Quasi-Monte Carlo Methods for Estimating Transient Measures of Discrete Time Markov Chains

Christian Lécot[1] and Bruno Tuffin[2]

[1] Laboratoire de Mathématiques
Université de Savoie
73376 Le Bourget-du-Lac Cedex, France
Christian.Lecot@univ-savoie.fr
[2] IRISA-INRIA
Campus universitaire de Beaulieu
35042 Rennes Cedex, France
Bruno.Tuffin@irisa.fr

Summary. We describe a new method for the transient simulation of discrete time Markov chains. It is a quasi-Monte Carlo method where different paths are simulated in parallel, but reordered at each step. We prove the convergence of the method, when the number of simulated paths increases. Using some numerical experiments, we illustrate that the error of the new algorithm is smaller than the error of standard Monte Carlo algorithms. Finally, we propose to analyze continuous time Markov chains by transforming them into a discrete time problem by using the uniformization technique.

Key words: Markov Chains, Quasi-Monte Carlo, Simulation

1 Introduction

In this presentation, we deal with the transient simulation of discrete time Markov chains which has applications in many fields such as queuing theory, telecommunications, reliability analysis, etc (see [Tri02] for instance).

Let us first recall the basics about discrete time Markov chains. Let Ω be a sample space and $\mathbb{P}$ a probability measure on it. A stochastic process $(Y_n)_{n\in\mathbb{N}}$ with discrete state space E is called a *Markov chain* if

$$\mathbb{P}[Y_{n+1} = j|Y_0, \cdots, Y_n] = \mathbb{P}[Y_{n+1} = j|Y_n]$$

for all $j \in E$ and $n \in \mathbb{N}$. We consider *time homogeneous* Markov chains, i.e., Markov chains such that

$$\mathbb{P}[Y_{n+1} = j | Y_n] = \mathbb{P}[Y_1 = j | Y_0]$$

for all $j \in E$ and $n \in \mathbb{N}$. To characterize these chains, it is then sufficient to have the initial distribution $\mu = (\mu\{i\} : i \in E)$ with $\mu\{i\} = \mathbb{P}[Y_0 = i]$ and the transition matrix $P = (p(i,j) : i,j \in E)$ with $p(i,j) = \mathbb{P}[Y_1 = j | Y_0 = i]$. In order to get the probability distribution at time n, μ_n (as a row vector), defined by $\mu_n\{i\} = \mathbb{P}[Y_n = i]$, we only need to compute μ_n by $\mu_n = \mu P^n$.

Due to complexity reasons, because this computation requires matrix multiplications, the state space cannot be very large ; unfortunately many applications require a prohibitively large number of states. Then, Monte Carlo (MC) methods [Fis96] are often the best solution to solve the problem.

Despite the versatility of MC methods, a drawback is their slow convergence. An approach to accelerate them is to change the choice of the random numbers that are used. Thus, quasi-Monte Carlo (QMC) methods use quasi-random numbers instead of pseudo-random numbers and can achieve a better convergence in certain cases [NS95, NHL98, NS00, FHN02].

The efficiency of a QMC method depends on the quality of the quasi-random points that are used. These points should form a low-discrepancy point set. We recall from [Nie92] some basic notations and concepts. Let $s \geq 1$ be a fixed dimension. For a point set $X = \{\mathbf{x}_0, \ldots, \mathbf{x}_{L-1}\}$, with $\mathbf{x}_\ell \in [0,1)^s$ $\forall 0 \leq \ell < L$, and for a Lebesgue-measurable subset Q of $[0,1)^s$ we define the *local discrepancy* by

$$D(Q;X) := \frac{1}{L} \sum_{0 \leq \ell < L} 1_Q(\mathbf{x}_\ell) - \int_{[0,1)^s} 1_Q(\mathbf{x}) d\mathbf{x},$$

where 1_Q is the characteristic function of Q. The *discrepancy* of the point set X is then defined by

$$D(X) := \sup_Q |D(Q;X)|,$$

the supremum being taken over all subintervals of $[0,1)^s$. Similarly, the *star discrepancy* of X is

$$D^\star(X) := \sup_{Q^\star} |D(Q^\star;X)|,$$

where $Q^\star$ runs through all subintervals of $[0,1)^s$ with one vertex at the origin. The idea of (t,q,s)-nets is to consider point sets X for which $D(Q;X) = 0$ for a large family of intervals Q. Such point sets should have a small discrepancy. For an integer $b \geq 2$, an *elementary interval in base* b is an interval of the form

