopolies to oligopolies. Since the Internet has moved from an academic network to a commercial one with providers fighting for customers by choosing the appropriate access price, competition issues in Internet access are highly relevant. Providers have to charge for access as a return on investment and want to maximize their profits. On the other hand, they have to take care of prices of competitors, since users can find a better combination of QoS and price with a competitor, and change providers. This kind of interaction is typical of non-cooperative game theory [4], and one usually tries to look for a *Nash equilibrium*, representing here a state where no provider can increase his revenue by an unilateral price change.

1.2. Goal

In this paper, we consider two providers in competition for customers. Users are assumed non-atomic, in the sense that their *individual* actions have no influence on the QoS of others. They are charged a fixed price per sent packet, so that the average price per served packet is the packet price charged divided by the probability of successful transmission. This way, a congestion cost is imposed thanks to the loss probability. Indeed, losses are frequently an issue in wireless networks, such as when dealing with WiFi for instance. Total demand, in terms of effective throughput, is assumed to be a decreasing function of the average price per served packet, that we call the *perceived* price. Each customer chooses the provider with the best -i.e., cheapest- perceived price. This results in a customer distribution equilibrium satisfying the Wardrop principle. That principle is widely used in transportation theory, an area closely related to telecommunications [5], and states in our context that within an area of competition between providers, the perceived price has to be the same at both providers provided they attract some demand; otherwise the highest charged users would have an interest in switching to the cheapest provider. The providers (which will be called provider 1 and provider 2) are assumed to have fixed (but possibly different) capacities, and operate in different areas. We assume that provider 2 operates in a sub-domain of provider 1's access area. Provider 1 could typically represent a WiMAX operator while the other proposes WiFi access. WiMAX can reach customers at a much longer distance than WiFi, and therefore runs a larger coverage area. We can then think of a WiMAX provider enduring competition on a fraction only of his customers, since the other part is not reachable from his competitor. The questions we aim at answering are:

- What is the strategy of each provider in terms of price setting, knowing what the user distribution would be (the Wardrop equilibrium) for any given couple of prices?
- Shall the (WiMAX) provider compete for demand on the common market, or shall he just focus on revenue on the monopolistic area to de-

termine his price so that all users in the common area could prefer to go to the (WiFi) competitor?

- Is there a (Nash) equilibrium in the price war? If it is the case, is it unique?

- What is the *price of anarchy* due to non-cooperation? The price of anarchy is a measure of the loss of efficiency due to actors' selfishness. This loss has been defined in [6] as the worst-case ratio comparing the global efficiency measure (that has to be chosen) at an outcome of the noncooperative game played among actors, to the optimal value of that efficiency measure. Similarly, what is the *price of stability*, measuring the loss of efficiency when the *best* Nash equilibrium is reached [7] (i.e., if we consider the socially optimal situation such that no actor will defect)?

1.3. Related work

Our work uses game theory to model competition among providers. Game theory [4] is a powerful tool for representing the interactions of selfish actors, and has been quite recently introduced in telecommunication networks; see [8] for a survey on the different types of problems that can be encountered. More specifically, our goal is to study pricing issues. Pricing [9, 10, 11] has been used in telecommunications to cope with congestion due to more and more demanding applications and an increasing number of customers; here typically, game theory is the natural tool to describe the interplay of selfish customers in front of a given pricing scheme. Providers use pricing to better control demand, differentiate services for different QoS-requiring users/applications, and/or provide return on investment.

On the other hand, most of the studies investigate the case of a single provider, a *monopoly*, and it is only recently that modeling the competition among providers has been introduced in networking. Competition may disrupt the monopoly-case behavior of some schemes such as the very promising *Paris Metro Pricing* (PMP) scheme, consisting in separating the network into disjoint networks served in the same manner but with different access prices. In that case, there is no guarantee that the QoS will be better at a subnetwork than at another, but it is expected that most expensive ones will be less congested due to the higher price. It is shown in [12] that such a simple and attractive scheme to differentiate service actually does not allow service differentiation under competition, since at equilibrium no provider has an interest in offering several classes. Other competition models, with less complexity than ours, have been studied. For example, [13] models competitive providers playing both on price and on a QoS parameter, but demand is there driven by an arbitrary function which does not depend on price and QoS at competitors and therefore does not cover the fact that users could switch to more attractive providers, if any. The Wardrop's principle we consider here precisely encompasses that aspect. [14] considers on the other hand a Wardrop equilibrium among users, but QoS does not depend on demand, a simplifying assumption we do not make here. In wireless networks, competition has been analyzed by several works in the case of a shared spectrum, in order to lead to a more efficient utilization than with potentially unused fixed licenses. For instance, [15] uses a more specific model than ours and shows that competition may increase users' acceptance probability for offered service. In [16], competition among selfish wireless providers is also considered, but their strategy space is only on the power of the pilot signals of their base stations,and does not include any pricing activity, a lever that should require attention. [17] studies the case where an operator can lease part of the bandwidth he owns from his license; a learning automaton is used to converge to an equilibrium, while in our model a direct proof of existence and uniqueness of an equilibrium is obtained. In a more general context, [18] studies competition in the case of uncertainty on demand, whereas in our case demand repartition among providers is obtained through a (deterministic) equilibrium among users. In [19], the pricing competition between a WiMAX and a WiFi community is investigated, but the externality is coverage instead of QoS here: the more customers the WiFi community has, the more connectivity it has. A model more closely related to ours is in [20], where atomic users can choose betwen two technologies operating on different ranges, typically a WAN and WiFi hotspots. Using a stochastic geometric model for the locations of customers and providers' access points and a greedy algorithm for the decision about which technology to use, multiple equilibria are found for a final selection. WAN and WiFi competition is analyzed in an asymptotic scenario where the service zones of WAN provider are much larger than those of WiFi access providers. Our model is different from the fact that users are assumed non-atomic. This drives to an analytical characterization of the equilibrium. Moreover, no asymptotic scenario is required for the analysis and we are able to precisely determine when both providers will attract customers in the common area.

Studying competition for customers when demand is distributed according to Wardrop's principle, was considered in [21, 22, 23], where the QoS externality is the expected delay, while it is the loss probability here, which seems more relevant for some wireless contexts. The price of anarchy, measuring the loss of efficiency due to competition with respect to cooperation, is determined, for fixed demand in [21] and random demand but linear delay in [23]; we do look at the price of anarchy too, showing for our model that competition does not lead to any loss, but also have a look at the price of stability. Moreover, we consider a more comprehensive model, by including the fact that part of customers are not accessible from one of the providers, thus competition is only on one part of demand. On the other hand, setting a price too high would also reduce (elastic) demand in the part where the WiMAX provider has a monopoly.

Note that competition can also occur in interdomain or multihop networks, where selfish providers need to send their traffic through competitors' networks to ensure end-to-end delivery, and pricing is a mean to produce such incentives [24, 25, 26]. The goal is different in this paper because we only look at direct competition for users between providers.

We have studied competition among providers in a previous work [27] using also loss probability as the externality, but for a specific network topology where all users have the choice among all providers. That could represent competition among access providers using the same technology, say WiFi, at a given hotspot or hotzone.

