VARIANCE REDUCTION TECHNIQUE FOR A CELLULAR SYSTEM WITH DYNAMIC RESOURCE SHARING

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ABSTRACT

A new algorithm is given to improve the simulation of a cellular system with dynamic resource sharing. This technique is based on low discrepancy sequences, used in quasi-Monte Carlo methods and which, combined with Monte Carlo techniques using importance sampling and control variates (already developed for such a system), gives the best known results.

INTRODUCTION

We consider a cellular system with dynamic resource sharing, used in digital cellular phone networks. The models used to evaluate such systems generally lead to product-form solutions, but analytic computation is intractable due to the size of the state space. An alternative way is the use of probabilistic methods, that is of Monte Carlo methods. Such techniques are commonly used in various research areas. In the same way, deterministic methods, called *quasi-Monte Carlo methods* use *low discrepancy sequences*. Unfortunately, in practice, an error bound for these approaches is uncomputable. We then use low discrepancy sequences as a variance reduction method.

In this paper, we first briefly describe the model and the solutions developed in (Fleming and Schaeffer 1995). Next, we develop a new variance reduction technique based on low discrepancy sequences. Finally, we give a numerical illustration of the improvement on a simple example.

MODEL DESCRIPTION

We consider a cell in a cellular phone system. The cell is divided in s sectors. For sector i ($i = 1, \cdots, s$), b_i channels are blocked for signaling or other overhead. Requests for channels in sector i occur as a Poisson process with rate λ_i and the different Poisson processes are mutually independent. Let $\lambda = \sum_{i=1}^{s} \lambda_i$ be the total arrival rate. The service times of calls are i.i.d and independent of the arrival processes. They are exponentially distributed with rate μ. The load factor on sector i is denoted by ρ_i which is given by $\rho_i = \lambda_i/\mu$. The channels are supposed to come in groups of c on a carrier and there are a_i carriers for sector i, which correspond to a discrete segmentation of the radio spectrum. It follows that the number m_i of channels available exclusively to sector i is $m_i = a_i c - b_i$. Moreover there are A carriers shared by all the sectors in a common pool. If a request for a channel occurs in sector i and all the carriers presently there are saturated, then a carrier from the common pool will be allocated to sector i. Finally, sector i is not allowed to borrow more than d_i carriers from the common pool. The maximum number of customers in sector i is then $r_i = d_i c + m_i$. A more complete description of the model can be found in (Fleming and Schaeffer 1995).

A state of the system is then given by $n = (n_1, \ldots, n_s)$ where n_i denotes the number of customers at sector i. The state space of the system is denoted by $\mathcal{R}$ and, according to the constraints described above, it is given by

$$\mathcal{R} = \left\{ n = (n_1, \ldots, n_s) \;\middle|\; \begin{array}{c} 0 \le n_i \le r_i,\ 1 \le i \le s \\ \displaystyle\sum_{i=1}^{s} \left\lceil \frac{n_i - m_i}{c} \right\rceil^{+} \le A \end{array} \right\},$$

where $x^{+} = \max(0, x)$ and $\lceil x \rceil$ is the ceiling function of x. For $n \in \mathcal{R}$, we denote by $P(n)$ the steady state probability to have n_i customers at sector i ($i = 1, \ldots, s$). It is shown in (Kelly 1986) that the model has a product-form solution given by

$$P(n) = \frac{\displaystyle\prod_{i=1}^{s} e^{-\rho_i} \frac{\rho_i^{n_i}}{n_i!}}{G},$$

where

$$G = \sum_{n \in \mathcal{R}} \prod_{i=1}^{s} e^{-\rho_i} \frac{\rho_i^{n_i}}{n_i!}$$

is the normalizing constant.

For most realistic systems, the size of the state space $\mathcal{R}$ is prohibitively large and a direct computation of G becomes impossible. We show in the following section how Monte Carlo methods can be used to evaluate performance measures of such a system.

PERFORMANCE EVALUATION

We consider in this section the steady state measures $E(N)$ and $E(N_i)$, $i = 1, \ldots, s$, which represent respectively the expected number of customers in the system and the expected number of customers in sector i. These measures are given by

$$E(N) = \sum_{n \in \mathcal{R}} (n_1 + \cdots + n_s) P(n),$$

and for $i = 1, \ldots, n$,

$$E(N_i) = \sum_{n \in \mathcal{R}} n_i P(n).$$

Other performance measures, as for instance system-blocking probability $P(B)$ or the blocking probability $P(B_i)$ in sector i, can be evaluated in the same way. But, as

$$\lambda(1 - P(B)) = E(N)\mu$$

and

$$\lambda_i(1 - P(B_i)) = E(N_i)\mu \text{ for } i = 1, \ldots, s$$

(balance equations), we concentrate our attention on $E(N)$ and $E(N_i)$, $(i = 1, \ldots, s)$.

