Multi-Bid Auctions for Bandwidth Allocation in Communication Networks

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***Abstract*— In this paper, we design a bandwidth pricing mechanism that solves congestion problems in communication networks. The scheme is based on second-price auctions, which are known to be incentive compatible when a single indivisible item is to be sold (users have no interest to lie about the price they are willing to pay for the resource) and to lead to an efficient allocation of resources in the sense that it maximizes social welfare. We prove these properties when an infinitely divisible resource (bandwidth on a communication link) is to be shared among users who are allowed to submit several bids when they want to establish a connection. Our scheme is highly related to the Progressive Second Price Auction of Lazar and Semret where players bid sequentially until an (optimal) equilibrium is reached. While keeping their incentive compatibility and efficiency properties, our scheme presents the advantage that the multi-bid is submitted once only, saving a lot of signalization overhead.**

***Index Terms*—Control theory, Economics**

I. Introduction

The demand for bandwidth in communication networks has been growing exponentially since the birth of worldwide networks, for the number of consumers has been soaring, and new applications, more and more bandwidth-needing (like video for instance), have been appearing. As a result, despite the efforts made to increase communication rates, the available capacities are often insufficient to satisfy all service requests, and situations of congestion occur frequently, meaning that some users' requests are rejected.

Currently, the pricing scheme for Internet communications, based on a fixed charge independent of use, does not take into account the negative externalities among users (a user consuming bandwidth may prevent another request from being treated successfully), and thus constitutes an incentive to overuse the network [1]. Designing new allocation and pricing schemes therefore appears as a solution for solving congestion problems, by inciting users to limit their consumption. Consequently, the pricing of network services is encountering a great interest, and many papers have been published on the subject (see [2], [3], [4], [5] and references therein).

Pricing network resources can have two different goals: reaching a maximum revenue for the network, or allocating efficiently the resource. We concentrate on the latter objective in this paper: taking into account the willingness-to-pay of users and considering efficiency as the criterion to optimize, the resource should go to people who value it most. In [6] (see also the extensions in [7], [8]), Lazar and Semret introduce the Progressive Second Price (PSP) Mechanism, an iterative auction scheme that allocates bandwidth on a single communication link[1] among users in a set $\mathcal{I}$. Players submit two-dimension bids of the form $s_i = (q_i, p_i)$, where q_i is the quantity of resource (bandwidth) asked by user (player) i, and p_i is the unit price that player i is willing to pay to obtain that quantity of resource. The allocations and prices to pay are then computed, based on the bids $s = (s_i)_{i\in\mathcal{I}}$ submitted by all the players. Users can modify their bid, knowing the bid submitted by the others, until an equilibrium is reached. Lazar and Semret model the users' preferences by a quasi-linear utility function, which is the difference between the price that player i is willing to pay for the quantity a_i he receives (his valuation, represented by a function θ_i) and the price c_i that he is charged for this quantity:

$$u_i(s) = \theta_i(a_i(s)) - c_i(s). \tag{1}$$

Lazar and Semret highlight the *incentive compatibility* property of their mechanism, i.e. bidding at each iteration with a price p_i equal to the marginal valuation $\theta_i'(q_i)$ brings the highest utility to player i. They prove that if

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[1] In [4], [9], an application of the PSP mechanism to the network case is proposed. However in this paper we focus on the single-link case.

players are informed of the other players' bids when they submit their own bids, the bid profile s converges after a finite time to a Nash equilibrium that corresponds to an efficient allocation of the resource.

The main drawback of this scheme is that the convergence phase can be quite long, and that it corresponds to a signaling burst (to send the necessary information to players) which may represent a non-negligible part of the available bandwidth. This is especially true if players are assumed to randomly enter or leave the game with time as in [10], meaning that the convergence phase will be repeated at each change of the set $\mathcal{I}$ of players.

The mechanism was modified by Delenda, who proposed in [11] a one-shot scheme: players are asked to submit their demand function, and the auctioneer directly computes the allocations and prices to pay without any convergence phase. The mechanism described in [11] is the continuous version of the Generalized Vickrey Auction (see [12], [13]). It is a *direct revelation* auction mechanism, meaning that players have to give their whole valuation function in their bid. However, communicating a general function is not feasible in practice, so there remains a signaling problem. Delenda suggests that only a finite number of demand functions be proposed, and that players choose among them. Nevertheless, this scheme supposes that the auctioneer has a idea of what the demand functions of users could be.

In this paper, we suggest an intermediate mechanism, which is still one-shot, but which does not suppose any knowledge about the demand functions. As in [4], we consider quasi-linear utility functions of the form (1), but we allow here players to submit several two-dimension bids (q_i, p_i) like in [14], and use an allocation and pricing scheme that is close to the one described in [11]. This mechanism will be called *multi-bid auction* scheme.

The paper is organized as follows. Section II describes in details the multi-bid scheme and gives the allocation and pricing rules. In Sections III through VI, we study this scheme as a *non-cooperative game* (see [15]) and establish some of its properties. Section III presents some basic and desirable properties of the allocation rule (monotony of the allocation with respect to the multi-bid, unit-cost higher than the reserve price, more players will increase the network revenue, individual rationality). We also prove in Section IV that this scheme is *incentive compatible*, in the sense that players have no interest in lying about their valuations when submitting their bids. We focus in Section V on the bid choice problem from the point of view of a user and argue that the players have an interest in choosing their bids as quantiles of their valuation function in order to minimize the difference between the real valuation function and the one derived from the multi-bids. We then prove the efficiency of the scheme, in terms of social welfare, in Section VI. Section VII is devoted to the determination of the number of bids that the network should allow each user to submit, in order to maximize its benefit. Finally, conclusions and future works are presented in Section VIII.

To derive our theoretical results, we will assume as in [4] that users have elastic demand such that

Assumption 1: For any $i \in \mathcal{I}$,
– θ_i is differentiable and $\theta_i(0) = 0$,
– θ_i' is positive, non-increasing and continuous
– $\exists \gamma_i > 0, \forall z \geq 0, \theta_i'(z) > 0 \Rightarrow \forall \eta < z, \theta_i'(z) \leq \theta_i'(\eta) - \gamma_i(z - \eta)$.

II. MULTI-BID AUCTIONS: ALLOCATION AND PRICING RULES

Let us consider a bottleneck communication link with available bandwidth (capacity) Q. We assume that this resource is infinitely divisible, and study a scheme to share it among all users.

Our goal here is to change the sequential (dynamic) bid process of [4], [6] into a one-shot multiple bid for each player in order to alleviate the bid-profile signalization overhead. Before the presentation of the allocation and charging rules in sub-sections II-C and II-D, we need to introduce the message process as well as some notations and basic definitions.

- When a player i enters the game (i.e. establishes a connection), he submits a set of M_i two-dimension bids $s_i = \{s_i^1, ..., s_i^{M_i}\}$, where for all $m, 1 \leq m \leq M_i, s_i^m = (q_i^m, p_i^m)$ as defined in the Progressive Second Price mechanism (see [6]): q_i^m represents a quantity of resource and p_i^m the unit price that player i is willing to pay to get this quantity. We assume without loss of generality that bids are sorted such that $p_i^1 \leq p_i^2 \leq ... \leq p_i^{M_i}$.
  Remark: Player i may submit no bid ($M_i = 0$). In this case we write $s_i = \emptyset$.
  In this paper, S denotes the set of multi-bids that a player can submit:
  $$S = \bigcup_{M \geq 0} \left(\mathbb{R}^+ \times \mathbb{R}^+\right)^M, \quad \text{with } \left(\mathbb{R}^+ \times \mathbb{R}^+\right)^0 = \emptyset.$$
- The auctioneer collects all multi-bids to form the *multi-bid profile* $s = (s_i)_{i \in \mathcal{I}}$. This profile will be used to compute the allocation a_i and the total price charged c_i for each player $i \in \mathcal{I}$.

Notice that, unlike in the PSP mechanism, we do not suppose here that players know the bids submitted by the others before bidding.

A. Reserve price p_0

Our model allows the auctioneer to fix a unit price $p_0 \geq 0$ under which she prefers not to sell the resource. This is equivalent to considering that the auctioneer may use the resource if it is not sold, with a valuation function $\theta_0(q) = p_0 q$. In the following, the auctioneer will be denoted player 0 ($0 \notin \mathcal{I}$), and p_0 will be called the *reserve price*. We suppose in this paper that this reserve price is known by all players.

We thus assume that a bid $s_0 = (q_0, p_0)$, with $q_0 > Q$ is introduced (Q is the total available capacity). Therefore, the set of bids that the auctioneer may submit is

$$S_0 = (Q, +\infty) \times \mathbb{R}^+.$$

Note that we have $M_0 = 1$, and $p_0^1 = p_0$.

B. Pseudo-demand function, pseudo-market clearing price

In this sub-section, we provide some definitions that will be helpful to understand the behaviour of the mechanism.