In this paper, we intend to model the competition in heterogeneous networks, i.e. for providers using different wireless technologies. Those technologies correspond to different coverage areas, and it therefore results in a model drastically different from the one studied in [27]. Indeed, the mathematical characterization of the user equilibrium changes completely since the different coverage zones have to be taken into account. Consequently, the higher level game played on prices by competing providers is much more complicated to study and all the required proofs are of different nature. On the other hand, we believe that studying the heterogeneousness we introduce here is a primal need, because it is a very important aspect of nowadays wireless networks.

1.4. Organization of the paper

The paper is organized as follows. Section 2 presents the mathematical model we will use to represent provider competition in heterogeneous networks, while Section 3 defines our social welfare measure as the sum of utilities of all actors (customers plus providers) and compute its maximum value; this will provide a reference to investigate the loss of efficiency due to competition. Section 4 discusses how demand is split -according to Wardrop's principle- between providers in both zones, the common one and the one where provider 1 is a monopoly. Section 5 then shows what the Nash equilibria are for the pricing game between providers, with an explicit characterization depending on the proportion of demand that is common. It is also shown that with social welfare as a global performance measure, the price of stability is one, meaning that there is no loss of efficiency by introducing competition when using the "best" Nash equilibrium. An argument is provided in favor of that particular equilibrium. The price of anarchy, when comparing social welfare at the optimal value and at the worst Nash equilibrium, is also computed. Finally, Section 6 concludes and gives directions for future research.

2. Model

2.1. Network topology and perceived prices

Consider two providers, denoted by 1 and 2, with provider 2 operating in a subdomain of provider 1, as illustrated in Figure 1. This is a typical situation

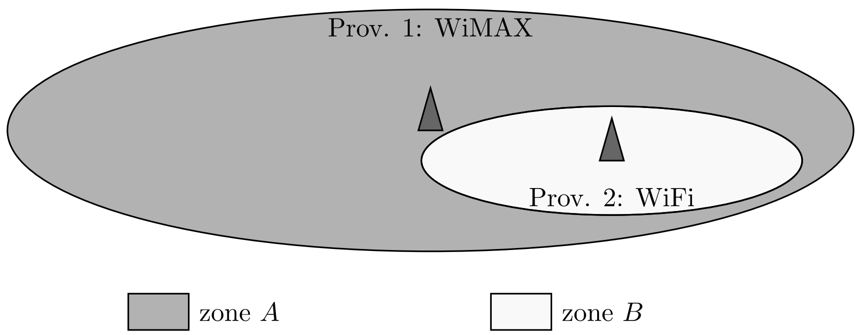

Figure 1: The competition framework

of a WiFi provider operating on smaller distances -tens of meters- than a WiMAX one -covering many kilometers-. As a consequence, competition only occurs in the domain of operator 2, while operator 1 has a monopoly in the remaining area. But operator 1 having a unique price, competition influences the optimal price in the monopoly area. As illustrated in Figure 1, we partition the total domain in

- zone A, the domain where only provider 1 operates, and
- zone B, the domain where both providers operate.

In order to analyze the outcome of competition, we consider a model where time is discretized, divided into slots. Provider i ($i \in \{1, 2\}$) is assumed to be able to serve C_i packets (or units, seen as a continuous number) per slot. Congestion is experienced at provider i if demand exceeds capacity, and demand in excess is lost, lost packets being chosen uniformly over the set of submitted ones. Formally, let d_i be the total demand at provider i. Then the number of packets served is $\min(d_i, C_i)$, meaning that packets are actually served with probability $\min(C_i/d_i, 1)$, i.e., packets are eventually served after a random time following a geometric distribution with parameter $\min(C_i/d_i, 1)$. Following an idea first introduced in [28], prices are per *submitted* packet rather than received one in order to prevent users from sending as many packets as possible, which would maximize their chance to be served. Charging on sent packets instead of successfully transmitted ones may seem unrealistic. However, that mechanism can be seen as a volume-based pricing scheme, with a congestion-dependent charge. Somewhat equivalently, it can also be seen as a consequence of the more frequently used time-based charging with a fixed price per time unit. Indeed, when congestion occurs on a network i and packets are lost, having to send them again multiplies the total transfer time (and thus the price paid) by $\max(1, d_i/C_i)$, the mean number of transmissions per packet. If each packet sent to provider i is charged p_i, the expected price $\bar{p}_i$ to successfully send a packet is therefore given by

$$\bar{p}_i = p_i / \min(C_i/d_i, 1) = p_i \max(d_i/C_i, 1), \qquad (1)$$

which will from now be called the *perceived price* per served traffic unit at provider i. Figure 2 plots that perceived price $\bar{p}_i$ depending on the demand d_i: $\bar{p}_i$ is constant while provider i is not saturated, and increases linearly when demand exceeds capacity C_i. Demand for provider 1 is decomposed into $d_{1,A}$,

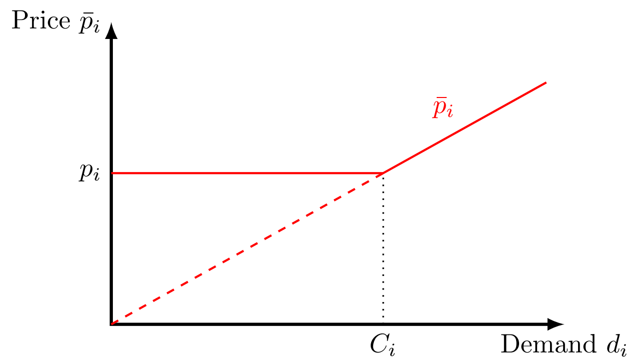

Figure 2: Perceived price at provider i versus demand d_i.

the demand in zone A, and $d_{1,B}$, the demand in zone B, with $d_1 = d_{1,A}+d_{1,B}$. Remark that our work does not deal with customer mobility: we assume that the (wireless) users do not move as soon as connected, a situation typical of most current WiFi users. Therefore coverage is not an issue for customers.

In a given zone $z \in \{A, B\}$ where the subset of operating service providers is $\mathcal{I}^z \subset \{1, 2\}$, the perceived price can be defined as

$$\bar{p}^z := \min_{i \in \mathcal{I}^z} \bar{p}_i.$$

This models the fact that users are only sensitive to the lowest perceived price available, since they choose the least expensive network.

2.2. User demand and valuation

In this paper, we assume that users are sensitive to the perceived price, in the sense that they reduce their demand when the perceived price increases. We model that effect using an aggregated demand function $D(\cdot)$.

Definition 1. *If the perceived price* $\bar{p}$ *were the same on the whole domain, then the total demand is a function* $D(\cdot)$ *of that perceived price* $\bar{p}$*. Let us denote by* $[0, p_{\max})$ *the support of* D*.*
The demand function D *is assumed to be continuous and strictly decreasing on its support, with* $D(p_{\max}) = 0$ *and possibly* $p_{\max} = +\infty$*, meaning that there is demand starvation when price is sufficiently high.*

In other words, $D(\bar{p})$ represents the number of users/packets having a willingness to pay larger than or equal to $\bar{p}$. To deal with the case where there actually is competition, we assume that there is not enough resource to satisfy all demand, i.e., $D(0) > C_1 + C_2$.

A useful function in the rest of the paper is the *marginal valuation function*, that is the generalized inverse of the demand function.