Let $X = (X_1, \cdots, X_s)$ be independent truncated Poisson random variables. X_i takes its values in $\{0, \cdots, \hat{n}_i\}$ where $\hat{n}_i$ is sufficiently large so that

$$\mathcal{R} \subset \hat{\mathcal{R}} = \{n\ : 0 \le n_i \le \hat{n}_i,\ i = 1 \ldots, s\},$$

and it has parameter ρ_i. Define $Z = \mathbf{1}_{X \in \mathcal{R}}$, $Y_i = X_i Z$ and $Y = (X_1 + \cdots + X_s)Z$. We have

$$E(N) = \frac{E(Y)}{E(Z)} \text{ and } E(N_i) = \frac{E(Y_i)}{E(Z)}.$$

In what follows, we only deal with the evaluation of $E(N)$. The same results easily apply for $E(N_i)$. To estimate the value of $E(N)$ using a standard simulation, we generate M i.i.d. replications of the random variables Y and Z. We consider their means $\bar{Y}$ and $\bar{Z}$; the central limit theorem gives the following confidence interval with confidence level 0.95:

$$\left] \bar{N} - 1.96 \frac{\bar{S}}{\sqrt{M}}, \bar{N} + 1.96 \frac{\bar{S}}{\sqrt{M}} \right[,$$

where $\bar{N} = \bar{Y}/\bar{Z}$ and $\bar{S}$ are respectively the estimator of $E(N)$ and the estimator of the standard deviation of N.

Previous works

Let us recall in this section the two variance reduction methods described in (Fleming and Schaeffer 1995). In the first one, we choose a new sampling measure by changing the mean of the Poisson random variables from ρ_i to ρ_i'. Thus, if $\rho = (\rho_1, \cdots, \rho_s)$ and if $E_\rho()$ designates the expectation with respect to the

Poisson probability measure with parameter ρ, it follows that

$$\begin{aligned} E_\rho(N) &= \frac{E_\rho(Y)}{E_\rho(Z)} \\ &= \frac{E_{\rho'}\left(Y\prod_{i=1}^{s}\left(\frac{\rho_i}{\rho'_i}\right)^{X_i}\right)}{E_{\rho'}\left(Z\prod_{i=1}^{s}\left(\frac{\rho_i}{\rho'_i}\right)^{X_i}\right)} \\ &= \frac{E_{\rho'}(YL)}{E_{\rho'}(ZL)} \end{aligned}$$

with $L = \prod_{i=1}^{s}(\rho_i/\rho'_i)^{X_i}$. The new estimator of $E(N)$ is then denoted by $\bar{N}_{\rho'}$.

The second method is based on control variates. The estimator of $E(N)$ is chosen to be

$$\bar{N}^{(1)} = \frac{\bar{Y}}{\bar{Z}} + \beta_1\left(\frac{\bar{Y}'}{\bar{Z}'} - N'\right) + \beta_2\left(\frac{\bar{Y}''}{\bar{Z}''} - N''\right),$$

where $\bar{Y}'$, $\bar{Z}'$ and $\bar{Y}''$, $\bar{Z}''$ are defined as $\bar{Y}$ and $\bar{Z}$ but for different domains $\mathcal{R}'$ and $\mathcal{R}''$ defined by

$$\mathcal{R}' = \{n : 0 \le n_i \le r_i\},$$

$$\mathcal{R}'' = \left\{n : n_i \ge 0, i = 1, \cdots, s \text{ and } \sum_{i=1}^{s} n_i \le A''\right\}$$

with $A'' = cA + \sum_{i=1}^{s} m_i$. In that case, N' and N'' are explicitly known. More precisely,

$$N' = \sum_{i=1}^{s}\frac{\sum_{k=1}^{r_i}\frac{\rho_i^k}{(k-1)!}}{\sum_{k=0}^{r_i}\frac{\rho_i^k}{k!}}$$

and

$$N'' = \frac{\sum_{k=1}^{A''}\frac{(\sum_{j=1}^{s}\rho_j)^k}{(k-1)!}}{\sum_{k=0}^{A''}\frac{(\sum_{j=1}^{s}\rho_j)^k}{k!}}.$$

The parameters β_1 and β_2 are chosen to minimize the variance of $\bar{N}^{(1)}$. We can remark that $\hat{\mathcal{R}}$ has to be sufficiently large (i.e. $\hat{n}_i$, $i = 1, \ldots s$) to have

$$\mathcal{R}' \subset \hat{\mathcal{R}} \text{ and } \mathcal{R}'' \subset \hat{\mathcal{R}}.$$

The best estimator in (Fleming and Schaeffer 1995) is the one obtained by combining both variance reduction methods. We obtain the new estimator

$$\bar{N}^{(2)}_{\rho''} = \frac{\bar{Y}_{\rho''}}{\bar{Z}_{\rho''}} + \gamma_1\left(\frac{\bar{Y}'_{\rho''}}{\bar{Z}'_{\rho''}} - N'\right) + \gamma_2\left(\frac{\bar{Y}''_{\rho''}}{\bar{Z}''_{\rho''}} - N''\right),$$

where the optimal parameters ρ'', γ_1 and γ_2 are different from ρ', β_1 and β_2.

