Revisited Progressive Second Price Auction for Charging Telecommunication Networks

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Flat-rate charging of telecommunication networks does not seem relevant anymore as it creates congestion (like in the Internet). Among usage-based charging schemes, auctionning for bandwidth looks very promising. We review here the tremendous work called progressive second price auction and give some new results about users' behaviour in order to prove the results published in the literature.

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1. Introduction

Pricing is one of the big challenges of the emerging networks such as ATM or the Next-Generation Internet. Indeed, the current charging mode of the Internet, mainly based on a subscription fee and an unlimited connection time, is not viable if we wish to give different quality of services for different kinds of applications, like video, telephony, html... Charging becomes tricky due to this large range of applications, whereas in the previous networks, we had only one or very few different applications. Moreover, as some of these applications require a big amount of bandwidth with respect to the others, it seems natural to charge them more, especially in case of congestion.

The new trend is then to introduce a usage-based charging scheme. In the literature, telecommunication network pricing is the subject of a tremendous work. Pricing with bandwidth reservation has been studied for instance in [3,12]. Pricing without bandwidth reservation has been studied in several directions. A proposal called Paris Metro Pricing (PMP) [11] suggests to decompose the network in several separate networks with different connection fees and such that each network works like the current one. It is expected that the most expensive networks will be less congested. In another proposition [1,2,4,10], the user chooses a priority class (with a given price), and packets are served according to this priority. Also, transfer rates can be adjusted according to the willingness to pay of the user and according to the network congestion [5,6,8]. Finally, auctioning for priority in a model called "smart market" has also proved to be an interesting solution [9].

In [7,13–15], the costly auctions (in signalling procedures) for individual packets are replaced by auctions for bandwidth during intervals of time. Unlike in traditional auctions, the resource (i.e., the bandwidth) is divisible. A good analysis of this scheme based on game theory is provided, including fairness properties. The behavior of the system is essentially real-time, and not model-based. The work made in [7,13–15] is tremendous, ranging from mathematical theory for a single resource and, more impressive, a whole network, to the implementation, resulting in two patents.

Nevertheless, we argue that a technical mistake is included in the theory. In the present paper, we point out this problem, but we show (as a consequence of Theorem 3) that the results remain valid anyway. This additional theorem about PSP auction argues that no player will bid an already bid unit price if all the required bandwidth is not available (wihch was including the technical mistake). This also proves what was only stated as a footnote in [7,13]: in practice, the equal bid case does not matter in the allocation rule.

The layout is the following. Section 2 recalls the main results about progressive second price auctions and their application to telecomminication networks, in order to make the paper self-contained. Section 3 is devoted to the new results previously described.

2. Progressive Second Price (PSP) Auctions

PSP auctions have been designed in order to alleviate the signalling burden of packet auctions in smart markets [9] (by placing bids at fixed intervals of times).

To explain how the auction works, consider a single ressource of capacity Q and I players competing for it. Player i's bid is $s_i = (q_i, p_i)$ where q_i is the capacity player i is asking and p_i is the unit price he is proposing. A bid profile is $s = (s_1, \cdots, s_I)$. Let $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_I)$ be the profile where player i's bid is excluded from the game. We will sometimes write s as $s = (s_i; s_{-i})$ in order to emphasize player i's bid. For $y \geq 0$ define

$$\underline{Q}_i(y; s_{-i}) = \left[Q - \sum_{k \neq i\ : p_k \geq y} q_k \right]^+$$

as the remaining capacity when player i is excluded and all bids at unit-price strictly higher than y are served. The progressive second price allocation rule [7,13] gives to player i a bandwidth

$$a_i(s) = \min(q_i, \underline{Q}_i(p_i; s_{-i})) \tag{1}$$

and set the total cost to

$$c_i(s) = \sum_{j \neq i} p_j [a_j(0; s_{-i}) - a_j(s_i; s_{-i})]. \tag{2}$$

Thus the highest bids are allocated the desired quantity and the cost is given by the declared willingness to pay (bids) of the users who are excluded by i's presence.