Definition 1: A player $i \in \mathcal{I}$ is said to submit a *truthful* multi-bid $s_i \in S$ if $s_i = \emptyset$, or if

$$\forall m, 1 \leq m \leq M_i,\ p_i^m = \theta_i'(q_i^m).$$

We write S_i^T the set of truthful multi-bids that can be submitted by player i. We also denote

$$S_i^T(p_0) = \{\emptyset\} \cup \{s \in S_i^T : p_i^1 \geq p_0\} \tag{2}$$

the set of truthful multi-bids for which all prices are above the reserve price.

Definition 2: Under Assumption 1, we define the *demand function* of player $i \in \mathcal{I}$ as the function $d_i(p) = (\theta_i')^{-1}(p)$ if $0 < p \leq \theta_i'(0)$ and 0 otherwise: $d_i(p)$ is the quantity player i would buy if the resource were sold at the unit price p, in order to maximize his utility.

Note that Assumption 1 implies that the demand function is non-increasing.

Definition 3: Consider a player $i \in \mathcal{I} \cup \{0\}$ having submitted a multi-bid $s_i \in S$.

We call *pseudo-demand function* of i associated with s_i the function $\bar{d}_i : \mathbb{R}^+ \to \mathbb{R}^+$, defined by

$$\bar{d}_i(p) = \begin{cases} 0 & \text{if } s_i = \emptyset \text{ or } p_i^{M_i} < p \\ \max\limits_{1 \leq m \leq M_i} \{q_i^m : p_i^m \geq p\} & \text{otherwise.} \end{cases} \tag{3}$$

Definition 4: Consider a player $i \in \mathcal{I} \cup \{0\}$, and $s_i \in S$ a multi-bid submitted by i. We call *pseudo-marginal valuation function* of i, associated with s_i, the function $\bar{\theta}_i' : \mathbb{R}^+ \to \mathbb{R}^+$, defined by

$$\bar{\theta}_i'(q) = \begin{cases} 0 & \text{if } s_i = \emptyset \text{ or } q_i^1 < q \\ \max\limits_{1 \leq m \leq M_i} \{p_i^m : q_i^m \geq q\} & \text{otherwise.} \end{cases} \tag{4}$$

Remark: The auctioneer's bid leads to the pseudo-demand function $\bar{d}_0(p) = q_0 \mathbb{1}_{p \leq p_0}$ and the pseudo-marginal valuation function $\bar{\theta}_0'(q) = p_0 \mathbb{1}_{q \leq q_0}$.

The demand, pseudo-demand, marginal valuation and pseudo-marginal valuation functions are illustrated in Fig. 1, with truthful bids.

Remark: Both pseudo-demand and pseudo-marginal valuation functions are positive, stair-step, non-increasing and left-continuous.

We now derive a property stating that the pseudo-demand and pseudo-marginal valuation functions are smaller than their "real" counterparts:

Lemma 1: Under Assumption 1, if player $i \in \mathcal{I}$ submits a truthful multi-bid s_i then

$$\bar{d}_i \leq d_i \tag{5}$$
$$\bar{\theta}_i' \leq \theta_i'. \tag{6}$$

Proof: Let $x \in \mathbb{R}^+$. If $\bar{d}_i(x) = 0$ then $\bar{d}_i(x) \leq d_i(x)$ is trivial, since $d_i \geq 0$. If we assume that $\bar{d}_i(x) > 0$, then $s_i \neq \emptyset$ and

$$\begin{aligned} \bar{d}_i(x) &= \max_{1 \leq m \leq M_i} \{q_i^m : p_i^m \geq x\} \\ &= q_i^{m_0} \text{ with } p_i^{m_0} \geq x \\ &= d_i(p_i^{m_0}) \leq d_i(x) \end{aligned}$$

where the non-increasingness of d_i is used. Relation (5) is then proved.

Relation (6) is established exactly the same way by inverting the roles of prices and quantities. ∎

Fig. 1 illustrates this result.

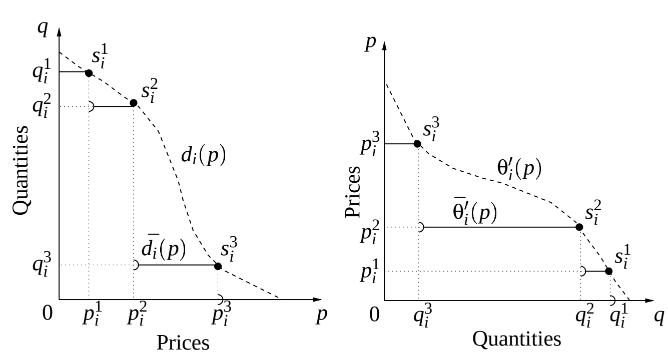

Fig. 1. Demand and pseudo-demand functions *(left)*, marginal valuation and pseudo-marginal valuation functions *(right)* for $M_i = 3$ and a truthful multi-bid.

Definition 5: Consider a set of players $i \in \mathcal{I}$, each submitting a multi-bid $s_i \in S$. We call *agregated pseudo-demand function* associated with the profile $s = (s_i)_{i\in\mathcal{I}\cup\{0\}}$ the function $\bar{d} : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\bar{d}(p) = \sum_{i\in\mathcal{I}\cup\{0\}} \bar{d}_i(p). \quad (7)$$

When the objective of the allocation problem is to maximize the *efficiency* $\sum_{i\in\mathcal{I}\cup\{0\}} \theta_i(a_i)$, it can be proved that, under Assumption 1, the optimal allocation is such that $\forall i \in \mathcal{I}, a_i = d_i(u)$, where u is the *market clearing price*, i.e. the unique price such that $\sum_{i\in\mathcal{I}} d_i(u) = Q$ if $\sum_{i\in\mathcal{I}} d_i(p_0) > Q$ and p_0 otherwise [16].

Remark: Note that the efficiency measure that we use corresponds to the usual *social welfare* criterion $\sum_i u_i(s)$ (see [4]): since the utility of the seller is $u_0(s) = \theta_0(a_0(s)) + \sum_{i\in\mathcal{I}} c_i(s)$, we have

$$\begin{aligned}\sum_i u_i(s) &= u_0(s) + \sum_{i\in\mathcal{I}} u_i(s)\\ &= \theta_0(a_0(s)) + \sum_{i\in\mathcal{I}} c_i(s) + \sum_{i\in\mathcal{I}} (\theta_i(a_i(s)) - c_i(s))\\ &= \sum_i \theta_i(a_i(s)).\end{aligned}$$

Here the auctioneer cannot compute the market clearing price, for she does not know the agregated demand function. Nevertheless, she can estimate the clearing price thanks to the agregated pseudo-demand.

Definition 6: Consider a multi-bid profile $s = (s_i)_{i\in\mathcal{I}\cup\{0\}}$. Denoting by $\bar{d}$ the agregated pseudo-demand function associated with this profile, we define the *pseudo-market clearing price* $\bar{u}$ by

$$\bar{u} = \sup\left\{p : \bar{d}(p) > Q\right\}. \quad (8)$$

Such a $\bar{u}$ always exists since $\bar{d}(0) \geq \bar{d}_0(0) = q_0 > Q$. Moreover $\bar{d}(p) = 0 \; \forall p > \max_{i\in\mathcal{I}\cup 0}(p_i^{M_i})$, which implies that $\bar{u} < +\infty$.

Remark: As $\bar{d}$ is a left-continuous stair-step function, the sup in (8) is actually a max. Thus we have

$$\bar{d}(\bar{u}) > Q. \quad (9)$$

Fig. 2 shows an example of an agregated pseudo-demand function and a pseudo-market clearing price.

C. Allocation rule

Given all these definitions, we are now ready to describe the allocation rule.

For every function $f : \mathbb{R} \to \mathbb{R}$ and all $x \in \mathbb{R}$, we define

$$f(x^+) = \lim_{z\to x, z>x} f(x) \quad (10)$$

when this limit exists.

We suggest that the resource be allocated the following way: if player i submits the multi-bid s_i (and thereby declares the associated functions $\bar{d}_i$ and $\bar{\theta}_i'$) then he receives a quantity $a_i(s_i, s_{-i})$, with

$$a_i(s_i, s_{-i}) = \bar{d}_i(\bar{u}^+) + \frac{\bar{d}_i(\bar{u}) - \bar{d}_i(\bar{u}^+)}{\bar{d}(\bar{u}) - \bar{d}(\bar{u}^+)}(Q - \bar{d}(\bar{u}^+)), \quad (11)$$

where $\bar{d}$ is the agregated pseudo-demand function associated with the bid profile s.

In other words:

- each player receives the quantity he asks at the lowest price $\bar{u}^+$ for which supply excesses pseudo-demand.
- If all the resource is not allocated yet, the surplus $Q - \bar{d}(\bar{u}^+)$ is shared among players who submitted a bid with price $\bar{u}$. This share is done with weights proportional to the "hops" of the pseudo-demand functions $\bar{d}_i(\bar{u}) - \bar{d}_i(\bar{u}^+)$.