$$\prod_{r=1}^{s} \left[\frac{a_r}{b^{q_r}}, \frac{a_r + 1}{b^{q_r}} \right),$$

with integers $q_r \geq 0$ and $0 \leq a_r < b^{q_r}$ for $1 \leq r \leq s$. If $0 \leq t \leq q$ are integers, a (t,q,s)-*net in base* b is a point set X consisting of b^q points in $[0,1)^s$ such that $D(Q;X) = 0$ for every elementary interval Q in base b with measure b^{t-q}. The analogous concept for an infinite sequence is defined as follows. If

$b \geq 2$ and $t \geq 0$ are integers, a sequence $\mathbf{x}_0, \mathbf{x}_1, \ldots$ of points in $[0,1)^s$ is a (t,s)-*sequence in base* b if, for all integers $n \geq 0$ and $q > t$, the points $\mathbf{x}_p$ with $nb^q \leq p < (n+1)b^q$ form a (t,q,s)-net in base b. For the construction of (t,s)-sequences, the reader can see, for instance, [Nie92]. The following result is shown in [Lec96].

Lemma 1 *Let X be a (t,q,s)-net in base b and $f : [0,1]^{s-1} \to [0,1]$ be of bounded variation in the sense of Hardy and Krause. Denote $Q_f := \{\mathbf{x} = (\mathbf{x}', x_s) \in [0,1)^s : x_s < f(\mathbf{x}')\}$. If $b^{t-q} \leq V(f)$, we have*

$$|D(Q_f; X)| \leq sV(f)b^{-\left\lfloor \frac{q-t}{s} + \frac{\log V(f)}{s \log b} \right\rfloor}.$$

The efficiency of QMC methods has limitations. They are valid for integration problems, and the induced correlation structure, necessary to improve MC simulation, makes them irrelevant when applied *directly* in a dynamic context. To our knowledge, the only attempt in the context of Markov chains is in [NG95], where QMC is used for regenerative simulation, but where the induced mathematical dimension is a number of steps of the paths, which render it inefficient, and where truncation has to be used in the path, introducing further approximations. This can be related to the so-called mixed sequences first introduced in [Spa95] and further studied in [Ökt96, Spa98], with applications in particle transport. In a mixed strategy, the initial decisions are made using quasi-random vector sequences while subsequent ones are made using a pseudo-random sequence (note also that Owen has suggested in [Owe00] to replace the pseudo-random vectors by a latin hypercube sample).

In this paper, we deal with the transient simulation of discrete time Markov chains. We propose to use a QMC method (in dimension 2) where different paths are simulated in parallel, but reordered at each step. The method may be viewed as an extension of a QMC simulation of random walk [CL98]. The same idea has been considered by several authors and applied to a number of other problems [MC93, Mor98, Mos95]. It can be related to the so-called scrambled strategy developed in [Spa95, Spa98] for particle transport problems. The aim there is pseudo-randomly to reorder at each time step a low dimensional quasi-random vector and use it, in place of a pseudo-random sequence, to make all the decisions according to the dynamics of the physical system. Our method could be formulated in a similar way by interpreting the reordering of paths after each step as a permutation applied to the quasi-random points (without reordering the paths). But our algorithm is purely deterministic and the motivation is to obtain an error bound (although this bound may be of little practical use). A systematic comparison between the scrambled strategy and our reordering technique certainly deserves attention but is left to future work.

The paper is organized as follows. In Section 2 we present the method in the context of Markov chains. In Section 3 we prove the convergence of

the method, as the number of simulated paths increases. Next, in Section 4 we illustrate the validity of the scheme on some simple discrete time models. In all our numerical illustrations, the low discrepancy sequences that will be used are (0,2)-Niederreiter sequences in base $b = 2$ [Nie92]. It means that, remarkably, the mathematical dimension of the sequence is 2, whatever the size of the simulated paths is. Section 5 deals with the analysis of continuous time Markov chains that are tranformed into discrete-time problems by using the uniformization technique. Finally, we conclude and give some directions for future research in Section 6.