Definition 2. *The maximum unit price at which a given quantity of traffic units can be sold is called the marginal valuation for that quantity. The marginal valuation is thus the application* $v : q \mapsto \min\{p : D(p) \leq q\}$*, with the convention* $\min \emptyset = 0$*.*
The sum of the marginal valuations of the q *units of users with largest willingness-to-pay is denoted by* $V(q)$*, and* $V(\cdot)$ *is called the global* valuation *function. Formally,*

$$V(q) := \int_{x=0}^{q} v(x)dx.$$

Notice that v is a nonincreasing function since D is nonincreasing. It is easy to see that

$$v(q) = \begin{cases} D^{-1}(q) & \text{if } q \in (0, D(0)) \\ p_{\max} & \text{if } q = 0 \\ 0 & \text{if } q \geq D(0). \end{cases} \tag{2}$$

Consequently, the valuation function V is nondecreasing and concave. $V(q)$ measures the "value" that the service has for the whole population, since it is the total price that the q units of demand with highest marginal valuation (i.e., those that actually accept to pay the unit price $v(q)$) are willing to pay to be served.

Since perceived prices on both zones may be different, we introduce a new parameter (namely, the proportion of the population covered by zone B) to express separately the demand in each zone, still using the aggregated demand function D.

Definition 3. *Let us denote by α the proportion of the population in zone B. We consider that users' willingness-to-pay across sub-domains A and B are equidistributed. Therefore, total demand in zone A (resp., zone B) is $(1-\alpha)D(\bar{p}_A)$ (resp., $\alpha D(\bar{p}_B)$ if the perceived price on that zone is $\bar{p}_A$ (resp., $\bar{p}_B$).*

If users are uniformly distributed over the domain, α is simply the proportion of the surface covered by provider 2 with respect to provider 1, but it can be more general if we assume a non-uniform repartition.

Most of our results hold under the following assumption on the influence of price on demand.

Assumption A. *Demand function* D *is differentiable, and price elasticity of demand* $-\frac{D'(p)p}{D(p)}$ *is strictly larger than* 1 *for all* $p \in [\hat{p}, p_{\max})$*, with* $\hat{p} \leq \min\left(v\left(\frac{C_1}{1-\alpha}\right), v\left(\frac{C_2}{\alpha}\right)\right)$*.*

The price elasticity of demand measures the percentage change in demanded quantity implied by a percentage change in perceived price. Values larger than 1 (leading to *relatively elastic* demand in economic terms) correspond to a quite high reactivity to a perceived price change.

Under Assumption A, the function $p \mapsto pD(p)$ is strictly decreasing on $[\hat{p}, p_{\max})$; this is a typical assumption in telecommunications ($\hat{p} = 0$ is often considered, our assumption here is weaker), confirmed by operators[1]. This property will be used in this paper to characterize the Nash equilibrium of the pricing game.

Assumption A can be interpreted as follows: if all users in zone B always choose provider 2 (or equivalently, if both zones were disjoint), then both providers have an interest in setting a price such that all of their capacity is used. Indeed, otherwise the revenue of provider i covering a proportion α_i of the population is $\alpha_i p_i D(p_i)$, which is strictly decreasing in p_i, thus provider i should decrease its price to maximize its revenue.

2.3. Methodology

Our analysis of the pricing game is decomposed into three steps:

1. We first study how, for fixed prices p_i ($i \in \{1, 2\}$), total demand is split among providers. This is described and discussed in Section 4 in terms

[1] From discussions at Orange Labs.

of a Wardrop equilibrium. The output, also called user equilibrium, consists in a demand distribution $\mathbf{d} := (d_{1,A}, d_{1,B}, d_2)$. Notice that we do have to consider the two different zones, each one impacting the other, when computing that equilibrium. As we will see, we may end up with different perceived prices on the two different zones.

2. Knowing how demand is distributed for fixed prices, each provider $i \in \{1,2\}$ tries to maximize his revenue

$$R_i(p_1, p_2) := p_i d_i$$

by playing with the price charged to customers. The strategy of a provider has an impact on the demand distribution, and therefore on the revenue of the other. In Section 5 we determine the Nash equilibria for the price game. Recall here the definition of a Nash equilibrium when applied to our problem

Definition 4. *A* Nash equilibrium *is a price vector* $\mathbf{p}^* := (p_1^*, p_2^*)$ *such that no provider can increase his own benefit by unilaterally changing his access price, i.e.,* $\forall p \geq 0$*,*

$$R_1(p_1^*, p_2^*) \geq R_1(p, p_2^*) \text{ and } R_2(p_1^*, p_2^*) \geq R_2(p_1^*, p).$$

3. In the same section, we show that among all the Nash equilibria, there is one corresponding to the socially-optimal situation, so that there is no loss of efficiency due to competition. We actually argue that this equilibrium is the most likely if (even if negligible and not counted here) management costs are involved. Therefore the price of stability is one. We also compute the price of anrachy if the worst Nash equilibrium, in terms of social welfare, is chosen.

3. Social welfare and optimal value

We define here social welfare (SW) as the sum of utilities of all agents (customers plus providers) in our specific context, and then study the optimal value that can be obtained.

If we consider only zone B, $v(q)$ is the price a user would pay to buy the $\alpha \times q$-th unit since only a proportion α of the population is in that zone. A customer buying the q-th unit of resource in zone B is therefore willing to pay $v(q/\alpha)$ to be served. If total demand in zone B is $d_{1,B} + d_2$, then the total price that users in zone B are willing to pay is

$$\int_{x=0}^{d_{1,B}+d_2} v(x/\alpha)dx = \alpha V\left(\frac{d_{1,B}+d_2}{\alpha}\right).$$

However, the demand $d_{1,B} + d_2$ might not totally be served due to capacity limitations. Consequently, reasonably assuming that packet loss are independent of user willingness-to-pay, the value that the service has to zone B users should include the average transmission success probability in zone B: that overall value is then

$$\frac{d_{1,B}\pi_1 + d_2\pi_2}{d_{1,B}+d_2}\alpha V\left(\frac{d_{1,B}+d_2}{\alpha}\right),$$

where $\pi_1 := \min\left(1, \frac{C_1}{d_{1,B}+d_{1,A}}\right)$ and $\pi_2 := \min\left(1, \frac{C_2}{d_2}\right)$ are the transmission success probabilities with provider 1 and provider 2, respectively. Similarly, the total value that the service has for zone A users is

$$\pi_1(1-\alpha)V\left(\frac{d_{1,A}}{1-\alpha}\right),$$

leading to the following definition of social welfare.

Definition 5. *For a demand configuration* $(d_{1,A}, d_{1,B}, d_2)$*, social welfare (sum of utilities of all actors) is*

$$SW(d_{1,A}, d_{1,B}, d_2) = \pi_1(1-\alpha)V\left(\frac{d_{1,A}}{1-\alpha}\right) + \frac{d_{1,B}\pi_1 + d_2\pi_2}{d_{1,B}+d_2}\alpha V\left(\frac{d_{1,B}+d_2}{\alpha}\right), \tag{3}$$

where $\pi_1 := \min\left(1, \frac{C_1}{d_{1,B}+d_{1,A}}\right)$ *and* $\pi_2 := \min\left(1, \frac{C_2}{d_2}\right)$.

Remark that social welfare depends only on $(d_{1,A}, d_{1,B}, d_2)$, but not on prices paid by users since SW is the sum of the utilities of all actors: customers (with their willingness to pay minus price paid) and providers (with the revenue they get from prices).