A new method

We consider here a new variance reduction technique based on low discrepancy sequences which is not still developed in the theory of variance reduction. A *low discrepancy sequence* $\mathcal{P} = (\xi^{(n)})_{n\ge 0}$ (Niederreiter 1992) is a sequence which is *well* distributed over a given integration region $[0,1]^s$. A measure of the equi-distribution, the *star discrepancy* of the N first points of $\mathcal{P}$ is

$$D^*_N(\mathcal{P}) = \sup_{x\in[0,1[^s}\left|\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{[0,x[}(\xi^{(n)}) - \prod_{i=1}^{s}x_i\right|,$$

where $[0,x[= \prod_{i=1}^{s}[0,x_i[$ for $x \in [0,1[^s$. Some other discrepancies are defined in (Bouleau and Lépingle 1993). The star discrepancy of a low discrepancy sequence is in $O(N^{-1}(\ln N)^s)$ in dimension s, which is the minimal known one. We can then approximate $\theta = \int_{[0,1]^s} f(x)dx$ by $\frac{1}{N}\sum_{n=1}^{N} f(\xi^{(n)})$. Such a deterministic method is called a *quasi-Monte Carlo method*. We have the following result:

Theorem 1 (Koksma-Hlawka)

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(\xi^{(n)}) - \int_{[0,1]^s} f(u)du\right| \le V(f)D^*_N(\mathcal{P})$$

where $V(f)$ is the total variation of f (Niederreiter 1992). The error obtained by this approximation is then in $O(N^{-1}(\ln N)^s)$, which is the best convergence rate (Beck and Chen 1987). Unfortunately, a good over-bound of this error is impossible to obtain in practice (Tuffin 1996b). As a matter of fact, the variation is more complicate to evaluate than the integral itself, and we have no good bound of the discrepancy for a fixed and "small" number of points. Then, to obtain an estimation of the error, we use low discrepancy sequences in Monte Carlo methods. Let U be uniformly distributed over$[0,1]^s$ and $\{x\}$ denote the fractional part of each coordinate of $x \in \mathbb{R}^s$. A new estimator of θ is then obtained from the I firsts terms of the sequence $\mathcal{P}$ by

$$\frac{1}{I}\sum_{i=1}^{I} f(\{U+\xi^{(i)}\}),$$

whose variance is in $O(I^{-2}(\ln I)^{2s})$. For a sufficiently large I, the variance is then smaller than σ^2/I, which is the variance of the mean of I independent random variables, each with variance σ^2.

We apply this method to our problem. An estimator $\bar{Z}$ of $E(Z)$ is then defined from a sequence of M i.i.d. vectors $U^{(m)} = (U_1^{(m)}, \cdots, U_s^{(m)})$, $m = 1, \ldots, M$, uniformly distributed over $[0,1]^s$, by

$$\bar{Z} = \frac{1}{M}\sum_{m=1}^{M} Z_m^I \tag{1}$$

with

$$Z_m^I =$$

$$\frac{1}{I}\sum_{i=1}^{I} \mathbf{1}_{\mathcal{R}}(\phi_1^{-1}(\{U_1^{(m)}+\xi_1^{(i)}\}, \cdots, \phi_s^{-1}(\{U_s^{(m)}+\xi_s^{(i)}\}),$$

where ϕ_i^{-1} is the pseudo-inverse of the distribution function of a truncated Poisson random random variable with parameter ρ_i. In the same way, we obtain an estimator of $E(Y)$ and then an estimator of $E(N)$.