Like in usual mathematical models in finance theory, a user i is characterized by a function θ_i representing the valuation that player i gives to his allocation. Acting selfishly, given s_{-i}, player i places bid s_i which maximizes his utility $u_i(s) = \theta_i(a_i(s)) - c_i(s)$. It is assumed that there is a bid fee ε each time a player submit a bid and that player i has a budget constraint b_i such that $c_i(s_i, s_{-i}) \leq b_i$. Let $\mathcal{S}_i(s_{-i})$ be the set of player i's bids verifying this constraint.

In order to obtain mathematical properties, the following smoothness assumptions on function θ_i are used:

- $\theta_i(0) = 0$,
- θ_i is differentiable,
- $\theta_i' \geq 0$, is non-increasing and continuous,
- $\exists \gamma_i > 0$, $\forall z \geq 0$, $\theta_i'(z) > 0 \Rightarrow \forall \eta < z, \theta_i'(z) < \theta_i'(\eta) - \gamma_i(z - \eta)$ (concavity assumption up to a point where the curve becomes flat).

The properties which need to be studied are stability, convergence and efficiency.

The stability issue which is studied is ε-Nash equilibrium, a more general notion than the usual Nash equilibrium. Basically, define player i's ε-best replies as the set

$$\mathcal{S}_i^\varepsilon(s_{-i}) = \{s_i \in \mathcal{S}_i(s_{-i}) : u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) - \varepsilon, \forall s_i' \in \mathcal{S}_i(s_{-i})\}.$$

An ε-Nash equilibrium is a fixed point of $\mathcal{S}^\varepsilon = \prod_{i=1}^I \mathcal{S}_i^\varepsilon(s_{-i})$. In an ε-Nash equilibrium, no player can increase his utility (provided that there is the bid-fee ε).

Define a truthful bid as a bid $s_i = (q_i, p_i)$ such that $p_i = \theta_i'(q_i)$, i.e., the price is equal to the marginal valuation. The key following property describes how a player chooses his bid, depending on the other bids.

Proposition 1. (Incentive compatibility) Define

$$Q_i(y; s_{-i}) = \left[Q - \sum_{k \neq i \ : p_k > y} q_k \right]^+$$

and

$$P_i(z, s_{-i}) = \inf\{y \geq 0 \ : \ Q_i(y, s_{-i}) \geq z\}.$$

Let

$$G_i(s_{-i}) = \sup\left\{z : z \leq Q_i(\theta_i'(z), s_{-i}) \text{ and } \int_0^z P_i(\eta, s_{-i}) d\eta \leq b_i\right\}.$$

Under the above assumptions on θ_i and $\forall 1 \leq i \leq I, \forall s_{-i}$ such that $Q_i(0, s_{-i}) = 0$, $\forall \varepsilon > 0$, there exists a truthful ε-best reply

$$t_i(s_{-i}) = (v_i = \left[G_i(s_{-i}) - \varepsilon/\theta_i'(0)\right]^+, \omega_i = \theta_i'(v_i)). \tag{3}$$

Introduce an additional player, labelled 0, which represents the minimal price that the seller is willing to buy the bandwidth from himself. His bid is $s_0 = (Q, p_0)$, meaning that he is not going to sell the bandwidth for less than the reserve price p_0.

The game is played dynamically. Each player makes his bid asynchronously each 1 every second for instance and submits a new bid if and only if he increases his utility by more than ε.

Then, (see [13]), we have the following result:

Proposition 2. Under the above assumptions, the dynamic game converges to a truthful ε-Nash equilibrium. Moreover, the procedure is efficient in the following way : Let s^* be an ε-Nash equilibrium and $a^* = a(s^*)$. Let $\mathcal{A} = \{a'(s^*) = (a_1'(q_1, p_1), \cdots, a_I'(q_I, p_I)) : \forall i, a_i'(q_i, p_i) \leq q_i$ and $a_i'(q_i) = q_i$ if $\sum_i q_i \leq Q$ and $\sum_i a_i'(q_i, p_i) = Q$ if $\sum_i q_i \geq Q\}$. If $b_i = \infty$, $\theta_i' < \infty$ and $\exists \kappa$ such that $\forall, z, z', z > z' \geq 0$, $\theta_i'(z) - \theta_i'(z') > -\kappa(z - z')$, we have

$$\max_{\mathcal{A}} \left(\sum_i \theta_i(a_i'(q_i, p_i)) - \sum_i \theta_i(a_i^*)\right) = O(\sqrt{\varepsilon\kappa}).$$

Note that when a player is allocated a bandwidth for the next one second time interval, he is not sure to keep it during that interval. Indeed, as the game is is played asynchronously, another player can bet during that interval and take all or a part of his bandwidth. The way the bandwidth is taken from the previous players has not been specified in [7,13–15] in the dynamic game but we can expect that it is done by removing the smallest bids.