2 The Method

Let E be a finite ($E := \{1, \ldots, N\}$) or countable ($E := \mathbb{N}$ or $E := \mathbb{Z}$) state-space. Consider a discrete-time Markov chain $(Y_n)_{n\in\mathbb{N}}$ with initial distribution $\mu = (\mu\{i\} : i \in E)$ and transition matrix $P = (p(i,j) : i,j \in E)$. We want to approximate the probability that after n steps the Markov chain is in a given state. Let δ_i be the row vector of unit mass at i.

$$\delta_i\{j\} = \begin{cases} 1 \text{ if } j = i, \\ 0 \text{ otherwise.} \end{cases}$$

An approximation of μP^n is

$$\mu^n := \frac{1}{L} \sum_{0\le\ell<L} \delta_{i_\ell^n},$$

for some integer L and judiciously chosen $i_0^n, \ldots, i_{L-1}^n \in E$. Let $b \ge 2$ and q be integers and put $L := b^q$. We shall use a low-discrepancy sequence $X = \{\mathbf{x}_0, \mathbf{x}_1, \ldots\} \subset [0,1)^2$ for QMC approximation. We assume that X is a $(t,2)$-sequence in base b for some $t \ge 0$. In addition, if X^n is the point set $\{\mathbf{x}_p : nL \le p < (n+1)L\}$ and if proj_1 and proj_2 denote the projections defined by $\mathrm{proj}_i(x_1, x_2) = x_i$, for $i = 1, 2$, we assume that

$$\mathrm{proj}_1 X^n \text{ is a } (0, q, 1)\text{-net in base } b, \tag{1}$$

and, if $E = \mathbb{Z}$,

$$0 \notin \mathrm{proj}_2 X. \tag{2}$$

Condition (1) ensures that each state i_ℓ^n changes at most once from step n to step $n+1$ and condition (2) allows to determine the new states i_ℓ^{n+1} when $E = \mathbb{Z}$ (see below).

A sample I^0 of L states $i_0^0, \ldots, i_{L-1}^0$ is chosen such that

$$\mu^0 := \frac{1}{L} \sum_{0\le\ell<L} \delta_{i_\ell^0} \approx \mu.$$

That means that the point set I^0 has a small star μ-discrepancy (see below). If we assume that we have calculated a set I^n of L states $i_0^n, \ldots, i_{L-1}^n$ such that

$$\mu^n := \frac{1}{L} \sum_{0 \le \ell < L} \delta_{i_\ell^n} \approx \mu P^n,$$

we compute μ^{n+1} in two steps.

- Relabeling the states.
$$i_0^n \le \cdots \le i_{L-1}^n. \tag{3}$$
This ensures convergence of the scheme.
- QMC integration for a transition. Let $\widetilde{\mu}^{n+1} := \mu^n P$, and so
$$\widetilde{\mu}^{n+1} u = \frac{1}{L} \sum_{0 \le \ell < L} \sum_{j \in E} p(i_\ell^n, j) u(j), \tag{4}$$
for any nonnegative and bounded sequence u. Let 1_ℓ be the characteristic function of
$$I_\ell := \left[\frac{\ell}{L}, \frac{\ell+1}{L} \right),$$
and $1_{i,j}$ denote the characteristic function of
$$I_{i,j} := \Big[\sum_{h<j} p(i,h), \sum_{h<j+1} p(i,h) \Big).$$
For any nonnegative and bounded sequence u, define
$$C^n u(\mathbf{x}) := \sum_{0 \le \ell < L} \sum_{j \in E} 1_\ell(x_1) 1_{i_\ell^n, j}(x_2) u(j), \quad \mathbf{x} \in [0,1)^2. \tag{5}$$
Then we have
$$\widetilde{\mu}^{n+1} u = \int_{[0,1)^2} C^n u(\mathbf{x}) d\mathbf{x}, \tag{6}$$
and the estimator μ^{n+1} of μP^{n+1} is defined by
$$\mu^{n+1} u = \frac{1}{L} \sum_{nL \le p < (n+1)L} C^n u(\mathbf{x}_p),$$
for any nonnegative and bounded sequence u.