From Definition 5, maximizing social welfare can be formally written as

$$\begin{aligned} &\max \mathrm{SW}(d_{1,A}, d_{1,B}, d_2) \\ s.t. \quad & d_{1,A} \geq 0, d_{1,B} \geq 0, d_2 \geq 0. \end{aligned} \tag{4}$$

We now solve that optimization problem.

Proposition 1. *The maximal value* SW^* *of social welfare is*

$$SW^* = \begin{cases} V(C_1 + C_2) & \text{if } \alpha C_1 \geq (1-\alpha)C_2, \\ (1-\alpha)V\left(\frac{C_1}{1-\alpha}\right) + \alpha V\left(\frac{C_2}{\alpha}\right) & \text{otherwise.} \end{cases}$$

The proof is provided in Appendix A.

4. Demand distribution

Let us now describe more clearly how demand distributes itself among providers. As in several other works where the number of users is large and no user has a significant weight with respect to the others [29, 30], we assume that users are infinitely small: their choices do not individually affect the demand levels (and therefore the perceived costs) of the different providers. Games involving infinitesimal users are called *nonatomic games* [31]. Under that nonatomicity assumption, an equilibrium among users follows Wardrop's principle [5] taken from road transportation: demand is distributed in such a way that all users choose the available provider with the least perceived price, and none if this perceived price is too expensive. That principle is formalized below.

Definition 6. *A Wardrop (or user) equilibrium is a triple* $(d_{1,A}, d_{1,B}, d_2)$ *that verifies the following system, where* $\bar{p}_i$ *stands for the perceived price at provider* $i \in \{1, 2\}$*.*

$$\bar{p}_1 = p_1 \max\left(1, \frac{d_{1,A} + d_{1,B}}{C_1}\right) \quad (5)$$

$$\bar{p}_2 = p_2 \max\left(1, \frac{d_2}{C_2}\right) \quad (6)$$

$$d_{1,A} \min\left(1, \frac{C_1}{d_{1,A} + d_{1,B}}\right) = (1-\alpha)D(\bar{p}_1) \quad (7)$$

$$d_{1,B} \min\left(1, \frac{C_1}{d_{1,A} + d_{1,B}}\right) + d_2 \min(1, C_2/d_2) = \alpha D(\min(\bar{p}_1, \bar{p}_2)) \quad (8)$$

$$\bar{p}_1 > \bar{p}_2 \;\Rightarrow\; d_{1,B} = 0 \quad (9)$$

$$\bar{p}_1 < \bar{p}_2 \;\Rightarrow\; d_2 = 0. \quad (10)$$

We now give the interpretations for those relations. (5) and (6) are simply (1) applied to provider 1 and 2, respectively. Relations (7) and (8) link demand (in terms of *effective* throughput, hence the multiplications by the success probabilities) to perceived prices in zones A and B. In zone B, where $100\alpha\%$ of the population is, the perceived price is $\bar{p}^B = \min(\bar{p}_1, \bar{p}_2)$. The other users (a proportion $100(1-\alpha)\%$) are in zone A, with perceived price $\bar{p}^A = \bar{p}_1$. As suggested in the definition of perceived prices per zone, the min in the right-hand side of (8) reflects the fact that users in zone B choose the cheapest provider (only provider 1 is available in zone A). Finally, relations (9) and (10) also represent user choices in zone B: if one provider is strictly more expensive than the other, then he gets no demand in that zone. An example of the situation faced by users is illustrated in Figure 3. To see how things happen in each zone, we artificially consider that demand

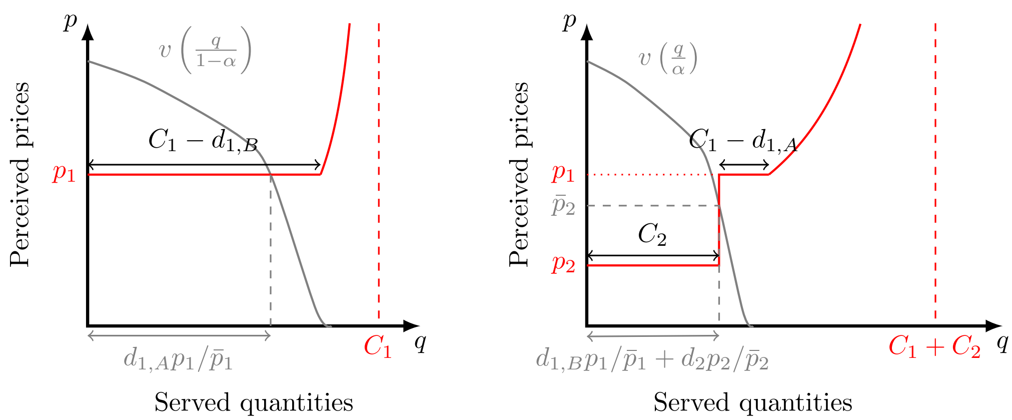

Figure 3: Demand repartition in zones A *(left)* and B *(right)*.

is fixed in the other zone. In zone A, while demand $d_{1,A} \leq C_1 - d_{1,B}$, then from (5) the perceived price is $\bar{p}_1 = p_1$. Then when $d_{1,A} > C_1 - d_{1,B}$ we get $\bar{p}_1 = p_1 \frac{d_{1,A}+d_{1,B}}{C_1}$. In that case, losses occur, so that for a fixed $d_{1,B}$, the perceived price to actually get a service rate q on zone A (i.e., because provider 1 is saturated, the total quantity served, C_1, is decomposed into $C_1 = q + d_{1,B}p_1/\bar{p}_1$ among the two zones) is $\bar{p}_1 = p_1 \frac{d_{1,B}}{C_1 - q}$. At a Wardrop equilibrium, from (7) in zone A the pair $(d_{1,A}, \bar{p}_1)$ is the (unique) intersection point of the functions $q \mapsto p_1 \max\left(1, \frac{d_{1,B}}{C_1-q}\right)$ and $q \mapsto v\left(\frac{q}{1-\alpha}\right)$.

In zone B, both providers are involved and users first choose the cheapest provider (here, provider 2) until it is saturated, then they continue choosing it, increasing the perceived price due to losses, until both providers have the same perceived price. Then some demand is served by provider 1 at a unit price p_1, until it gets saturated. Afterwards, if $d_{1,A}$ is fixed, then the perceived price in zone B to be served at a rate q is $\bar{p}_1 = \bar{p}_2 = p_1 \frac{d_{1,A}}{C_1+C_2-q}$. From (8), the pair $(d_{1,B} + d_2, \bar{p}_2)$ is the (unique) intersection point of that demand-price relation with the function $q \mapsto v(q/\alpha)$. In the example of Figure 3, we have $d_2 > C_2$ but $\bar{p}_2 < p_1$, thus $d_{1,B} = 0$ (i.e. all users in zone B choose provider 2 because demand is fulfilled in that zone before perceived price at provider 2 reaches p_1, the price at provider 1).

The difficulty of the Wardrop equilibrium is that both zones have to be combined: the demand $d_{1,A}$ in zone A must correspond to the values of $d_{1,B}$ and d_2 in zone B and vice-versa.

The following proposition gives insightful results about the existence and characterization of a Wardrop equilibrium.