The game is extended in [13,15] to networked auctions. In this extension, an auction is made at each link of the path (or of a subnetwork) a player is interested in, and the same mathematical properties are obtained. Remarkably, in this networked game, players can be raw bandwidth sellers, end-users or service providers buying and selling bandwidth to each others.

An implementation of this method has been performed, resulting in two patents.

3. Revisited PSP Auctions

In this Section, we show that a technical mistake was included in the proofs of [7,13–15]. It happens when two players bid the same unit price. We show here that, by modifying the allocation rule, this can be fixed. Nevertheless, our main result, Theorem 3, states that, in the algorithm where each player bids like described in Equation (3), the equal-bid case will never happen. Consequently, whatever the allocation rule is in this case, the results of [7,13–15] remain valid.

So, a problem occurs when it is said in [13] that it is readily apparent that

$$c_i(s) = \int_0^{a_i(s)} P_i(z, s_{-i})dz. \tag{4}$$

This equality is then used in most of the proofs about convergence, steady-state efficiency and stability of the allocation rule. However, we can point out that Equation (4) is not true in the case of players bidding the same amount per bandwidth unit. Indeed, considering the case $Q = 100$, $I = 4$, $s_1 = (60, 4)$, $s_2 = (70, 4)$, $s_3 = (40, 2)$ and $s_4 = (10, 1)$, we have $(a_1(s), c_1(s)) = (30, 180)$, $(a_2(s), c_2(s)) = (40, 200)$ and $(a_3(s), c_3(s)) = (a_4(s), c_4(s)) = (0, 0)$. As

$$Q_1(y, s_{-1}) = \begin{cases} 0 & \text{if } 0 \le y < 2 \\ 30 & \text{if } 2 \le y < 4 \\ 100 & \text{if } 4 \le y, \end{cases}$$

we have

$$P_1(z, s_{-1}) = \begin{cases} 0 \text{ if } z = 0 \\ 2 \text{ if } 0 < z \le 30 \\ 4 \text{ if } 30 < z \le 100. \end{cases}$$

Then

$$\int_0^{a_1(s)} P_1(z, s_{-1})dz = \int_0^{30} P_1(z, s_{-1})dz = 60 \ne c_1(s).$$

Moreover, the example shows that the allocation leads to an under-use of the bandwidth with respect to what was asked (the total demand was of 180 units of bandwidth and only 70 were allocated although the capacity is 100). It is stressed in [13] that not allocating the remaining bandwidth is just a technicallity since the users will prefer to change their bids so that it will not happen at equilibrium. Nevertheless, even if steady-state efficiency will be the same, we believe that a maximum bandwidth allocation rule in such "transient" situations is preferable, especially if it is at no additional cost.

A way to circumvent these problems is to make sure that, whatever the allocation rule is among the players betting the same amount, provided that the remaining bandwidth is allocated to them. A simple manner is by defining a new allocation rule:

$$a_i(s) = \min\left(q_i, \frac{q_i}{\sum_{k\ :p_k=p_i} q_k} Q_i(p_i; s_{-i})\right) \tag{5}$$

and we keep the total cost to

$$c_i(s) = \sum_{j \neq i} p_j[a_j(0; s_{-i}) - a_j(s_i; s_{-i})].$$

By doing this, the remaining bandwidth is shared between users betting the same unit price proportionally to their request. Another possibility would be to share that bandwidth equally.

We can remark that in the case of players betting different amounts per bandwidth unit, this allocation rule is equivalent to the one of Equation (1).

Using Equation (5) or any similar allocation that gives the maximum possible bandwidth, it is easy to show that Equation (4) is valid.