The last step of the algorithm may be summarized as follows. Denote

$$\ell(x) := \lfloor Lx \rfloor, \quad x \in [0,1).$$

According to (1) the function

$$p \in \{nL, nL+1, \cdots, (n+1)L-1\} \to \ell(x_{p,1}) \in \{0, 1, \cdots, L-1\}$$

is one-to-one. For $(i,x) \in E \times [0,1)$ (or $(i,x) \in E \times (0,1)$, if $E = \mathbb{Z}$ since it may happen that $\forall j \in E \ \sum_{h<j} p(i,h) > 0$) define $j(i,x) \in E$ by

$$x \in I_{i,j(i,x)}.$$

Then the L states $i_0^{n+1}, \ldots, i_{L-1}^{n+1}$ are computed according to:

$$i_{\ell(x_{p,1})}^{n+1} = j\big(i_{\ell(x_{p,1})}^{n}, x_{p,2}\big), \quad \text{for } nL \le p < (n+1)L. \tag{7}$$

That means that the projection of the quasi-random sequence on the first coordinate is used to select the state at step n while the projection on the second coordinate is used to determine the state at step $n+1$.

3 Convergence

We now establish a convergence result for the QMC algorithm. First we need to adapt the basic concepts of QMC methods to the present study. Let λ be a distribution on E and $I := \{i_0, \ldots, i_{L-1}\} \subset E$. For an arbitrary subset F of E we define the *local λ-discrepancy* by

$$D(F; I, \lambda) := \frac{1}{L} \sum_{0 \le \ell < L} 1_F(i_\ell) - \sum_{i \in F} \lambda\{i\},$$

where 1_F denotes the sequence

$$1_F(i) = \begin{cases} 1 \text{ if } i \in F, \\ 0 \text{ otherwise.} \end{cases}$$

The *star λ-discrepancy* of the point set I is defined by

$$D^\star(I, \lambda) := \sup_{k \in E} |D(F_k; I, \lambda)|,$$

where

$$F_k := \{i \in E : i < k\}.$$

The *variation* of a sequence u is defined by

$$V(u) := \sum_{i \in E'} |u(i+1) - u(i)|,$$

where $E' := \{i \in E : i + 1 \in E\}$.

The next Lemma is a version of the classical Koksma-Hlawka inequality. The proof follows the general outline of the proof of the Koksma-Hlawka inequality given in [Zar68].

Lemma 2 *Let λ be a distribution on E. If u is a sequence of bounded variation and if I is a point set consisting of $i_0, \ldots, i_{L-1} \in E$, then*

$$\left| \frac{1}{L} \sum_{0 \le \ell < L} u(i_\ell) - \sum_{i \in E} \lambda\{i\} u(i) \right| \le V(u) D^\star(I, \lambda).$$

We shall also use the following notations. If u is a nonnegative and bounded sequence, then

$$D(u; I, \lambda) := \frac{1}{L} \sum_{0 \le \ell < L} u(i_\ell) - \sum_{i \in E} \lambda\{i\} u(i),$$

so that $D(F; I, \lambda) = D(1_F; I, \lambda)$. Similarly, if $X = \{\mathbf{x}_0, \ldots, \mathbf{x}_{L-1}\} \subset [0,1)^s$ and if f is a nonnegative and bounded function defined on $[0,1)^s$, we put

$$D(f; X) := \frac{1}{L} \sum_{0 \le \ell < L} f(\mathbf{x}_\ell) - \int_{[0,1)^s} f(\mathbf{x}) d\mathbf{x},$$

so that $D(Q; X) = D(1_Q; X)$ for any $Q \subset [0,1)^s$.

We now go back to the convergence analysis of the QMC algorithm.

Proposition 1 *If the transition matrix P satisfies*

$$\forall k \in E \quad \sum_{i \in E'} \left| \sum_{j < k} p(i+1, j) - \sum_{j < k} p(i, j) \right| \le 1,$$

then, for $q \ge t$,

$$D^\star(I^n, \mu P^n) \le D^\star(I^0, \mu) + 2nb^{-\lfloor \frac{q-t}{2} \rfloor}. \tag{8}$$

Proof. For any nonnegative and bounded sequence u, we have

$$D(u; I^{n+1}, \mu P^{n+1}) = D(Pu; I^n, \mu P^n) + D(C^n u; X^n).$$

In particular, if we write s_k for 1_{F_k}, we obtain

$$D(F_k; I^{n+1}, \mu P^{n+1}) = D(Ps_k; I^n, \mu P^n) + D(C^n s_k; X^n).$$

By Lemma 2, we have

$$|D(Ps_k; I^n, \mu P^n)| \le V(Ps_k) D^\star(I^n, \mu P^n),$$

and since $V(Ps_k) \le 1$,

$$|D(Ps_k; I^n, \mu P^n)| \le D^\star(I^n, \mu P^n).$$

On the other hand, according to (2),

$$D(C^n s_k; X^n) = D(Q_k^n; X^n),$$

where

$$Q_k^n := \bigcup_{0 \le \ell < L} I_\ell \times \Big[0, \sum_{j<k} p(i_\ell^n, j)\Big).$$