Proposition 2. *For every price profile* (p_1, p_2) *with strictly positive prices, there exists at least a Wardrop equilibrium. Moreover, the corresponding perceived prices* $(\bar{p}_1, \bar{p}_2)$ *are unique.*

The proof is given in Appendix B.

Remark 1. *The uniqueness of perceived prices at a Wardrop equilibrium leads in most cases to a uniqueness of demands. Actually from (7) and (8),* $d_{1,A}$ *and* $d_{1,B}p_1/\bar{p}_1 + d_2p_2/\bar{p}_2$ *are unique. From (5) and (6), if* $\bar{p}_1 > p_1$ *or* $\bar{p}_2 > p_2$ *then demands are unique. Also, (9) and (10) imply that demands are also unique if* $\bar{p}_1 \neq \bar{p}_2$*. Therefore the only cases when demands might not be unique are when* $\bar{p}_1 = p_1 = p_2 = \bar{p}_2$*. Moreover, if* $d_1 + d_2 = C_1 + C_2$ *then demands are also unique (proof by contradiction: either* $d_1 = C_1$ *and* $d_2 = C_2$*, or from (1) one provider* $i \in \{1, 2\}$ *has* $\bar{p}_i > p_i$*). This will actually be the case for the Nash equilibrium of the pricing game: we will end up with* $d_1 + d_2 = C_1 + C_2$*, and a unique Wardrop equilibrium.*

We will see in the next section that even in a competitive context, situations with $\bar{p}_1 > \bar{p}_2$ can occur. In that case, all customers in zone B join provider 2, but the revenue that provider 1 gets from zone A exceeds what he could obtain by entering the price war on zone B.

5. Price war and Nash equilibrium

Knowing the above user equilibrium, we can discuss the pricing game between the two providers. Provider $i \in \{1, 2\}$ tries to maximize his revenue $R_i(p_1, p_2) = p_i d_i$ by playing with his price. Again, a price change modifies the Wardrop equilibrium, therefore the revenue of the competitor.

We give here a simple lemma regarding providers revenues.

Lemma 1. *For each provider* i*,* $i = 1, 2$*, we have at a Wardrop equilibrium* $R_i \leq \bar{p}_i C_i$*, and*

$$d_i \geq C_i \Leftrightarrow R_i = \bar{p}_i C_i. \tag{11}$$

As a consequence, we also have $\bar{p}_i > p_i \Rightarrow R_i = \bar{p}_i C_i$.

Proof: The lemma immediately follows from (5), (6) and the expressions of the revenues $R_i = p_i d_i$. □

We now show our main result, characterizing the set of Nash equilibria.

Proposition 3. *Under Assumption A, in the price war between providers there is a set of Nash equilibria $(0, p_1^*] \times (0, p_2^*]$ for the price profile (p_1, p_2), all yielding the same revenues $R_1^* = p_1^* C_1$ and $R_2^* = p_2^* C_2$. This set is characterized as follows.*

- *If $\frac{C_1}{1-\alpha} \leq \frac{C_2}{\alpha}$, that set of Nash equilibria is such that*

$$p_1^* = v\left(\frac{C_1}{1-\alpha}\right) \geq p_2^* = v\left(\frac{C_2}{\alpha}\right). \tag{12}$$

  We then have $d_{1,B} = 0$, meaning that zone B is left to provider 2 by provider 1.

- *If $\frac{C_1}{1-\alpha} > \frac{C_2}{\alpha}$, the set of Nash equilibria $(0, p_1^*] \times (0, p_2^*]$ is such that*

$$p_1^* = p_2^* = p^* = v(C_1 + C_2). \tag{13}$$

  In that case, zone B is shared by the providers.

The proof is given in Appendix C.

Remark 2. *The assumption $\frac{C_1}{1-\alpha} \leq \frac{C_2}{\alpha}$ means that the capacity per unit of surface for (the smaller-range) provider 2 is larger than that in the remaining area for provider 1. This can happen for fixed C_1 and C_2 if the proportion α of the common zone is small enough. As a consequence, at a Nash equilibrium,* *it is better for provider* 1 *to disregard potential revenue from zone* B*, and all users there go to the cheaper provider* 2*. We therefore end up with two monopolies in the different zones. On the other hand, if the assumption is not verified, zone* B *is too important for provider* 1*, and the price war is played. Both providers then share the area.*

Remark 3. *Among all the Nash equilibria, all yielding the same revenues, the price profile* (p_1^*, p_2^*) *is the one for which demand is the smallest, because price is the highest. We claim that it is the most likely situation since there is in this case less demand to manage, therefore less management costs, even if those costs are assumed negligible and not considered here. In the next proposition, we actually show that this equilibrium exactly corresponds to the socially-optimal situation.*

Corollary 4. *In this system, the Nash equilibrium* (p_1^*, p_2^*) *corresponds to the socially-optimal situation. As a consequence, the price of stability, defined as the best-case ratio comparing social welfare at the Nash equilibrium to the optimal value, is equal to one.*

Proof: This corollary is a direct consequence of the Nash equilibrium (p_1^*, p_2^*) demand repartition, that exactly corresponds to the socially optimum one computed in Section 3. □

However, if we consider any Nash equilibrium, then the performance of the system can be arbitrarily bad with respect to the socially optimal situation.

Corollary 5. *In this system, the price of anarchy is unbounded. Indeed, social welfare tends to* 0 *when the prices fixed by providers tend to* 0 *(that situation being a Nash equilibrium of the pricing game).*

Proof: As seen in the proof of Proposition 3, when prices (p_1, p_2) set by providers are sufficiently small then (p_1, p_2) is a Nash equilibrium, and $\bar{p}_i = p_i^*$ for p_i^* given in (12)-(13). From (5)-(6) and (7), this means that demands $d_{1,A}$ and d_2 tend to infinity. Now remark that due to the concavity and increasingness of V,

$$\lim_{x\to\infty} V(x)/x = \lim_{x\to\infty} v(x) = 0,$$

where the last equality is a consequence of D being bounded for strictly positive prices.

Consequently, using $d_i\pi_i \leq C_i$ in the social welfare expression, we have when prices tend to 0:

$$\mathrm{SW} \leq C_1 \underbrace{\frac{1-\alpha}{d_{1,A}} V\left(\frac{d_{1,A}}{1-\alpha}\right)}_{\to 0} + (C_1 + C_2) \underbrace{\frac{\alpha}{d_{1,B}+d_2} V\left(\frac{d_{1,B}+d_2}{\alpha}\right)}_{\to 0}, \tag{14}$$

which concludes the proof. □

6. Conclusion

In this paper, we have studied a pricing game between two wireless access providers, one of the two (say, with WiFi technology) operating only in a sub-area of the other (say, with WiMAX technology). Demand is driven by the perceived price, being the price charged per packet sent divided by the probability to be served (i.e. the average price per served unit). Users are assumed to choose the cheapest available provider, or none if both are too expensive. We have explained how demand is distributed according to Wardrop's principle. Knowing this distribution, providers play a pricing game in order to maximize their revenue. We have characterized explicitly all Nash equilibria for that game. Moreover, if the capacity per user offered in the WiFi hotzone exceeds the capacity per user of the WiMAX access in the remaining zone, then the WiMAX provider leaves the common area to the WiFi provider and only takes care of the region where he is the only provider available. Otherwise, the providers share the common area. A last contribution is to study whether competition brings a loss in terms of social welfare with respect to the cooperative case. We have shown that the price of stability, when looking at the Nash equilibrium yielding the largest welfare, is one, and remarked that this situation is actually a likely one.