The low discrepancy sequences that we use are those described in (Tuffin 1996a), a permuted version of Halton sequences. Let $p \in \mathbb{N}$. We represent the digit expansion of $n \in \mathbb{N}$ in base p by

$$n = a_j p^j + \cdots + a_1 p + a_0 \text{ with } a_i \in \{0, \cdots, p-1\}.$$

Thus, we set

$$S_p(n) = \frac{\pi_p(a_0)}{p} + \frac{\pi_p(a_1)}{p^2} + \cdots + \frac{\pi_p(a_j)}{p^{j+1}}$$

where π_p is a permutation of $\{0, \ldots, p\}$ such that $\pi_p(0) = 0$. The sequence $\mathcal{P} = (\xi^{(n)})_{n\in\mathbb{N}}$ in dimension s is defined by

$$\xi^{(n)} = (S_{p_1}(n), \cdots, S_{p_s}(n)).$$

The integers p_i $(i = 1, \ldots, s)$ have to be mutually prime. The choice $\pi_{p_i} = Id$ gives the Halton sequences. Another choice is given in (Braaten and Weller 1979). Our choice is made from an algorithm given in (Tuffin 1996a) which is more efficient than those proposed in (Braaten and Weller 1979).

We can combine this technique with the importance sampling method and the control variates method described in previous section. Then

$$\bar{N}_{\rho^*}^{(3)} = \frac{\bar{Y}_{\rho^*}}{\bar{Z}_{\rho^*}} + \eta_1\left(\frac{\bar{Y}'_{\rho^*}}{\bar{Z}'_{\rho^*}} - N'\right) + \eta_2\left(\frac{\bar{Y}''_{\rho^*}}{\bar{Z}''_{\rho^*}} - N''\right),$$

where $\bar{Y}_{\rho^*}$, $\bar{Z}_{\rho^*}$, $\bar{Y}'_{\rho^*}$, $\bar{Z}'_{\rho^*}$, $\bar{Y}''_{\rho^*}$ and $\bar{Z}''_{\rho^*}$ are defined as in (1), but respectively for functions $L(X_1 + \cdots + X_s)\mathbf{1}_{X\in\mathcal{R}}$, $L\mathbf{1}_{X\in\mathcal{R}}$, $L(X_1 + \cdots + X_s)\mathbf{1}_{X\in\mathcal{R}'}$, $L\mathbf{1}_{X\in\mathcal{R}'}$, $L(X_1 + \cdots + X_s)\mathbf{1}_{X\in\mathcal{R}''}$ and $L\mathbf{1}_{X\in\mathcal{R}''}$ instead of $\mathbf{1}_{X\in\mathcal{R}}$. The function ϕ_i used in (1) becomes the pseudo-inverse function of a Poisson random variable with parameter ρ_i^*. The values ρ^*, η_1 and η_2 are once again chosen to minimize the variance.

In practice, although the integer I has to be as large as possible to obtain the largest variance reduction, we must use a sufficiently large number of iterations M to guarantee a good normal approximation when using the central limit theorem. Moreover we can remark that the smallest the dimension s is, the larger is the variance reduction.

NUMERICAL RESULTS

We illustrate here, by means of a simple example, the variance reductions. Let us set $s = 3$, $\mu = 1.0$, $c = 6$, $A = 5$, $\lambda_i = 30$, $a_i = 3$, $b_i = 3$, $d_i = 2$ and $\rho'_i = 20$ for all i. In Table 1, for 10^6 iterations with importance sampling and control variates (which is the best known method), we obtain an interval width of $5.3e-06$. On the other hand, in Table 2, our method for 10^2 iterations and $I = 10^4$ gives an interval width of $2.9e-06$. Then, it takes about $(1.85)^2 = 3.4$ times more iterations without low discrepancy sequences to obtain the same interval width.

On the other hand our method needs much less computational time. Table 3 shows the different times for both simulation techniques on a Sun Sparcstation.

Variate	Estimation	Conf. Interv.Width
$E(N)$	6.868858e+01	5.275635e-02
$E(N_1)$	2.290045e+01	5.334733e-02
$E(N_2)$	2.289710e+01	5.399780e-02
$E(N_3)$	2.289103e+01	5.436225e-02

Table 1: Results with the best method in (Fleming and Schaeffer 1995) or 10^6 iterations.

Variate	Estimation	Conf. interv. Width
$E(N)$	6.868352e+01	2.875900e-02
$E(N_1)$	2.290406e+01	2.930439e-02
$E(N_2)$	2.288951e+01	2.656419e-02
$E(N_3)$	2.289118e+01	3.030912e-02

Table 2: Results with our method for 10^2 iterations and $I = 10^4$.

We can compare the efficiency $1/(\bar{S}^2 \times t)$ of the algo-

Simulation method	Time (Sec.)
Fleming and Schaeffer	3.9990e+03
New method	1.1564e+03

Table 3: Computational time comparison for both algorithms.

rithms. We observe that the new method is about 11.9 times more efficient than the best previous one. Let us remark that this number is considerably increased if we compare with a standard simulation.

CONCLUSION

We describe here a new variance reduction technique for a cellular system with dynamic resource sharing. This method uses low discrepancy sequences. Combined with importance sampling and control variates, it improves the results obtained in (Fleming and Schaeffer 1995).

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