Let f_k^n denote the function

$$f_k^n(x) := \sum_{0 \le \ell < L} \sum_{j<k} p(i_\ell^n, j) 1_\ell(x), \quad x \in [0,1).$$

Then

$$Q_k^n = \{\mathbf{x} \in [0,1)^2 : x_2 < f_k^n(x_1)\}.$$

In view of (3) we have

$$V(f_k^n) \le \sum_{i \in E'} \Big| \sum_{j<k} p(i+1, j) - \sum_{j<k} p(i,j) \Big|,$$

and so $V(f_k^n) \le 1$. Since X^n is a $(t, q, 2)$-net in base b, an application of Lemma 1 yields

$$|D(Q_k^n; X^n)| \le 2b^{-\lfloor \frac{q-t}{2} \rfloor}.$$

The desired result follows by induction. □

Corollary 1 *If we wish to estimate the mean* $\mathbb{E}(u(Y_n))$ *of a sequence* u *such that* $V(u) < \infty$ *at the* n^{th} *step of a discrete-time Markov chain* $(Y_n)_{n \in \mathbb{N}}$*, the above estimation converges while the number* L *of paths tends to infinity.*

Similarly, if we wish to estimate the cumulative reward $\sum_{m=1}^n \mathbb{E}(u(Y_m))/n$ *up to time* n *instead of just at time* n*, the method converges under the same assumptions.*

Proof. The first part is a direct consequence of Lemma 2 and Proposition 1. The second part follows by averaging over the first n steps. □

4 Numerical Results

In all our numerical illustrations, the low discrepancy sequences that will be used are (0,2)-Niederreiter sequences in base $b = 2$.

4.1 A simple discrete time queue : the Geo/Geo/15 queue

We consider a small example, so that we can compute the exact solution, in order to show the kind of improvement that QMC can bring with respect to MC. Consider a (discrete) Geo/Geo/15 queue (see [Tak93] for a better understanding of these queues) where the queue is empty at the initial time, where each customer completes its service during a slot with probability 0.5, and

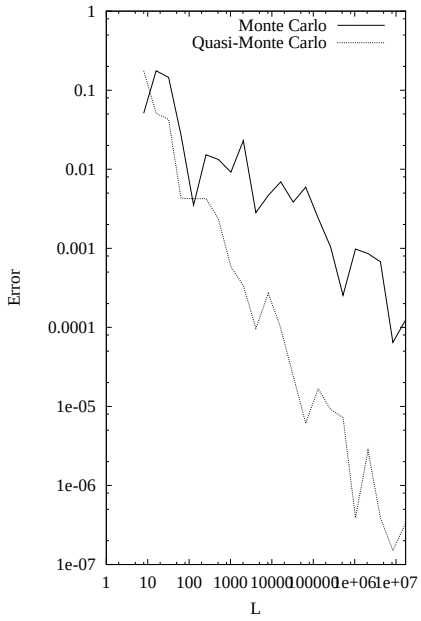

Fig. 1. Convergence with respect to L for the Geo/Geo/15 queue

where one customer can arrive with probability 0.6 in each slot. We are looking for the mean number of customers in the queue at time $n = 10$. Figure 1 shows the errors obtained for MC and QMC methods with respect to the number of chains that are used. QMC is showing a faster convergence. Formally, using regression analysis, we obtain the following convergence speeds :

$$\mathrm{Err}_{\mathrm{MC}}(L) = 0.22L^{-0.42},$$
$$\mathrm{Err}_{\mathrm{QMC}}(L) = 0.50L^{-0.91},$$

showing a strong improvement when using quasi-Monte Carlo.

Considering the average number of customers in the queue up to time $n = 10$, Fig. 2 shows the errors obtained for MC and QMC methods with respect to the number of chains used. The following convergence speeds are observed :

$$\mathrm{Err}_{\mathrm{MC}}(L) = 0.11L^{-0.51},$$
$$\mathrm{Err}_{\mathrm{QMC}}(L) = 0.18L^{-0.85},$$

showing here too the effectiveness of QMC methods.