As directions for future research, we plan to look at several issues. First, the case of more than two providers would be interesting to study, but much more complex. Adding demand uncertainty, and/or other externalities than loss probability such as delay [22], to the model could highlight more complex provider strategies and increase the price of stability. Also, considering that providers can not only play with their price but also with their capacity or the area they can reach would be of interest: in wireless networks, this could typically mean playing with the transmission power of the antennas (or base stations), similarly to [33]. Those points would help understand better the providers behavior in a competitive wireless network environment.

Acknowledgment

This work was supported by Euro-NF Network of Excellence and the French research agency through the CAPTURES project for both authors, and the WINEM project for the second author.

A. Proof of Proposition 1

Proof: We are looking for nonnegative values $(d_{1,A}^*, d_{1,B}^*, d_2^*)$ that maximize the objective function $\mathrm{SW}(d_{1,A}, d_{1,B}, d_2)$ given in (3). To do so, we prove two intermediate results.

a) We can take $d_2^* = C_2$, since SW is nondecreasing in d_2 if $d_2 \leq C_2$, and nonincreasing if $d_2 \geq C_2$.

Indeed, remark that only the second term in the sum in (3) depends on d_2, so we only focus on that term. Remark also that π_1 does not depend on d_2, while $\pi_2 = \min(1, C_2/d_2)$.

- If $d_2 \geq C_2$ then $\pi_2 = C_2/d_2$, so that the second term of SW is $\frac{d_{1,B}\pi_1 + C_2}{d_{1,B}+d_2} V\left(\frac{d_{1,B}+d_2}{\alpha}\right)$. The valuation function V being concave with $V(0) = 0$, $z \mapsto V(z)/z$ is nonincreasing, which implies SW being nonincreasing in d_2.

- If $d_2 \leq C_2$ then the second term of SW is $\frac{d_{1,B}\pi_1 + d_2}{d_{1,B}+d_2} V\left(\frac{d_{1,B}+d_2}{\alpha}\right)$, which is a product of two terms that are nondecreasing in d_2, since $\pi_1 \leq 1$ and V is nondecreasing.

b) We can also take $d_{1,A}^* + d_{1,B}^* = C_1$:

- If $d_{1,A} + d_{1,B} \leq C_1$ then $\pi_1 = 1$, so that only the first term in (3) depends on $d_{1,A}$. Thus SW is nondecreasing due to the nondecreasingness of V, and consequently we can consider that $d_{1,A}^* + d_{1,B}^* \geq C_1$.

- If $d_{1,A} + d_{1,B} \geq C_1$, then $\pi_1 = \frac{C_1}{d_1}$ with $d_1 = d_{1,A} + d_{1,B}$. We then define $\beta := \frac{d_{1,A}}{d_1}$, so that $d_{1,A} = \beta d_1$ and $d_{1,B} = (1-\beta)d_1$. The objective function SW can then be written

$$(1-\alpha)\frac{C_1}{d_1}V\left(\frac{\beta d_1}{1-\alpha}\right)+\frac{(1-\beta)C_1+d_2\pi_2}{(1-\beta)d_1+d_2}\alpha V\left(\frac{(1-\beta)d_1+d_2}{\alpha}\right),$$

where both terms in the sum are nonincreasing in d_1 because V is concave and $V(0)=0$.

As a result, we can find some nonnegative values $(d^*_{1,A}, d^*_{1,B}, d^*_2)$ maximizing SW, and that are such that $d^*_2 = C_2$ and $d^*_{1,A} + d^*_{1,B} = C_1$. Remark that $\pi_1 = \pi_2 = 1$ in that case. There just remains one parameter to find, say $d^*_{1,B}$ (since $d^*_{1,A} = C_1 - d^*_{1,B}$) to obtain the maximum value of social welfare. That value is thus the solution of the problem

$$\begin{aligned} \max_y \quad & f(y) \\ s.t. \quad & 0 \le y \le C_1, \end{aligned}$$

where $f(y) := (1-\alpha)V\left(\frac{C_1-y}{1-\alpha}\right) + \alpha V\left(\frac{C_2+y}{\alpha}\right)$ is differentiable. The marginal valuation function v being strictly decreasing, $f'(y) = v\left(\frac{C_2+y}{\alpha}\right) - v(\frac{C_1-y}{1-\alpha})$ verifies

- $f'(y) > 0 \Leftrightarrow \frac{C_1-y}{1-\alpha} > \frac{C_2+y}{\alpha}$, and
- $f'(y) < 0 \Leftrightarrow \frac{C_1-y}{1-\alpha} < \frac{C_2+y}{\alpha}$.

Using the constraints over y, function f reaches its maximum at $y = \max(0, \alpha C_1 - (1-\alpha)C_2)$.

We therefore have proved the proposition, by just inserting the values in the expression of social welfare. □

B. Proof of Proposition 2

Proof: A very general proof of the existence of a Nash equilibrium in nonatomic games (what we call here a Wardrop equilibrium) was provided by Schmeidler [32]. Therefore a solution to the system (5)-(10) exists.

We now establish the uniqueness of perceived prices at a user equilibrium. We will use the fact that (5) and (6) respectively imply

$$\begin{aligned} \min\left(1, \frac{C_1}{d_{1,A}+d_{1,B}}\right) &= \frac{p_1}{\bar{p}_1} \\ \text{and} \qquad \min\left(1, \frac{C_2}{d_2}\right) &= \frac{p_2}{\bar{p}_2}. \end{aligned}$$

Assume two user equilibria $(d_{1,A}, d_{1,B}, d_2)$ and $(\tilde{d}_{1,A}, \tilde{d}_{1,B}, \tilde{d}_2)$ with different perceived prices $(\bar{p}_1, \bar{p}_2)$ and $(\tilde{p}_1, \tilde{p}_2)$ exist for a given price profile (p_1, p_2), and suppose that $\tilde{p}_1 > \bar{p}_1$. Then (5) implies that

$$\tilde{d}_{1,A}\frac{p_1}{\tilde{p}_1} + \tilde{d}_{1,B}\frac{p_1}{\tilde{p}_1} = C_1 \geq d_{1,A}\frac{p_1}{\bar{p}_1} + d_{1,B}\frac{p_1}{\bar{p}_1}. \tag{15}$$

On the other hand, (7) yields[2] $\tilde{d}_{1,A}\frac{p_1}{\tilde{p}_1} < d_{1,A}\frac{p_1}{\bar{p}_1}$, therefore (15) gives

$$\tilde{d}_{1,B}\frac{p_1}{\tilde{p}_1} > \frac{p_1}{\bar{p}_1}d_{1,B}. \tag{16}$$

Thus $\tilde{d}_{1,B} > 0$, and from (9) we have $\tilde{p}_1 \leq \tilde{p}_2$. Now applying (8) twice gives

$$\tilde{d}_{1,B}\frac{p_1}{\tilde{p}_1} + \tilde{d}_2\frac{p_2}{\tilde{p}_2} = \alpha D(\tilde{p}_1) < \alpha D(\bar{p}_1) \leq \alpha D(\min(\bar{p}_1, \bar{p}_2)) = d_{1,B}\frac{p_1}{\bar{p}_1} + d_2\frac{p_2}{\bar{p}_2}.$$