Remark: We know from (9) that the denominator in (11) is always strictly positive, since $\bar{d}(\bar{u}^+) \leq Q$ and $\bar{d}(\bar{u}) > Q$. Consequently $0 \leq \frac{Q - \bar{d}(\bar{u}^+)}{\bar{d}(\bar{u}) - \bar{d}(\bar{u}^+)} \leq 1$ and we have

$$\bar{d}_i(\bar{u}^+) \leq a_i(s) \leq \bar{d}_i(\bar{u}). \quad (12)$$

Remark: As noticed by Tuffin in [7], the PSP allocation rule does not always allocate all the bandwith even if demand exceeds supply. Here, the term $\frac{\bar{d}_i(\bar{u}) - \bar{d}_i(\bar{u}^+)}{\bar{d}(\bar{u}) - \bar{d}(\bar{u}^+)}(Q - \bar{d}(\bar{u}^+))$ ensures that all the resource is allocated (see Property 3).

D. Pricing rule

Each player $i \in \mathcal{I}$ is charged a *total price* $c_i(s)$, where

$$c_i(s_i, s_{-i}) = \sum_{j\in\mathcal{I}\cup\{0\}, j\neq i} \int_{a_j(s)}^{a_j(s_{-i})} \bar{\theta}_j'(q)dq. \quad (13)$$

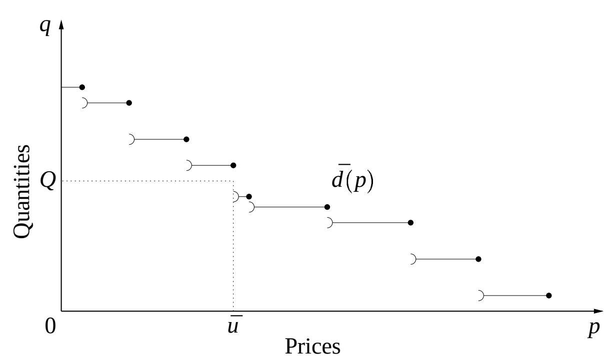

Fig. 2. The pseudo-market clearing price $\bar{u}$

The intuition behind this pricing rule is an *exclusion-compensation* principle, which lies behind all second-price mechanisms [17]: player i pays so as to cover the "social opportunity cost", that is to say the loss of utility he imposes on all other users by his presence. Actually, applying directly this principle would lead to $c_i = \sum_{j\in\mathcal{I}\cup\{0\},j\neq i}\theta_j(a_j(s_{-i})) - \theta_j(a_j(s)) = \sum_{j\in\mathcal{I}\cup\{0\},j\neq i}\int_{a_j(s)}^{a_j(s_{-i})}\theta_j'$, as in [11]. However, in our case the auctioneer does not know the valuation functions θ_j nor the marginal valuation functions $\theta_j', j\in\mathcal{I}$. That is the reason why we use the pseudo-marginal valuation functions instead, that are computed thanks to submitted bids.

Note here that we do not define c_0, since we consider that the auctioneer cannot buy resource to herself.

We now give a lemma that will be used in the rest of the paper.

Lemma 2: $\forall i\in\mathcal{I}\cup\{0\},\forall s_i\in S,\forall x,y\in\mathbb{R}^+$,

$$\bar{d}_i(x^+)=\begin{cases}0 & \text{if } s_i=\emptyset \text{ or } p_i^{M_i}\leq x,\\ \max\limits_{1\leq m\leq M_i}\{q_i^m : p_i^m > x\} & \text{otherwise.}\end{cases}$$

$$\bar{\theta}_i'(y^+)=\begin{cases}0 & \text{if } s_i=\emptyset \text{ or } q_i^1\leq y,\\ \max\limits_{1\leq m\leq M_i}\{p_i^m : q_i^m > y\} & \text{otherwise.}\end{cases}$$

Consequently we have $\forall i\in\mathcal{I}\cup\{0\},\forall s_i\in S$,

$$\forall x\in\mathbb{R}^+,\quad \bar{\theta}_i'(\bar{d}_i(x^+)^+)\leq x \tag{14}$$

$$s_i\neq\emptyset\Rightarrow\forall x\in[0,p_i^{M_i}],\quad \bar{\theta}_i'(\bar{d}_i(x))\geq x. \tag{15}$$

A proof of Lemma 2 is given in Appendix I.

III. Properties of the multi-bid mechanism

In this section, we establish some basic properties of the multi-bid mechanism, showing its interest, before dealing in next sections with the central notions of incentive compatibility and efficiency.

Property 3: All the resource is allocated, i.e. for all multi-bid profile $s=(s_0,(s_i)_{i\in\mathcal{I}})\in S_0\times S^{|\mathcal{I}|}$,

$$\sum_{i\in\mathcal{I}\cup\{0\}} a_i(s) = Q, \tag{16}$$

where $|\mathcal{I}|$ is the number of elements in set $\mathcal{I}$.

Proof:

$$\sum_{i\in\mathcal{I}\cup\{0\}} a_i(s) = \sum_{i\in\mathcal{I}\cup\{0\}}\bar{d}_i(\bar{u}^+) + \frac{\bar{d}_i(\bar{u})-\bar{d}_i(\bar{u}^+)}{\bar{d}(\bar{u})-\bar{d}(\bar{u}^+)}(Q-\bar{d}(\bar{u}^+))$$
$$= \bar{d}(\bar{u}^+) + \frac{\bar{d}(\bar{u})-\bar{d}(\bar{u}^+)}{\bar{d}(\bar{u})-\bar{d}(\bar{u}^+)}(Q-\bar{d}(\bar{u}^+)) = Q.$$

■

In the following, s denotes the multi-bid profile, i.e. $s=(s_0,(s_i)_{i\in\mathcal{I}})$. For $i\in\mathcal{I}$, we also write $s_{-i}=(s_0,(s_j)_{j\in\mathcal{I},j\neq i})$ the multi-bid profile without player i's multi-bid. Therefore $s=(s_i,s_{-i})$.

Property 4: $\forall(s_i)_{i\in\mathcal{I}},\forall i\in\mathcal{I}$,

$$a_i(s_i,s_{-i}) = \sum_{j\in\mathcal{I}\cup\{0\},j\neq i}\left[a_j(\emptyset,s_{-i}) - a_j(s_i,s_{-i})\right],$$

meaning that player i's allocation is the difference between what other players would have obtained if player i was not part of the game and what they actually obtain.

Proof: Apply (16) to the multi-bid profiles (s_i,s_{-i}) and $(\emptyset,s_{-i})$, and remark that $a_i(\emptyset,s_{-i})=0$. ■

Property 5: A player increases his allocation by declaring a higher pseudo-demand function. More precisely, consider a player $i\in\mathcal{I}$ and two multi-bids $s_i,\tilde{s}_i\in S$ (which may not have the same number of bids). We denote $\bar{d}_i$ and $\tilde{d}_i$ the associated pseudo-demand functions. Then $\forall s_{-i}\in S_0\times S^{|\mathcal{I}|-1},\forall j\in\mathcal{I}\cup\{0\}\setminus\{i\}$,

$$\bar{d}_i\leq\tilde{d}_i\Rightarrow\begin{cases}a_i(s_i,s_{-i})\leq a_i(\tilde{s}_i,s_{-i})\\ a_j(s_i,s_{-i})\geq a_j(\tilde{s}_i,s_{-i}).\end{cases} \tag{17}$$

A proof of Property 5 is provided in Appendix II.

Property 6: When a player $i\in\mathcal{I}$ leaves the game, the allocations of all other players in the game increase. Formally,
$\forall i\in\mathcal{I},\forall j\in\mathcal{I}\cup\{0\}\setminus\{j\},\forall s\in S^{|\mathcal{I}|}$,

$$a_j(\emptyset,s_{-i})\geq a_j(s_i,s_{-i}).$$

Proof: Just apply (17) to pseudo-demand functions $\tilde{d}_i$ (associated with the multi-bid s_i) and $\bar{d}_i=0$ (that corresponds to the multi-bid $\emptyset$, following (3)). ■

Property 7: If a player declares a pseudo-demand function that is higher than the pseudo-demand function of another player, then he obtains more bandwidth. Formally, if $\forall i,j\in\mathcal{I}\cup\{0\},\forall s\in S_0\times S^{|\mathcal{I}|}$ and $\bar{d}_i$ and $\bar{d}_j$ are the pseudo-demand functions associated with multi-bids s_i and s_j, we have

$$\bar{d}_i\leq\bar{d}_j\quad\Rightarrow\quad a_i(s)\leq a_j(s).$$

Proof: We write $\bar{d}$ the agregated pseudo-demand function computed using the multi-bid profile s and $\bar{u}$ the corresponding pseudo-market clearing price. Thus

$$a_j(s)-a_i(s) = \left(\bar{d}_j(\bar{u}^+)-\bar{d}_i(\bar{u}^+)\right)\left(1-\frac{Q-\bar{d}(\bar{u}^+)}{\bar{d}(\bar{u})-\bar{d}(\bar{u}^+)}\right)+$$
$$+\left(\bar{d}_j(\bar{u})-\bar{d}_i(\bar{u})\right)\frac{Q-\bar{d}(\bar{u}^+)}{\bar{d}(\bar{u})-\bar{d}(\bar{u}^+)}$$
$$\geq 0,$$

where we have used $0\leq\frac{Q-\bar{d}(\bar{u}^+)}{\bar{d}(\bar{u})-\bar{d}(\bar{u}^+)}\leq 1$ and $\bar{d}_i\leq\bar{d}_j$. ■

Property 8: (*reserve price*). The reserve price p_0 that the auctioneer declares in her bid ensures her that the resource is sold at a unit price higher than p_0:

$$\forall s\in S_0\times S^{|\mathcal{I}|},\quad c_i(s)\geq p_0 a_i(s). \tag{18}$$

A proof of Property 8 is given in Appendix III.