However, the key following theorem will be of direct consequence on the validity of the properties of the algorithm.

Theorem 3. If each player places his bid as described in Equation (3), the equal-bid case (when the total required bandwidth at this unit price is not available) does not occur in the PSP scheme with any supplemental equal-bid allocation algorithms. This is true for the old PSP algorithm, the proposed proportional allocation, the equal allocation...

As a direct consequence, the properties of the algorithm, given in Proposition 2, remain valid whatever the equal-bid allocation is.

Proof: Assume that players 1 to $I-1$ have already placed their bids (q_i, ω_i) and that they are sorted such that $\omega_1 > \cdots > \omega_{I-1}$ (if an equal price is given for several players, we just aggregate the bids and consider that we have less players).

Let $J = \min\{i : Q \leq \sum_{j=1}^{i} q_j\}$ be the number of players who are given some bandwidth before player I places his bid. We then have

$$Q_I(\theta_I'(z), s_{-I}) = \begin{cases} Q & \text{if } \theta_I'(z) \geq \omega_1 \\ Q - \sum_{j=1}^{i} q_j & \text{if } \omega_i > \theta_I'(z) \geq \omega_{i+1}\ (1 \leq i < J) \\ 0 & \text{if } \theta_I'(z) < \omega_J. \end{cases}$$

From the assumptions given at the beginning of the section, θ_I' is invertible on the set $[0, z_I^*]$, z_I^* being the point where θ_I begins to flatten (if any). We assume that $\omega_1 \leq \theta_I'(0)$ in order that $\theta_I'^{-1}(\omega_1)$ is defined. If it is not the case, just remove the players i such that $\omega_i > \theta_I'(0)$ and replace each time Q by $Q - q_i$, because these players are "untouchable" for player I. Hence, $\forall z \in [0, z_I^*]$,

$$Q_I(\theta_I'(z), s_{-I}) = \begin{cases} Q & \text{if } z \leq \theta_I^{'-1}(\omega_1) \\ Q - \sum_{j=1}^{i} q_j & \text{if } \theta_I^{'-1}(\omega_i) < z \leq \theta_I^{'-1}(\omega_{i+1}) \ (1 \leq i < J) \\ 0 & \text{if } z > \theta_I^{'-1}(\omega_J). \end{cases}$$

Let $G_I'(s_{-I}) = \sup\{z \ : z \leq Q_I(\theta_I'(z), s_{-I})\}$ be defined as $G_I(s_{-I})$ but without the budget constraint. Define also $G_I''(s_{-I})$ as the maximum bandwidth that player I can get with his budget constraint b_I. We the have

$$G_I(s_{-I}) = \min(G_I'(s_{-I}), G_I''(s_{-I})).$$

Try to compute $G_I'(s_{-I})$ first. We have two cases shown in Figure 3, depending on whether $Q_I(\theta_I'(z), s_{-I})$ meets the line $y = z$ or not. Define k_I as the integer