Relation (16) then yields

$$\tilde{d}_2\frac{p_2}{\tilde{p}_2} < d_2\frac{p_2}{\bar{p}_2} \leq C_2, \tag{17}$$

[2] Notice that $D(\tilde{p}_1) > 0$, otherwise one can check that we would get $\tilde{d}_{1,A} = \tilde{d}_{1,B} = 0$, a contradiction with (15). Therefore D is strictly decreasing on $[\bar{p}_1, \tilde{p}_1]$ and $D(\tilde{p}_1) < D(\bar{p}_1)$.

where the second inequality comes from (6). This implies that $d_2 > 0$, and thus $\bar{p}_2 \leq \bar{p}_1$ from (10). Summarizing our results we get $\bar{p}_2 \leq \bar{p}_1 < \tilde{p}_1 \leq \tilde{p}_2$, thus $\tilde{p}_2 > \bar{p}_2$. From (6) this means $\tilde{d}_2 > C_2$ and therefore $\tilde{d}_2 \frac{p_2}{\tilde{p}_2} = C_2$, a contradiction with (17). Therefore the perceived price $\bar{p}_1$ is unique.

Likewise, knowing that $\tilde{p}_1 = \bar{p}_1$, assume $\tilde{p}_2 > \bar{p}_2$. Then from (6) we get $\tilde{d}_2 = C_2 \frac{\tilde{p}_2}{p_2}$. Therefore $\tilde{d}_2 > 0$, and (10) implies $\tilde{p}_2 \leq \tilde{p}_1 = \bar{p}_1$, thus

$$\bar{p}_2 < \bar{p}_1. \tag{18}$$

Now applying (8) we obtain[3]

$$\tilde{d}_{1,B} \frac{p_1}{\tilde{p}_1} + \underbrace{\tilde{d}_2 \frac{p_2}{\tilde{p}_2}}_{=C_2} = \alpha D(\tilde{p}_2) < \alpha D(\bar{p}_2) \leq \alpha D(\min(\bar{p}_1, \bar{p}_2)) = d_{1,B} \frac{p_1}{\bar{p}_1} + \underbrace{d_2 \frac{p_2}{\bar{p}_2}}_{\leq C_2},$$

therefore $d_{1,B} > 0$, and thus $\bar{p}_1 \leq \bar{p}_2$ from (9), which is a contradiction with (18) and proves the uniqueness of $\bar{p}_2$. □

C. Proof of Proposition 3

We distinguish the two cases that appeared when computing the welfare maximizing situation.

C.1. Case $\alpha C_1 \leq (1-\alpha)C_2$

Lemma 2. *Consider that Assumption A holds, and assume that $\alpha C_1 \leq (1-\alpha)C_2$. For any price $p_1 > 0$, any price $p_2 \in (0, p_2^*]$ ensures provider 2 a revenue $R_2 = p_2^* C_2$, while any other price yields a strictly lower revenue.*

[3]Again, since $\tilde{d}_2 > 0$, from (8) we are in the zone where D is strictly positive, thus strictly decreasing.

Proof: We consider a strictly positive price p_1 and a price $p_2 \in (0, p_2^*]$, and we proceed in several steps to prove that $R_2 = p_2^* C_2$.

1. We have $\bar{p}_1 \geq p_2^*$: if not, (5) and (7) would imply

$$C_1 \geq d_{1,A} \frac{p_1}{\bar{p}_1} = (1-\alpha) D(\bar{p}_1) > (1-\alpha) D(p_2^*) \geq (1-\alpha) D(p_1^*) = C_1,$$

   a contradiction.

2. Also, $\bar{p}_2 \geq p_2^*$: otherwise from step 1 and (9) we would have $d_{1,B} = 0$, and (8) and (6) would give

$$C_2 \geq d_2 \frac{p_2}{\bar{p}_2} = \alpha D(\bar{p}_2) > \alpha D(p_2^*) = C_2,$$

   another contradiction.

3. But on the other hand, $\bar{p}_2 \leq p_2^*$: otherwise we would have $\bar{p}_2 > p_2$, and thus $d_2 p_2 / \bar{p}_2 = C_2$ from (6). Then (10) would yield $\bar{p}_1 \geq \bar{p}_2$, and applying (8) would give

$$d_{1,B} \frac{p_1}{\bar{p}_1} + C_2 = \alpha D(\bar{p}_2) < \alpha D(p_2^*) = C_2,$$

   another contradiction. As a consequence of this and of previous result, $\bar{p}_2 = p_2^*$.

4. Finally, $R_2 = p_2^* C_2$: we use results from the previous steps, and distinguish two cases.

   - if $\bar{p}_1 = p_2^*$, then adding (7) and (8) gives

$$\underbrace{d_1 \frac{p_1}{\bar{p}_1}}_{\leq C_1 \text{ from (5)}} + \underbrace{d_2 \frac{p_2}{p_2^*}}_{\leq C_2 \text{ from (6)}} = \alpha D(p_2^*) + (1-\alpha) D(p_2^*) \geq C_1 + C_2,$$

     thus all inequalities are equalities, and in particular $d_2 p_2 = p_2^* C_2$.

- if $\bar{p}_1 > p_2^*$ then (9) and (8) directly give $p_2 d_2 = p_2^* C_2$.

Now we prove that any price $p_2 > p_2^*$ corresponds to a revenue $R_2 < p_2^* C_2$. From (10):

- either $d_2 = 0$, and therefore $R_2 = 0$;
- or $\bar{p}_2 \leq \bar{p}_1$, which from (8) implies that

$$d_2 p_2 / \bar{p}_2 \leq \alpha D(\bar{p}_2) \leq \alpha D(p_2) < \alpha D(p_2^*) = C_2,$$

  and from (6) yields $\bar{p}_2 = p_2$. Then applying (8) again, we have $d_2 \leq \alpha D(p_2)$, and

$$R_2 \leq \alpha p_2 D(p_2) < \alpha p_2^* D(p_2^*) = p_2^* C_2,$$

  where the last inequality comes from Assumption A. This concludes the proof.

□

Lemma 3. *Consider that Assumption A holds. For any fixed price* $p_2 \in (0, p_2^*]$*, any price* $p_1 \in (0, p_1^*]$ *ensures provider* 1 *a revenue* $R_1 = p_1^* C_1$*, while any other price yields a strictly lower revenue.*

Proof: Fix $p_2 \in (0, p_2^*]$. As seen in the proof of Lemma 2 we have $\bar{p}_2 = p_2^*$ and $d_2 p_2 / \bar{p}_2 = C_2$ whatever the value of p_1, which from (10) gives $\bar{p}_2 \leq \bar{p}_1$, and from (8) implies that $d_{1,B} = 0$. As a result, the total demand for provider 1 is in zone A, and is given by (7). Then,

- If provider 1 sets $p_1 \leq p_1^*$, then (5) and (7) give

$$\underbrace{C_1}_{(1-\alpha)D(p_1^*)} \geq d_1 \frac{p_1}{\bar{p}_1} = (1-\alpha) D(\bar{p}_1), \tag{19}$$

thus $\bar{p}_1 \geq p_1^*$. But actually $\bar{p}_1 = p_1^*$, otherwise (5) and (7) would give

$$C_1 = d_1 \frac{p_1}{\bar{p}_1} = (1-\alpha)D(\bar{p}_1) < (1-\alpha)D(p_1^*) = C_1,$$

a contradiction. As a result, from (19) we have $R_1 = p_1 d_1 = p_1^* C_1$.