Remark: Notice that a bid (q_i^m, p_i^m) does not affect $\bar{d}_i(p)$ for $p > p_i^m$, and that we always have $\bar{u} \geq p_0$ since $\bar{d}(p_0) \geq q_0 > Q$. For that reason, submitting a multi-bid s_i with $p_i^1 < p_0$ is useless for player $i \in \mathcal{I}$: he would get exactly the same utility as if he had submitted the multi-bid $(s_i^2, ..., s_i^{M_i})$. Therefore we can consider that players do not submit such bids, i.e. $\forall i \in \mathcal{I} : s_i \neq \emptyset, p_i^1 \geq p_0$.

In the next property, we show that the arrival of a new player i with the multi-bid s_i corresponds to an increase of the network revenue. This result implies that the network will not deter any player from entering the auction game.

Property 9: The seller's revenue is always greater with all players than when a player is excluded from the game: $\forall s \in S_0 \times S^{|\mathcal{I}|}, \forall i \in \mathcal{I}$,

$$\sum_{j \in \mathcal{I}} c_j(s) \geq \sum_{j \in \mathcal{I} \setminus \{i\}} c_j(s_{-i}). \tag{19}$$

More precisely, if the seller has a marginal valuation p_0 of the resource, (i.e. $\theta_0(q) = p_0 q$), then the seller's net utility is larger when all players are in the game than in the case when a player is excluded: $\forall s \in S_0 \times S^{|\mathcal{I}|}, \forall i \in \mathcal{I}$,

$$p_0 a_0(s) + \sum_{j \in \mathcal{I}} c_j(s) \geq p_0 a_0(s_{-i}) + \sum_{j \in \mathcal{I} \setminus \{i\}} c_j(s_{-i}).$$

We provide a proof of Property 9 in Appendix IV.

Let us now introduce our last property. A mechanism is said to be *individually rational* if no player can be worse off from participating in the auction than if he had declined to participate [12]. The following property states that this holds for truthful bidders, since the price a player is charged is lower than his declared willingness-to-pay.

Property 10: (individual rationality)

$$\forall i \in \mathcal{I}, \forall s \in S_0 \times S^{|\mathcal{I}|}, \quad c_i(s) \leq \int_0^{a_i(s)} \bar{\theta}_i'(q) dq. \tag{20}$$

Moreover, if player i submits a truthful multi-bid ($s_i \in S_i^T$), then

$$c_i \leq \int_0^{a_i(s)} \theta_i'(q) dq = \theta_i(a_i(s)), \tag{21}$$

which means that $u_i(s) \geq 0$.
Appendix V gives a proof of Property 10.

IV. Incentive Compatibility

In this section, we focus on the problem of the multi-bid choice by a player. In particular, we prove that a player cannot do much better than simply reveal his true valuation, which in our context means bidding with prices equal to the marginal valuations: $\forall m, 1 \leq m \leq M_i, p_i^m = \theta_i'(q_i^m)$.

Proposition 1: If a player $i \in \mathcal{I}$ submits a truthful multi-bid $s_i \neq \emptyset$, then every other multi-bid $\tilde{s}_i$ (truthful or not) necessarily corresponds to an increase of utility that is less than $\int_{\bar{d}_i(\bar{u}+)}^{d_i(\bar{u})} (\theta_i'(q) - \bar{u}) dq$.

Formally, $\forall s_i \in S_i^T, \forall \tilde{s}_i \in S, \forall s_{-i} \in S_0 \times S^{|\mathcal{I}|-1}$,

$$u_i(s_i, s_{-i}) \geq u_i(\tilde{s}_i, s_{-i}) - \int_{\bar{d}_i(\bar{u}+)}^{d_i(\bar{u})} (\theta_i'(q) - \bar{u}) dq. \tag{22}$$

A proof of Proposition 1 is given in Appendix VI.

This result is illustrated in Fig. 3 where the shaded area corresponds to the maximum utility gain player i could expect by submitting a different multi-bid.

Since the pseudo-market clearing price is necessarily higher than p_0, we can therefore also note in Fig. 3 that, when $p_i^1 \geq p_0$, the quantity

$$\int_{\bar{d}_i(\bar{u}+)}^{d_i(\bar{u})} (\theta_i'(q) - \bar{u}) dq$$

is always less than

$$\max_{0 \leq m \leq M_i} \left\{ \int_{d_i(p_i^{m+1})}^{d_i(p_i^m)} (\theta_i'(q) - p_i^m) dq \right\},$$

where $p_i^{M_i+1} = \theta_i'(0)$ and $p_i^0 = p_0$.

This last quantity is the largest shaded area in Fig. 4. The following proposition is then straightforward:

Proposition 2: Under Assumption 1 we have $\forall i \in \mathcal{I}, \forall s_i \in S_i^T(p_0) \setminus \emptyset, \forall \tilde{s}_i \in S, \forall s_{-i} \in S_0 \times S^{|\mathcal{I}|-1}$,

$$u_i(s_i, s_{-i}) \geq u_i(\tilde{s}_i, s_{-i}) - C_i, \tag{23}$$

where $S_i^T(p_0)$ is defined in (2), and where

$$C_i = \max_{0 \leq m \leq M_i} \left\{ \int_{d_i(p_i^{m+1})}^{d_i(p_i^m)} (\theta_i'(q) - p_i^m) dq \right\} \tag{24}$$

with $p_i^{M_i+1} = \theta_i'(0)$ and $p_i^0 = p_0$.

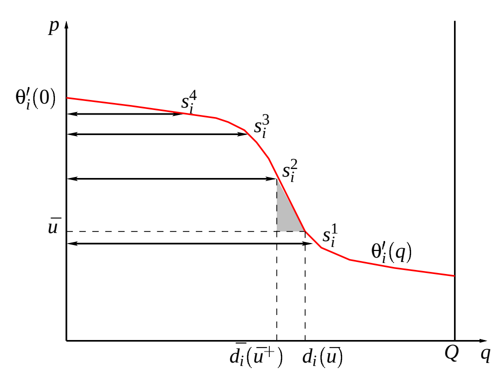

Fig. 3. The multi-bid $s_i = (s_i^1, s_i^2, s_i^3, s_i^4)$ is optimal for player i up to the value $\int_{\bar{d}_i(\bar{u}+)}^{d_i(\bar{u})} (\theta_i'(q) - \bar{u}) dq$ of shaded surface

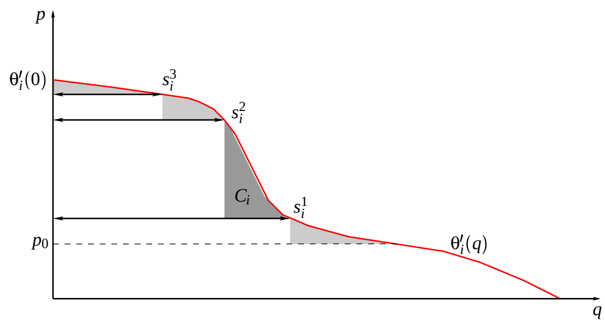

Fig. 4. The multi-bid $s_i = (s_i^1, s_i^2, s_i^3)$ is optimal for player i up to a constant C_i, *whatever the multi-bids submitted by others s_{-i} be*. C_i is the surface of the darkest shaded area.

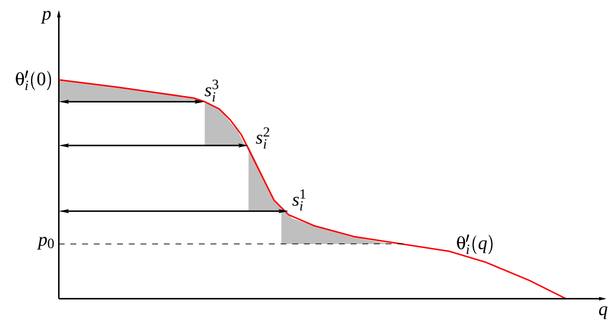

Fig. 5. *Quantile uniform* repartition of bids for $M_i = 3$: the four shaded zones have the same surface.

Note that C_i can also be written

$$C_i = \max_{0\leq m\leq M_i}\left\{\int_{q_i^{m+1}}^{q_i^m}(\theta_i'(q)-\theta_i'(q_i^m))dq\right\}$$
$$\text{with } q_i^{M_i+1}=0 \text{ and } q_i^0 = d_i(p_0).$$

Proposition 2 implies that a player i who knows the reserve price p_0 can give a truthful multi-bid that brings him the best utility possible, up to a value C_i that can be controlled through the choice of the bids s_i^m on the demand curve. One important point is that this value does not depend on the number of other players, nor on the multi-bid they submit.