Another interesting subject is to check whether the error increases with time (meaning n) as bound (8) could suggest. Fig. 3 shows that this is not the case.

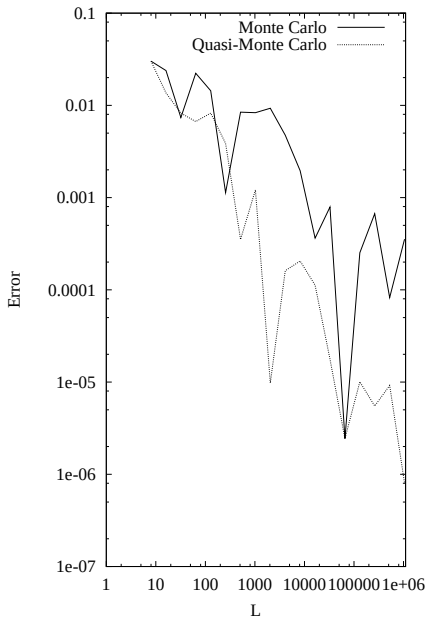

Fig. 2. Convergence with respect to L for the Geo/Geo/15 queue, in the average case

4.2 A larger example

We consider a gambler going to a casino for four hours. He plans to play to the same game every ten seconds, meaning that he will play 1440 times. In this game, for each dollar that you bid, you get 0 with probability 0.9 and k from 1 to 10 with probability 0.01 each. The gambler policy is the following : if he has more than 100 dollars, he plays 2 dollars, but if he has 100 dollars or less, he plays only 1. To make sure that he can play during the four hours, he brings 2780 dollars with him. The model is a discrete Markov chain on state space $E = \{0, 1, \ldots, 28\,700\}$. We wish to compute the mean amount of money the gambler is supposed to keep at the end of the day. Figure 4 shows the errors for both MC and QMC methods. The improvement is obvious, and QMC often gives the exact value.

5 Continuous Time Markov Chains

In this section, we explain how some continuous time Markov problems can be transformed into discrete time ones, and then solved by our method.

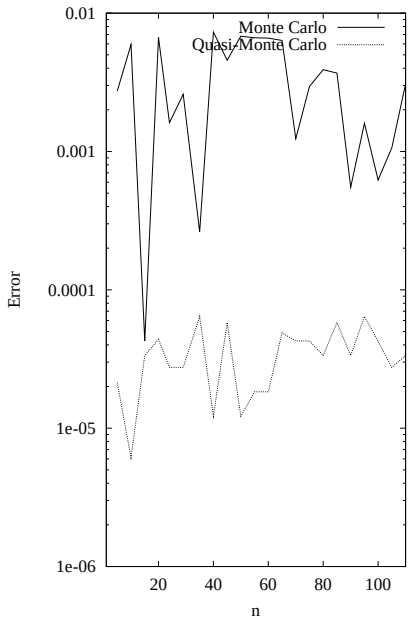

Fig. 3. Convergence with respect to n for the Geo/Geo/15 queue, with $L = 65\,536$

5.1 Uniformization

Assume that we have a continuous time Markov chain with initial distribution μ and infinitesimal generator A. The transient probability vector $p(t) = (p_i(t))_{i \in E}$ at time t is given by

$$p(t) = \mu e^{At}.$$

Assume that the Markov chain can be uniformized, i.e., that $\Lambda = \sup_i a_{i,i} < \infty$. Then, as $P = I + A/\Lambda$ is a stochastic matrix, the uniformization method transforms $p(t)$ into

$$p(t) = \sum_{k=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^k}{k!} q(k)$$

where

$$q(k) = \mu P^k$$

is the probability distribution at the k^{th} step of a discrete time Markov chain with initial vector distribution μ and probability matrix P.