• If provider 1 sets $p_1 > p_1^*$, then $\bar{p}_1 > p_1^*$ from (5), and (7) yields

$$d_1 \frac{p_1}{\bar{p}_1} = (1-\alpha)D(\bar{p}_1) > (1-\alpha)D(p_1^*) = C_1,$$

thus from (5), $d_1 < C_1$ and $\bar{p}_1 = p_1$. As a result, (5) implies

$$d_1 p_1 = (1-\alpha)p_1 D(p_1) < (1-\alpha)p_1^* D(p_1^*) = p_1^* C_1,$$

where the inequality comes from Assumption A. □

The fact that any price (p_1, p_2) with $p_i \in (0, p_i^*], i = 1, 2$ is a Nash equilibrium of the price game is a direct consequence of Lemmas 2 and 3.

C.1.1. Case $\alpha C_1 > (1-\alpha)C_2$

Recall that in that case we have

$$\frac{C_2}{\alpha} < D(p^*) = C_1 + C_2 < \frac{C_1}{1-\alpha}. \tag{20}$$

Lemma 4. *Consider that Assumption A holds. All price profiles $(p_1, p_2) \in (0, p^*]^2$ form a Nash equilibrium. Those profiles are the only Nash equilibria of the pricing game, and the corresponding revenue for each provider $i = 1, 2$ is $R_i = p^* C_i$.*

Proof: The proof follows three steps:

1. We first prove that when both providers set a price below p^* then each provider k gets a revenue $p^* C_k$.

2. Then we show that if only one provider j sets a price $p_j > p^*$ he gets a strictly smaller revenue, while his opponent i gets at least the same revenue $p^* C_i$.

3. Finally, we prove that if both providers were to set a price strictly above p^*, then at least one provider i would obtain strictly less than $p^* C_i$, and thus from the previous point he would be better off reducing his price below p^*.

Step 1. Consider a price profile (p_1, p_2) with $p_k \in (0, p^*], k = 1, 2$. Then adding (7) and (8) gives

$$\underbrace{d_1 \frac{p_1}{\bar{p}_1}}_{\leq C_1 \text{ from (5)}} + \underbrace{d_2 \frac{p_2}{\bar{p}_2}}_{\leq C_2 \text{ from (6)}} \geq D(\bar{p}_1),$$

thus $\bar{p}_1 \geq p^*$, due to the nonincreasingness of D. Now, we also have $\bar{p}_2 \geq p^*$, otherwise (9) would imply $d_{1,B} = 0$, and (6) and (8) would give

$$C_2 \geq d_2 \frac{p_2}{\bar{p}_2} = \alpha D(\bar{p}_2) > \alpha D(p^*) > C_2,$$

a contradiction. Now we prove that we actually have $\bar{p}_1 = \bar{p}_2 = p^*$. Assume $\bar{p}_1 > p^*$: then $d_1 p_1 / \bar{p}_1 = C_1$ from (5), and adding (7) and (8) would give

$$C_1 + d_2 \frac{p_2}{\bar{p}_2} < D(p^*) = C_1 + C_2,$$

thus $d_2 p_2 / \bar{p}_2 < C_2$, and $\bar{p}_2 = p_2$ from (6). Since $p_2 \leq p^*$ and $\bar{p}_2 \geq p^*$, we would have $\bar{p}_2 = p^*$. But then (10) and (7) would imply

$$C_1 = (1 - \alpha) D(\bar{p}_1) < (1 - \alpha) D(p^*) < C_1,$$

a contradiction. We therefore have $\bar{p}_1 = p^*$, which implies $\bar{p}_2 \leq p^*$ (otherwise (10) and (6) would lead to a contradiction). Summarizing, we have $\bar{p}_1 = \bar{p}_2 = p^*$. Adding again (7) and (8) now gives

$$\underbrace{d_1 \frac{p_1}{p^*}}_{\leq C_1 \text{ from (5)}} + \underbrace{d_2 \frac{p_2}{p^*}}_{\leq C_2 \text{ from (6)}} = C_1 + C_2,$$

implying $d_1 p_1 / p^* = C_1$ and $d_2 p_2 / p^* = C_2$, which gives $d_k p_k = p^* C_k$ for $k = 1, 2$ and establishes the first step of the proof.

Step 2. Now consider a provider i setting $p_i \in (0, p^*]$, while his opponent j sets a price $p_j > p^*$. Then we prove that $R_j < p^* C_j$ and $R_i \geq p^* C_i$.

- If $\bar{p}_i < \bar{p}_j$, then using (9) or (10), and adding (7) and (8), we have

$$\underbrace{d_i \frac{p_i}{\bar{p}_i}}_{\leq C_i \text{ from (1)}} = D(\bar{p}_i).$$

  Thus $\bar{p}_i \geq v(C_i) > p^* \geq p_i$, and therefore from Lemma 1, $R_i = \bar{p}_i C_i > p^* C_i$. To study R_j we distinguish two cases.

  – Case $i = 1$: (10) directly gives $R_2 = 0$;

  – Case $i = 2$: (9) implies $d_{1,B} = 0$, and (7) yields

$$d_1 p_1 = (1 - \alpha) \bar{p}_1 D(\bar{p}_1) < (1 - \alpha) p^* D(p^*) < p^* C_1,$$

  where the first inequality comes from Assumption A, and the second one from (20).

- If $\bar{p}_i \geq \bar{p}_j$, then from (1) this implies $d_i p_i / \bar{p}_i = C_i$ and directly gives $R_i > p^* C_i$. Moreover, adding (7) and (8) we obtain

$$d_j \frac{p_j}{\bar{p}_j} + C_i \leq D(\bar{p}_j),$$

which gives

$$d_j p_j \leq \bar{p}_j (D(\bar{p}_j) - C_i) < p^* (D(p^*) - C_i) = p^* C_j,$$

where we used Assumption A and the fact that $\bar{p}_j > p^*$.

This concludes the second step of the proof.

Step 3. Assume now that both providers set a price strictly above p^*. We index the providers such that $\bar{p}_i \geq \bar{p}_j$ at the Wardrop equilibrium.

- If $\bar{p}_i > \bar{p}_j$ then $R_i < p^* C_i$ for the same reasons as in the previous step ($R_i = 0$ if $i = 2$, and $R_i = (1 - \alpha) p^* D(p^*) < p^* C_1$ if $i = 1$).

- If $\bar{p}_i = \bar{p}_j$, then adding (7) and (8) we have

$$d_i \frac{p_i}{\bar{p}_i} + d_j \frac{p_j}{\bar{p}_j} = \frac{R_i + R_j}{\bar{p}_i} = D(\bar{p}_i),$$

and from Assumption A we obtain

$$R_i + R_j = \bar{p}_i D(\bar{p}_i) < p^* D(p^*) = p^* (C_i + C_j),$$

which implies that either $R_i < p^* C_i$, or $R_j < p^* C_j$.

Those three steps completely characterize the Nash equilibria of the pricing game: if a provider (or both) sets his price strictly above p^* then at least one provider is strictly better off reducing his price, while when both providers set their price below p^* no provider can strictly improve his revenue by a unilateral price change. □

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