Remark: Since submitting a truthful multi-bid is a C_i-best action for player i *independently of the other players' actions*, we can say that submitting such a bid is an *ex post C_i-dominant strategy* for i (see [18]). Therefore, the situation where all players submit truthful multi-bids is an *ex post K-Nash equilibrium*, with $K = \max_{i\in\mathcal{I}} C_i$, in the sense that no player could have improved his utility by more than K if he had submitted a different multi-bid.

Remark: Note that if $\theta_i'(Q) > p_0$, player i can also ensure that $\bar{u} \geq p_i^1 \geq p_0$ with $p_i^1 \leq \theta_i'(Q)$ by submitting the truthful bid $s_i^1 = (q_1, \theta_i'(q_1) = p_1)$ with $d_i(p_0) \geq q_1 > Q$. Choosing a bid sufficiently close to $(Q, \theta_i'(Q))$, player i may reduce his constant C_i.

V. "Quantile uniform" choice of bids

It is reasonable to assume that each user i intends to ensure a utility that is as close as possible to the maximum. To do so, it would be of interest to consider that players have beliefs (*a priori* probability distributions, as in [19]) on the number of users in the game and on their preferences, in order to deduce a probability distribution of the pseudo-market price $\bar{u}$. (In this sense, the auction game can be seen as a game with population uncertainty [20].) Given this distribution, player i may use Proposition 1 to choose his bids so as to minimize $\mathbb{E}\left[\int_{\bar{d}_i(\bar{u}^+)}^{d_i(\bar{u})}(\theta_i'(q)-\bar{u})dq\right]$. However, estimating such a distribution on the pseudo-market price may imply high cost for information-gathering and market appraisal [17].

For this reason, and for sake of simplicity, we assume that players have no idea of what the pseudo-market price will be, except that it will not be below p_0. With this in mind, the simplest way to choose a multi-bid that would be almost optimum, whatever the multi-bid profile is, is to minimize the quantity C_i of Proposition 2. Nevertheless, if player i is allowed to submit as many bids as he wants in his multi-bid, he will give a number M_i of bids as large as possible, in order to make C_i tend to zero. We therefore focus here on the choice of the multi-bid by player i, *after the number of bids M_i is determined*, for instance in the case where it is fixed by the auctioneer (see also Section VII). It is then clear that for a fixed M_i, the multi-bid $(s_i^1, ..., s_i^{M_i})$ that minimizes C_i is such that $\forall m, n, 0 \leq m, n \leq M_i$,

$$\int_{d_i(p_i^{m+1})}^{d_i(p_i^m)}(\theta_i'(q)-p_i^m)dq = \int_{d_i(p_i^{n+1})}^{d_i(p_i^n)}(\theta_i'(q)-p_i^n)dq$$
$$\text{with } p_i^{M_i+1} = \theta_i'(0) \text{ and } p_i^0 = p_0,$$

i.e. all the shaded areas are equal. In the following, we will call *quantile uniform* this bid repartition. An example of quantile uniform repartition of bids is presented in Fig. 5.

Example: For parabolic valuation functions, i.e. of the form

$$\theta_i(q) = \alpha\left[-(q\wedge\bar{q}_i)^2/2 + \bar{q}_i(q\wedge\bar{q}_i)\right]$$

with parameters α and $\bar{q}_i$, the marginal valuation function is linear:

$$\theta_i'(q) = \alpha\left[\bar{q}_i - q\right]^+ .$$

When $\theta_i'(0) > p_0$, the quantile uniform repartition of bids is easy to compute: prices $p_i^m, 1 \leq m \leq M_i$ are such that

$$p_i^m = p_0 + m\frac{\theta_i'(0)-p_0}{M_i+1} = p_0 + m\frac{\alpha\bar{q}_i - p_0}{M_i+1}.$$

VI. Efficiency

One goal of the auction is to make sure that the capacity goes to users who value it most. In this section, we prove that the multi-bid second-price mechanism verifies this property.

To obtain this result, we make another regularity assumption on the valuation functions:

Assumption 2: $\exists \kappa > 0, \forall i \in \mathcal{I}$,
– $\theta_i'(0) < +\infty$
– $\forall z, z', z > z' \geq 0, \theta_i'(z) - \theta_i'(z') > -\kappa(z - z')$.

Notice that this assumption is also needed to prove the efficiency of the PSP (see [6]).

Proposition 3: If Assumptions 1 and 2 hold, then $\forall s \in S_0 \times S^{|\mathcal{I}|}$,

$$\max_{\mathcal{A}} \sum_i \theta_i(\tilde{a}_i) - \sum_i \theta_i(a_i(s)) \leq Q\sqrt{\kappa \max_{i \in \mathcal{I}} C_i},$$

where $\mathcal{A} = \left\{\tilde{a} \in [0, Q]^{|\mathcal{I}|+1} : \sum_i \tilde{a}_i \leq Q\right\}$.
We provide a proof of Proposition 3 in Appendix VII.

Remark: This proposition states that the allocation $a(s)$ is optimal up to a certain value, in terms of efficiency as well as social welfare (see the remark in sub-section II-B).

VII. Determination of the number of bids admitted by the auctioneer

In this section, we assume that the auctioneer imposes the number of bids for all players $i \in \mathcal{I}$ to be M. We further suppose that players, knowing the reserve price p_0 and the number M of bids allowed, choose their multi-bid according to the quantile uniform distribution, as mentionned in Section IV.

Since increasing the value of M increases the signaling overhead, the memory storage and the complexity of all underlying allocation and price computations, we introduce a *cost function* $C(M, \mathcal{I})$ that models these negative effects, which the auctioneer will have to take into account when calculating her benefit $B(M, \mathcal{I})$:

$$B(M, \mathcal{I}) = p_0 a_0 + \sum_{i \in \mathcal{I}} c_i - C(M, \mathcal{I}),$$

where allocations and prices correspond to the situation when each user submits exactly M bids.

We denote $\mathcal{T}$ the set of possible player types, characterizing the valuation function (in other words, a type-t player has valuation function $\theta_{(t)}$). We model the auctioneer's beliefs about the number of players of each type by an *a priori* law $\mathbb{P}_{\mathcal{T}}$ on $\mathbb{N}^{\mathcal{T}}$. Therefore, the auctioneer can estimate her expected revenue $\mathbb{E}_{\mathcal{I}}[R_M]$ when players submit M bids. We have

$$\begin{aligned} \mathbb{E}_{\mathcal{I}}[R_M] &= \mathbb{E}_{\mathcal{I}}\left[p_0 a_0 + \sum_{i \in \mathcal{I}} c_i \middle| M\right] \\ &= \int_{\mathcal{I} \in \mathbb{N}^{\mathcal{T}}} \left(p_0 a_0 + \sum_{i \in \mathcal{I}} c_i \middle| M\right) d\mathbb{P}_{\mathcal{T}}(\mathcal{I}). \end{aligned}$$

We now make an assumption about the cost function $C(M, \mathcal{I})$:

Assumption 3: The expected cost $\mathbb{E}_{\mathcal{I}}[C](M) = \int_{\mathcal{I} \in \mathbb{N}^{\mathcal{T}}} C(M, \mathcal{I}) d\mathbb{P}_{\mathcal{T}}(\mathcal{I})$ is non-decreasing, and tends to infinity when M tends to infinity:

$$\lim_{M \to +\infty} \mathbb{E}_{\mathcal{I}} C(M) = +\infty.$$

This assumption seems intuitive. Actually, if we deal with memory costs, we have $C(M, \mathcal{I}) = M|\mathcal{I}|$, so Assumption 3 is verified as soon as $\mathbb{E}_{\mathcal{I}}[|\mathcal{I}|] > 0$. Considering computation or signaling costs would also lead to a cost function that would verify Assumption 3.

The following result gives an idea on how the auctioneer may choose M:

Proposition 4: If the marginal valuation functions $\left(\theta'_{(t)}\right)_{t \in \mathcal{T}}$ are uniformly bounded by a value $p_{\max}$ (that is $\forall t \in \mathcal{T}, \theta'_{(t)}(0) \leq p_{\max}$), then under Assumption 3 there exists a finite M that maximizes the expected net benefit of the seller, i.e. that maximizes

$$\mathbb{E}_{\mathcal{I}}\left[p_0 a_0 + \sum_{i \in \mathcal{I}} c_i \middle| M\right] - \mathbb{E}_{\mathcal{I}}\left[C(M, \mathcal{I})\right].$$

Proof: We assume without loss of generality that $p_{\max} \geq p_0$. Applying Property 10, we have $\forall \mathcal{I}, \forall M$,

$$\begin{aligned} p_0 a_0 + \sum_{i \in \mathcal{I}} c_i &\leq \sum_{i \in \mathcal{I} \cup \{0\}} \theta_i(a_i) \leq \sum_{i \in \mathcal{I} \cup \{0\}} a_i \theta_i'(0) \\ &\leq p_{\max} \sum_{i \in \mathcal{I} \cup \{0\}} a_i = p_{\max} Q. \end{aligned}$$

Consequently $\mathbb{E}_{\mathcal{I}}\left[p_0 a_0 + \sum_{i \in \mathcal{I}} c_i \middle| M\right] \leq p_{\max} Q$ for all $M \in \mathbb{N}$. Therefore

$$\lim_{M \to +\infty} \mathbb{E}_{\mathcal{I}}\left[p_0 a_0 + \sum_{i \in \mathcal{I}} c_i - C(M, \mathcal{I}) \middle| M\right] = -\infty,$$

which ensures us that there exists a finite M that maximizes the expected net benefit $\mathbb{E}_{\mathcal{I}}\left[p_0 a_0 + \sum_{i \in \mathcal{I}} c_i \middle| M\right] - \mathbb{E}_{\mathcal{I}}\left[C(M, \mathcal{I})\right]$. ■

Remark: It is possible that the expected net benefit be non-positive for all $M \geq 1$. This means that organizing the auction is too expensive for the seller of the resource. In that case the owner will choose $M = 0$, i.e. she prefers not to sell the resource.