An approximation of $p_i(t)$ with error ε is

$$p_i(t) \approx \sum_{k=0}^{K} e^{-\Lambda t} \frac{(\Lambda t)^k}{k\ !} q_i(k), \quad i \in E$$

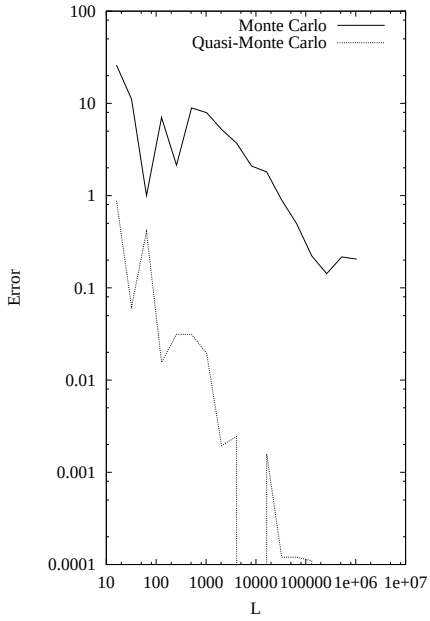

Fig. 4. Convergence with respect to L for the gambler game

where

$$K = \min\left\{ N \ : \sum_{k=0}^{N} e^{-\Lambda t} \frac{(\Lambda t)^k}{k\ !} > 1 - \varepsilon \right\}.$$

An estimator is then

$$\frac{1}{L} \sum_{i=1}^{L} Z_i$$

where each Z_i is a random variable on a path of the discrete time Markov chain $(Y_k)_{0 \le k \le K}$ defined by

$$Z_i = \sum_{k=0}^{K} e^{-\Lambda t} \frac{(\Lambda t)^k}{k!} 1_{\{Y_k = i\}}.$$

Each path is then of length $n = K$.

5.2 A M/M/1/40 queue

Consider the M/M/1/40 queue with arrival rate 0.97 and service rate 1.0 where we wish to compute the loss probability at horizon time $t = 150$. Using uniformization (which gives $K = 411$ for a precision $\varepsilon = 10^{-10}$), we obtain in

Fig. 5 the errors for MC and QMC methods with respect to the number of chains used. The following convergence speeds are observed:

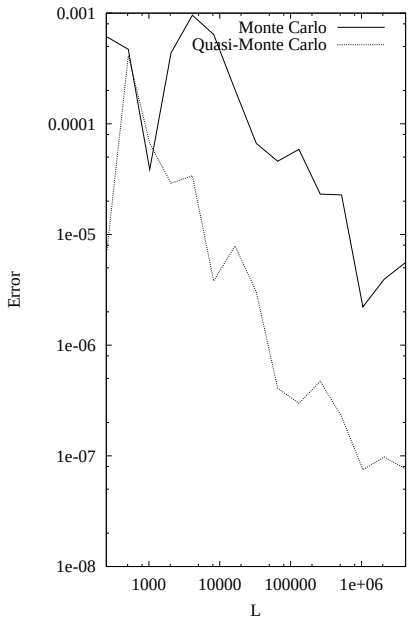

Fig. 5. Convergence with respect to L for the M/M/1/40 queue

$$\mathrm{Err}_{\mathrm{MC}}(L) = 0.020L^{-0.55}$$
$$\mathrm{Err}_{\mathrm{QMC}}(L) = 0.012L^{-0.83},$$

showing an accelerated convergence for QMC.

5.3 A BMAP/M/1/40 queue

Consider now the same queue but where each arrival (with arrival rate still 0.97) is a batch between 1 and 8 with uniform probability 0.125. Using uniformization again (we also have $K = 411$ for a precision $\varepsilon = 10^{-10}$), we obtain in Fig. 6 the errors for MC and QMC methods with respect to the number of chains used, while computing the loss probability with the same parameters than in the previous case. The following convergence speeds are observed:

$$\mathrm{Err}_{\mathrm{MC}}(L) = 0.0080L^{-0.37}$$
$$\mathrm{Err}_{\mathrm{QMC}}(L) = 0.019L^{-0.92}.$$

Here also, the convergence is strongly sped up with QMC.

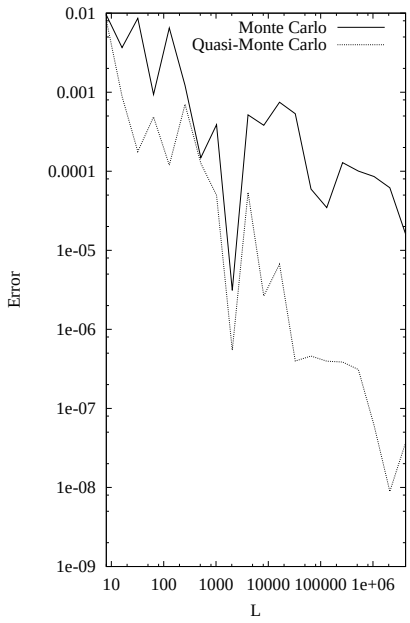

Fig. 6. Convergence with respect to L for the BMAP/M/1/40 queue

6 Conclusions

In this paper, we have devised a QMC method for the transient simulation of discrete time Markov chains. Indeed, when applied directly, QMC does not converge. But by relabelling the paths, we have been able to prove the convergence. We have also illustrated the degree of improvement that can be obtained.