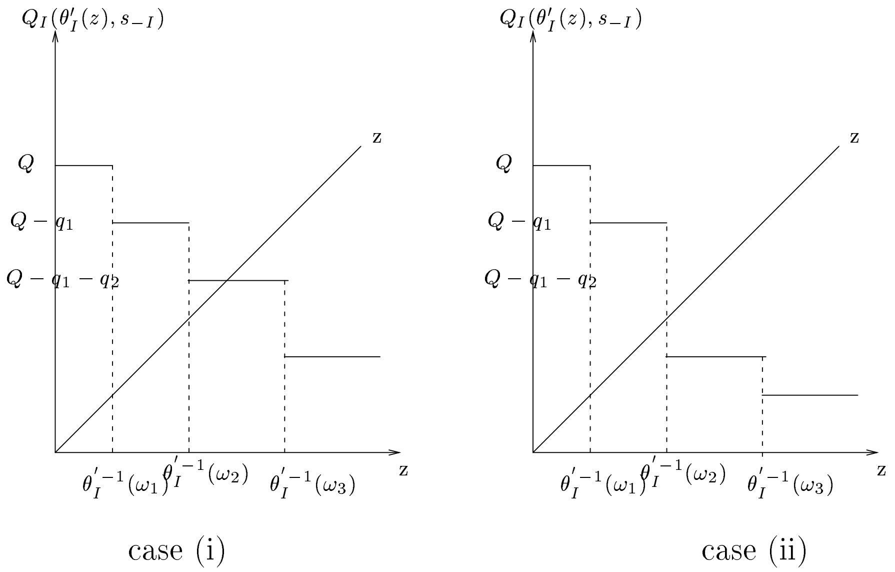

Figure 1. Two situations when computing $G_I'(s_{-I})$

such that $k_I = 0$ if $\theta_I^{'-1}(\omega_1) > Q - q_1$ and

$$k_I = \max\left\{ i \ : \theta_I^{'-1}(\omega_i) \leq Q - \sum_{j=1}^{i} q_j \right\}$$

otherwise. The two cases of Figure 3 can be translated to

- case (i) : $\theta_I^{'-1}(\omega_{k_I+1}) \geq Q - \sum_{j=1}^{k_I} q_j$;

- case (ii) : $\theta_I^{'-1}(\omega_{k_I+1}) < Q - \sum_{j=1}^{k_I} q_j$.

Then

$$G'_I(s_{-I}) = \begin{cases} Q - \sum_{j=1}^{k_I} q_j \text{ in case (i)} \\ \theta_I^{'-1}(\omega_{k_I+1}) \text{ in case (ii).} \end{cases}$$

Assume first that $G''_I(s_{-I}) \geq G'_I(s_{-I})$ so that $G_I(s_{-I}) = G'_I(s_{-I})$. In case (i), $G_I(s_{-I}) = Q - \sum_{j=1}^{k_I} q_j$, so

$$v_I = Q - \sum_{j=1}^{k_I} q_j - \varepsilon/\theta'_I(0) \text{ and } \omega_I = \theta'_I\left(Q - \sum_{j=1}^{k_I} q_j - \varepsilon/\theta'_I(0)\right)$$

(the case $v_I = 0$ does not need to be considered for our problem). If there exists j, $1 \leq j < I$ such that $\omega_I = \omega_j$, then necessarily $j \leq k_I$, but then the total required bandwidth at price $\omega_I = \omega_j$ or higher is $v_I + \sum_{i=1}^{j} q_i < Q$, so that the situation of concern (not enough bandwidth for all users betting the same unit price) does not occur. Consider now case (ii), i.e., $G_I(s_{-I}) = \theta_I^{'-1}(\omega_{k_I+1})$. Then

$$v_I = \theta_I^{'-1}(\omega_{k_I+1}) - \varepsilon/\theta'_I(0) \text{ and } \omega_I = \theta'_I(\theta_I^{'-1}(\omega_{k_I+1}) - \varepsilon/\theta'_I(0)) > \omega_{k_I+1}.$$

In this case also, if there exists j, $1 \leq j < I$, such that $\omega_I = \omega_j$, then necessarily $j \leq k_I$ and the total required bandwidth at price ω_I or higher verifies $v_I + \sum_{i=1}^{j} q_i < Q$.

If the budget constraint is used, i.e., $G''_I(s_{-I}) < G'_I(s_{-I})$, then

$$v_I = G''_I(s_{-I}) - \varepsilon/\theta'_I(0) < G'_I(s_{-I}) - \varepsilon/\theta'_I(0).$$

The bandwidth allocated to player I is then less than the one allocated in the previous case, hence the total required bandwidth at price ω_I or higher is less than Q, which completes the proof. It is impossible that two players bid the same unit price and the aggregated required bandwidth exceeds the remaining capacity. ∎

As already remarked, the old and new allocation rules differ only when two players bet the same amount per bandwidth unit *and the required bandwidth is not available for both of them.* Thus the proof of Proposition 1 is the one given in [7,13] because the above situation does not occur.

By using this theorem, the properties of stability, convergence to an ε-Nash equilibrium and efficiency of [7,13–15] are conserved.

Moreover, by the same kind of argument than the one used in Theorem 3, the results remain true when switching to our allocation rule, the main differences being at the beginning of the proof of Proposition 4 of [15] (which is also Proposition 7 of [13]).

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