Remark: We considered in this section the case when the auctioneer chooses the number M of bids allowed. However, one can imagine a high authority which may want to ensure a certain efficiency of the allocation. This authority could then fix M so as to reach a good trade-off between efficiency and costs, and subsidize the auctioneer in order to compensate her net benefit loss.

VIII. Conclusions and perspectives

In this paper, we have designed and studied a one-shot auction-based mechanism for sharing an arbitrarily divisible resource, like the capacity of a communication link. This mechanism requires each user i to submit M_i two-dimension bids when entering the auction. With respect to the progressive second price (PSP) auction of the literature, our mechanism saves a lot of signaling overhead.

Considering an elastic-demand model of user preferences, we have proved that our rule incites players to submit truthful bids. We have highlighted a reasonable behavior for users, assuming that they know their own preferences, but do not have beliefs about the other players (neither about their numbers nor about their valuation functions). We have shown that the rule leads to an efficient allocation of resource, and have given some hints to understand how the number of bids can be chosen.

This mechanism is particularly well-suited for pricing bandwidth on a communication link, for it implies relatively low computational costs and signaling traffic, and adapts instantly to a change in the user set (start or end of a connection).

Future work in that context will essentially focus on the application of our mechanism to the network case, i.e. to the situation where users want to buy bandwidth on several links from one point to another. The PSP mechanism was studied in the network case in [4], [9], with the same signaling overhead than in the single link case. We are currently working on a scheme based on multi-bids that would still be efficient in a network context.

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Appendix

I. Proof of Lemma 2

Proof: The equations giving $\bar{d}_i(.^+)$ and $\bar{\theta}_i'(.^+)$ are straightforward.

We focus here on (14):

- if $s_i = \emptyset$ then $\bar{\theta}_i'(.) = 0 \le x$, and (14) is verified.
- If $s_i \neq \emptyset$ and $p_i^{M_i} \le x$, (14) comes from $\bar{\theta}_i'(.) \le p_i^{M_i}$.
- If $s_i \neq \emptyset$ and $p_i^{M_i} > x$ then

$$\bar{d}_i(x^+) = \max_{1 \le m \le M_i} \{q_i^m : p_i^m > x\}. \quad (25)$$

Now let us assume that $\bar{\theta}'_i(\bar{d}_i(x^+)^+) > x$ and see that we arrive at a contradiction. The fact that $\bar{\theta}'_i(\bar{d}_i(x^+)^+) > 0$ implies that we are in the case when

$$\bar{\theta}'_i(\bar{d}_i(x^+)^+) = \max_{1 \le m \le M_i} \{p_i^m : q_i^m > \bar{d}_i(x^+)\}.$$

Since $\bar{\theta}'_i(\bar{d}_i(x^+)^+) > x$,

$$\exists m_1, 1 \le m_1 \le M_i : \begin{cases} s_i^{m_1} = (q_i^{m_1}, p_i^{m_1}) \in s_i \\ p_i^{m_1} = \bar{\theta}'_i(\bar{d}_i(x^+)^+) > x \\ q_i^{m_1} > \bar{d}_i(x^+). \end{cases}$$

We now remark that this contradicts the definition of $\bar{d}_i(x^+)$ (Eq. (25)). Therefore (14) is verified.

Now we establish (15). By definition, $\bar{d}_i(x) = \max_{1 \le m \le M_i}\{q_i^m : p_i^m \ge x\}$. This means that there exists $m_0 \le M_i$ such that $\bar{d}_i(x) = q_i^{m_0}$ and $p_i^{m_0} \ge x$.

Consequently we have

$$\bar{\theta}'_i(\bar{d}_i(x)) = \max_{1 \le m \le M_i} \{p_i^m : q_i^m \ge q_i^{m_0}\} \ge p_i^{m_0} \ge x,$$

which gives (15). ■

II. Proof of Property 5

Proof: We just need to prove the result for players $j \neq i$, the inequality $a_i(s_i, s_{-i}) \le a_i(\tilde{s}_i, s_{-i})$ coming directly from (16). To simplify the notations we will write $\mathcal{I}_0 = \mathcal{I} \cup \{0\}$.

So consider $j \neq i$. We write $\bar{u}$ (respectively $\tilde{u}$) the pseudo-market clearing price when the multi-bid profile is $s = (s_i, s_{-i})$ (respectively $(\tilde{s}_i, s_{-i})$). We also denote the respective agregated pseudo-demand functions by $\bar{d}$ and $\tilde{d}$, and we define $\bar{d}_{-i} = \sum_{k \in \mathcal{I}_0 \setminus \{i\}} \bar{d}_k$. We then have $a_j(s) - a_j(\tilde{s}_i, s_{-i})$

$$= \bar{d}_j(\bar{u}^+) - \bar{d}_j(\tilde{u}^+) + \frac{\bar{d}_j(\bar{u}) - \bar{d}_j(\bar{u}^+)}{\bar{d}(\bar{u}) - \bar{d}(\bar{u}^+)}(Q - \bar{d}(\bar{u}^+)) - \frac{\bar{d}_j(\tilde{u}) - \bar{d}_j(\tilde{u}^+)}{\tilde{d}(\tilde{u}) - \tilde{d}(\tilde{u}^+)}(Q - \tilde{d}(\tilde{u}^+)). \quad (26)$$

Since by hypothesis $\bar{d}_i \le \tilde{d}_i$, then $\bar{d} = \bar{d}_i + \bar{d}_{-i} \le \tilde{d} = \tilde{d}_i + \bar{d}_{-i}$, and necessarily $\bar{u} \le \tilde{u}$.

We distinguish two cases:

- if $\bar{u} < \tilde{u}$, then since $\frac{\bar{d}_j(\bar{u}) - \bar{d}_j(\bar{u}^+)}{\bar{d}(\bar{u}) - \bar{d}(\bar{u}^+)}(Q - \bar{d}(\bar{u}^+)) \ge 0$ and $\frac{\bar{d}_j(\tilde{u}) - \bar{d}_j(\tilde{u}^+)}{\tilde{d}(\tilde{u}) - \tilde{d}(\tilde{u}^+)}(Q - \tilde{d}(\tilde{u}^+)) \le \bar{d}_j(\tilde{u}) - \bar{d}_j(\tilde{u}^+)$, (26) implies

$$\begin{aligned} a_j(s) - a_j(\tilde{s}_i, s_{-i}) &\ge \bar{d}_j(\bar{u}^+) - \bar{d}_j(\tilde{u}^+) - \bar{d}_j(\tilde{u}) + \bar{d}_j(\tilde{u}^+) \\ &\ge \bar{d}_j(\bar{u}^+) - \bar{d}_j(\tilde{u}) \ge 0, \end{aligned}$$

where the last inequality stems from $\bar{u} < \tilde{u}$ and from the non-increasingness of $\bar{d}_j$.

- if $\bar{u} = \tilde{u}$, then (26) becomes $a_j(s) - a_j(\tilde{s}_i, s_{-i}) =$

$$(\bar{d}_j(\bar{u})) - \bar{d}_j(\bar{u}^+)) \left(\frac{Q - \bar{d}(\bar{u}^+)}{\bar{d}(\bar{u}) - \bar{d}(\bar{u}^+)} - \frac{Q - \tilde{d}(\bar{u}^+)}{\tilde{d}(\bar{u}) - \tilde{d}(\bar{u}^+)} \right).$$

Since $\bar{d}_j(\bar{u}) - \bar{d}_j(\bar{u}^+) \ge 0$, we only need to show that $A = (\tilde{d}(\bar{u}) - \tilde{d}(\bar{u}^+))(Q - \bar{d}(\bar{u}^+)) - (\bar{d}(\bar{u}) - \bar{d}(\bar{u}^+))(Q - \tilde{d}(\bar{u}^+))$ is non-negative.