As a direction for future research, we can try to extend the method to continuous time Markov chains (not using uniformization as a transformation) or to steady-state simulation. The pitfall we have to cope with is that the number of steps of each sample path is not a constant, which complicates the application of the method.

References

[CL98] Coulibaly, I., Lécot, C. (1998) Simulation of diffusion using quasi-random walk methods. Math. Comput. Simul. **47**, 153–163

[FHN02] Fang, K.-T., Hickernell, F.J., and Niederreiter, H. (Eds.) (2002) Monte Carlo and Quasi-Monte Carlo Methods 2000. Springer, Berlin

[Fis96] Fishman, G.S. (1996) Monte Carlo: Concepts, Algorithms, and Applications. Springer, New York

[Lec96] Lécot, C. (1996) Error bounds for quasi-Monte Carlo integration with nets. Math. Comp. **65**, 179–187

[Mor98] Morokoff, W. J. (1998) Generating quasi-random paths for stochastic processes. SIAM Rev. **40**, 765-788

[MC93] Morokoff, W. J., Caflisch, R. E. (1993) A quasi-Monte Carlo approach to particle simulation of the heat equation. SIAM J. Numer. Anal. **30**, 1558–1573

[Mos95] Moskowitz, B. (1995) Quasirandom diffusion Monte Carlo. In Niederreiter, H., Shiue, P.J.-S. (Eds.) Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing, Springer, New York, 278–298

[NG95] Nananukul, S., Gong, W.B. (1995) The quasi-Monte Carlo method for regenerative simulation. Proceedings of the 34th Conference on Decision and Control, New Orleans, 1964–1969

[Nie92] Niederreiter, H. (1992) Random Number Generation and Quasi-Monte Carlo Methods. SIAM, Philadelphia

[NS95] Niederreiter, H., Shiue, P.J.-S. (Eds.) (1995) Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing. Springer, New York

[NHL98] Niederreiter, H., Hellekalek, P., Larcher, G., and Zinterhof, P. (Eds.) (1998) Monte Carlo and Quasi-Monte Carlo Methods 1996. Springer, New York

[NS00] Niederreiter, H., Spanier, J. (Eds.) (2000) Monte Carlo and Quasi-Monte Carlo Methods 1998. Springer, Berlin

[Ökt96] Ökten, G. (1996) A probabilistic result on the discrepancy of a hybrid-Monte Carlo sequence and applications. Monte Carlo Methods and Appl. **2**, 255–270

[Owe00] Owen, A.B. (2000) Monte Carlo, quasi-Monte Carlo, and randomized quasi-Monte Carlo. In Niederreiter, H., Spanier, J. (Eds.) Monte Carlo and Quasi-Monte Carlo Methods 1998, Springer, Berlin, 86–97

[Spa95] Spanier, J. (1995) Quasi-Monte Carlo methods for particle transport problems. In Niederreiter, H., Shiue, P.J.-S. (Eds.) Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing, Springer, New York, 121–148

[Spa98] Spanier, J., Li, L. (1998) Quasi-Monte Carlo methods for integral equations. In Niederreiter, H., Hellekalek, P., Larcher, G., and Zinterhof, P. (Eds.) Monte Carlo and Quasi-Monte Carlo Methods 1996, Springer, New York, 398–414

[Tak93] Takagi, H. (1993) Queueing Analysis. A Foundation of Performance Evaluation. Volume 3: Discrete-Time Systems. North-Holland, Amsterdam

[Tri02] Trivedi, K.S. (2002) Probability and Statistics with Reliability, Queuing, and Computer Science Applications. John Wiley & Sons, 2nd ed., New York

[Zar68] Zaremba, S.K. (1968) Some applications of multidimensional integration by parts. Ann. Pol. Math. **21**, 85-96