We have

$$\begin{aligned} A &= (\underbrace{\tilde{d}(\bar{u})}_{\ge \bar{d}(\bar{u})} - \tilde{d}(\bar{u}^+)) \underbrace{(Q - \bar{d}(\bar{u}^+))}_{\ge 0} - \\ &\quad -(\bar{d}(\bar{u}) - \bar{d}(\bar{u}^+))(Q - \tilde{d}(\bar{u}^+)) \\ &\ge (\bar{d}(\bar{u}) - Q)(\tilde{d}(\bar{u}^+) - \bar{d}(\bar{u}^+)) \ge 0 \end{aligned}$$

where we have used (9) and the assumption $\tilde{d}_i \ge \bar{d}_i$, which gives $\tilde{d} \ge \bar{d}$.

Finally we have proved that in all cases, $a_j(s) \ge a_j(\tilde{s}_i, s_{-i})$, which establishes the property. ■

III. Proof of Property 8

Proof: We first show that $\forall \mathcal{I}, \forall s \in S_0 \times S^{|\mathcal{I}|}, \forall j \in \mathcal{I}_0, \forall y \in \mathbb{R}^+$,

$$\int_y^{a_j(s)} \bar{\theta}'_j \ge \bar{u}(a_j(s) - y) \quad (27)$$

where $\bar{u}$ is the pseudo-market clearing price corresponding to the multi-bid profile s.

- If $y \le a_j(s)$, then $\int_y^{a_j(s)} \bar{\theta}'_j \ge \bar{\theta}'_j(a_j(s))(a_j(s) - y)$. Since we know from (12) that $a_j(s) \le \bar{d}_j(\bar{u})$, the non-increasingness of $\bar{\theta}'_j$ implies that $\bar{\theta}'_j(a_j(s)) \ge \bar{\theta}'_j(\bar{d}_j(\bar{u})) \ge \bar{u}$, by applying (15). Finally, $a_i(s) - y \ge 0$ gives (27).
- If $y > a_j(s)$, then $\int_{a_j(s)}^y \bar{\theta}'_j \le \bar{\theta}'_j(a_j(s)^+)(y - a_j(s))$. We now use (15) and $a_j(s) \ge \bar{d}_j(\bar{u}^+)$ to obtain $\bar{\theta}'_j(a_j(s)^+) \le \bar{\theta}'_j(\bar{d}_j(\bar{u}^+)^+) \le \bar{u}$, which leads to (27).

To prove Property 8, we apply (27) to the bid profile s_{-i} and get

$$\begin{aligned} c_i(s) &= \sum_{j \in \mathcal{I}_0, j \neq i} \int_{a_j(s)}^{a_j(s_{-i})} \bar{\theta}'_j \\ &\ge \sum_{j \in \mathcal{I}_0, j \neq i} \bar{u}_{-i}(a_j(s_{-i}) - a_j(s)) = \bar{u}_{-i} a_i(s) \end{aligned}$$

where $\bar{u}_{-i}$ denotes the pseudo-market clearing price corresponding to the multi-bid profile s_{-i}, and where Property 6 has been used. Since $\bar{d}_0(p_0) = q_0 > Q$, then $\bar{u}_{-i} \ge p_0$, establishing Property 8. ■

IV. Proof of Property 9

Proof: To simplify the notations, we omit to precise the multi-bid profile when it is s, i.e. we denote $a_j(s) = a_j$ for all $j \in \mathcal{I}_0$.

The difference between the two terms of (19) is $\sum_{j\in\mathcal{I}} c_j - \sum_{j\in\mathcal{I}\setminus\{i\}} c_j(s_{-i})$

$$\begin{aligned} &= c_i + \sum_{j\in\mathcal{I}\setminus\{i\}} [c_j - c_j(s_{-i})] \\ &= \sum_{j\in(\mathcal{I}_0)\setminus\{i\}} \int_{a_j}^{a_j(s_{-i})} \bar{\theta}'_j + \sum_{j\in\mathcal{I}\setminus\{i\}} [c_j - c_j(s_{-i})] \\ &= p_0(a_0(s_{-i}) - a_0) + \sum_{j\in\mathcal{I}\setminus\{i\}} A_j \end{aligned} \tag{28}$$

with

$$\begin{aligned} A_j &= \int_{a_j}^{a_j(s_{-i})} \bar{\theta}'_j + c_j - c_j(s_{-i}) \\ &= \int_{a_j}^{a_j(s_{-i})} \bar{\theta}'_j + \sum_{k\in\mathcal{I}_0\setminus\{j\}} \int_{a_k}^{a_k(s_{-j})} \bar{\theta}'_k - \sum_{k\in\mathcal{I}_0\setminus\{i,j\}} \int_{a_k(s_{-i})}^{a_k(s_{-i,j})} \bar{\theta}'_k \\ &= \int_{a_j}^{a_j(s_{-i})} \bar{\theta}'_j + \int_{a_i}^{a_i(s_{-j})} \bar{\theta}'_i + \\ &\quad + \sum_{k\in\mathcal{I}_0\setminus\{i,j\}} \left[\int_{a_k}^{a_k(s_{-j})} \bar{\theta}'_k - \int_{a_k(s_{-i})}^{a_k(s_{-i,j})} \bar{\theta}'_k \right]. \end{aligned} \tag{29}$$

To show the proposition, we just need to prove that $A_j \geq 0$ for all $j \in \mathcal{I} \setminus \{i\}$. In order to do that, we denote $\bar{u}_{-i}$ (respectively $\bar{u}_{-j}$) the pseudo-market clearing price associated with the multi-bid profile s_{-i} (respectively s_{-j}), and consider two cases:

- if $\bar{u}_{-j} \geq \bar{u}_{-i}$ then (27) applied to the profiles s_{-i} and s_{-j} gives

$$\begin{aligned} A_j \geq\ & \bar{u}_{-i}\left[a_j(s_{-i}) - a_j\right] + \bar{u}_{-j}\left[a_i(s_{-j}) - a_i\right] + \\ & + \sum_{k\in\mathcal{I}_0\setminus\{i,j\}} \bar{u}_{-j}(a_k(s_{-j}) - a_k) + \\ & + \sum_{k\in\mathcal{I}_0\setminus\{i,j\}} \bar{u}_{-i}(a_k(s_{-i}) - a_k(s_{-i,j})) \\ \geq\ & \bar{u}_{-i}\left(\sum_{k\in\mathcal{I}_0\setminus\{i\}} a_k(s_{-i}) - \sum_{k\in\mathcal{I}_0\setminus\{i,j\}} a_k(s_{-i,j}) \right) + \\ & + \bar{u}_{-j}\left(\sum_{k\in\mathcal{I}_0\setminus\{j\}} a_k(s_{-j}) - \sum_{k\in\mathcal{I}_0} a_k \right) + \\ & + (\bar{u}_{-j} - \bar{u}_{-i})a_j \\ \geq\ & 0 \quad \text{(Property 3)}; \end{aligned}$$

- if $\bar{u}_{-j} < \bar{u}_{-i}$ then we use the non-increasingness of the pseudo-marginal valuation functions $\bar{\theta}'_k$ to modify the integration bounds in (29):

$$\begin{aligned} & \int_{a_k}^{a_k(s_{-j})} \bar{\theta}'_k - \int_{a_k(s_{-i})}^{a_k(s_{-i,j})} \bar{\theta}'_k \\ & \qquad \geq \int_{a_k}^{a_k(s_{-j})} \bar{\theta}'_k - \int_{a_k}^{a_k(s_{-i,j})+a_k-a_k(s_{-i})} \bar{\theta}'_k \\ & \qquad \geq \int_{a_k(s_{-i,j})+a_k-a_k(s_{-i})}^{a_k(s_{-j})} \bar{\theta}'_k. \end{aligned}$$

By applying (27) we obtain

$$\begin{aligned} A_j \geq\ & \bar{u}_{-i}(a_j(s_{-i}) - a_j + \bar{u}_{-j}(a_i(s_{-j}) - a_i) + \\ & + \bar{u}_{-j} \sum_{k\in\mathcal{I}_0\setminus\{i,j\}} (a_k(s_{-j}) + a_k(s_{-i}) - a_k - a_k(s_{-i,j})) \\ \geq\ & \bar{u}_{-j}\left(\sum_{k\in\mathcal{I}_0\setminus\{i\}} a_k(s_{-i}) - \sum_{k\in\mathcal{I}_0\setminus\{i,j\}} a_k(s_{-i,j}) \right) + \\ & + \bar{u}_{-j}\left(\sum_{k\in\mathcal{I}_0\setminus\{j\}} a_k(s_{-j}) - \sum_{k\in\mathcal{I}_0} a_k \right) + \\ & + (\bar{u}_{-i} - \bar{u}_{-j})(a_j(s_{-i}) - a_j) \\ \geq\ & 0. \end{aligned}$$

Finally we always have $\forall j \in \mathcal{I} \setminus \{i\}, A_j \geq 0$, and (28) then gives the second part of the proposition. The first part immediately follows, using the inequality $a_0 \leq a_0(s_{-i})$ (Property 6). ∎

V. Proof of Property 10

Proof: Applying (27) we get

$$\begin{aligned} c_i(s) &= \sum_{j\in\mathcal{I}_0\setminus\{i\}} \int_{a_j(s)}^{a_j(\emptyset,s_{-i})} \bar{\theta}'_j \\ &\leq \sum_{j\in\mathcal{I}_0\setminus\{i\}} \bar{u}(a_j(\emptyset, s_{-i}) - a_j(s)) \\ &\leq a_i(s)\bar{u}, \end{aligned}$$

where the last line comes from Property 4.

Now we consider two cases:

- if $a_i(s) = 0$ then $c_i(s) \leq 0$ and (20) is established.
- if $a_i(s) > 0$ then necessarily $\bar{d}_i(\bar{u}) > 0$, thus $s_i \neq \emptyset$ and $p_i^{M_i} \geq \bar{u}$. Lemma 2 then implies $\bar{\theta}'_i(\bar{d}_i(\bar{u})) \geq \bar{u}$. Moreover, since $\bar{\theta}'_i$ is non-increasing and $a_i(s) \leq \bar{d}_i(\bar{u})$,

$$\int_0^{a_i(s)} \bar{\theta}'_i \geq a_i(s)\bar{\theta}'_i(a_i(s)) \geq a_i(s)\bar{\theta}'_i(\bar{d}_i(\bar{u})) \geq a_i(s)\bar{u},$$

which gives (20).

Proving (21) is then straightforward by applying (20) and Lemma 1. ∎

VI. Proof of Proposition 1

Proof: Consider the difference of the prices charged to player i depending on his multi-bid:

$$\begin{aligned} c_i(\tilde{s}_i, s_{-i}) - c_i(s) &= \sum_{j \in \mathcal{I}_0 \setminus \{i\}} \int_{a_j(\tilde{s}_i, s_{-i})}^{a_j(s)} \bar{\theta}'_j \\ &\geq \bar{u} \sum_{j \in \mathcal{I}_0 \setminus \{i\}} (a_j(s) - a_j(\tilde{s}_i, s_{-i})) \\ &\geq \bar{u}(a_i(\tilde{s}_i, s_{-i}) - a_i(s)), \end{aligned} \tag{30}$$

where the first inequality comes from (27) and the last one from Property 4.

On the other hand, consider the difference of valuations $D_{\theta_i} := \theta_i(a_i(s)) - \theta_i(a_i(\tilde{s}_i, s_{-i}))$.
We distinguish several cases:

- if $a_i(s) > a_i(\tilde{s}_i, s_{-i})$, then

$$\begin{aligned} D_{\theta_i} = \int_{a_i(\tilde{s}_i, s_{-i})}^{a_i(s)} \theta' &\geq \int_{a_i(\tilde{s}_i, s_{-i})}^{a_i(s)} \bar{\theta}'_i \\ &\geq \bar{u}(a_i(s) - a_i(\tilde{s}_i, s_{-i})) \end{aligned}$$

from inequalities (5) and (27).

- If $a_i(s) \leq a_i(\tilde{s}_i, s_{-i})$ and $\bar{u} \geq \theta'_i(0)$, then

$$\begin{aligned} D_{\theta_i} = \int_{a_i(\tilde{s}_i, s_{-i})}^{a_i(s)} \theta' &\geq \theta'_i(0)(a_i(s) - a_i(\tilde{s}_i, s_{-i})) \\ &\geq \bar{u}(a_i(s) - a_i(\tilde{s}_i, s_{-i})). \end{aligned}$$

- If $a_i(s) \leq a_i(\tilde{s}_i, s_{-i})$ and $\bar{u} < \theta'_i(0)$, then $\theta'_i(d_i(\bar{u})) = \bar{u}$ and

$$\begin{aligned} D_{\theta_i} &= \theta_i(a_i(s)) - \theta_i(d_i(\bar{u})) + \\ &\quad + \theta_i(d_i(\bar{u})) - \theta_i(a_i(\tilde{s}_i, s_{-i})) \\ &\geq \int_{d_i(\bar{u})}^{a_i(s)} (\theta'_i(q) - \bar{u}) dq + \bar{u}(a_i(s) - d_i(\bar{u})) + \\ &\quad + \bar{u}(d_i(\bar{u}) - a_i(\tilde{s}_i, s_{-i})) \\ &\geq \int_{d_i(\bar{u})}^{\bar{d}_i(\bar{u}^+)} (\theta'_i(q) - \bar{u}) dq + \bar{u}(a_i(s) - a_i(\tilde{s}_i, s_{-i})) \end{aligned}$$

where the last line comes from (12) and from the fact that $\theta'_i(q) - \bar{u} \geq 0$ for all $q \leq d_i(\bar{u})$.

Finally we always have

$$D_{\theta_i} \geq \bar{u}(a_i(s) - a_i(\tilde{s}_i, s_{-i})) - \int_{\bar{d}_i(\bar{u}^+)}^{d_i(\bar{u})} (\theta'_i(q) - \bar{u}) dq. \tag{31}$$

To conclude the proof, from (30) and (31), we get

$$\begin{aligned} u_i(s) - u_i(\tilde{s}_i, s_{-i}) &= D_{\theta_i} + c_i(\tilde{s}_i, s_{-i}) - c_i(s) \\ &\geq -\int_{\bar{d}_i(\bar{u}^+)}^{d_i(\bar{u})} (\theta'_i(q) - \bar{u}) dq. \end{aligned}$$

■

VII. Proof of Proposition 3

Proof: First notice that if Assumptions 1 and 2 hold, then $\forall i \in \mathcal{I}, \forall e, f : 0 < e \leq f \leq \theta'_i(0)$,

$$d_i(e) - d_i(f) \geq \frac{f - e}{\kappa}. \tag{32}$$

Consider a player $i \in \mathcal{I}$ such that $a_i(s) > 0$. Since $a_i(s) \leq \bar{d}_i(\bar{u})$, we have $d_i(\bar{u}) \geq \bar{d}_i(\bar{u}) > 0$. Thus $\theta'_i(d_i(\bar{u})) = \bar{u}$, and

$$\theta'_i(a_i(s)) \geq \theta'_i(\bar{d}_i(\bar{u})) \geq \theta'_i(d_i(\bar{u})) = \bar{u}.$$

On the other hand, we have

$$\theta'_i(a_i(s)) \leq \theta'_i(\bar{d}_i(\bar{u}^+)). \tag{33}$$

- If $\theta'_i(0) \leq \bar{u}$ then $\theta'_i(a_i) \leq \bar{u}$.
- If $\theta'_i(0) > \bar{u}$ then

$$\begin{aligned} \theta'_i(\bar{d}_i(\bar{u}^+)) &= \min_{1 \leq m \leq M_i + 1} \{p_i^m : p_i^m > \bar{u}\} \\ &\leq \bar{u} + \max_{0 \leq m \leq M_i} (p_i^{m+1} - p_i^m). \end{aligned} \tag{34}$$

with $p_i^{M_i+1} = \theta'_i(0)$ and $p_i^0 = p_0$.

Now we remark that for all $m, 0 \leq m \leq M_i$

$$\begin{aligned} \int_{d_i(p_i^{m+1})}^{d_i(p_i^m)} (\theta'_i(q) - p_i^m) dq &= \int_{p_i^m}^{p_i^{m+1}} (d_i(p) - d_i(p_i^{m+1})) dp \\ &\geq \int_{p_i^m}^{p_i^{m+1}} \frac{p_i^{m+1} - p_i^m}{\kappa} dp \\ &\geq \frac{(p_i^{m+1} - p_i^m)^2}{\kappa} \end{aligned}$$

where the second line comes from (32).

Finally, (24) implies that $\forall m, 0 \leq m \leq M_i$, $p_i^{m+1} - p_i^m \leq \sqrt{\kappa C_i}$. Therefore (34) gives

$$\theta'_i(a_i(s)) \leq \bar{u} + \sqrt{\kappa C_i}.$$

Define $\mathcal{A} = \left\{\tilde{a} \in [0, Q]^{|\mathcal{I}|+1} : \sum_i \tilde{a}_i \leq Q\right\}$, and take any $\tilde{a} \in \mathcal{A}$. Let $\mathcal{I}_+ = \{k : \tilde{a}_k \geq a_k(s)\}$ and $\mathcal{I}_- = \{k : \tilde{a}_k < a_k(s)\}$. For $i \in \mathcal{I}_-$, we have $a_k(s) > \tilde{a}_k \geq 0$, and therefore (33) implies $\theta'_i(a_i(s)) \geq \bar{u}$. Applying (34), we then have

$$\begin{aligned} &\sum_i \theta_i(\tilde{a}_i) - \theta_i(a_i(s)) \\ &\leq \sum_{\mathcal{I}_+} \theta'_i(a_i(s))(\tilde{a}_i - a_i(s)) - \sum_{\mathcal{I}_-} \theta'_i(a_i(s))(a_i(s) - \tilde{a}_i) \\ &\leq \bar{u} \sum_i (\tilde{a}_i - a_i(s)) + \sum_{\mathcal{I}_+} \sqrt{\kappa C_i} (\tilde{a}_i - a_i(s)) \\ &\leq Q \sqrt{\kappa \max_{i \in \mathcal{I}} C_i}, \end{aligned}$$

which establishes the proposition. ■