A central limit theorem and improved error bounds for a hybrid-Monte Carlo sequence with applications in computational finance

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Abstract

In problems of moderate dimensions, the quasi-Monte Carlo method usually provides better estimates than the Monte Carlo method. However, as the dimension of the problem increases, the advantages of the quasi-Monte Carlo method diminish quickly. A remedy for this problem is to use hybrid sequences; sequences that combine pseudorandom and low-discrepancy vectors. In this paper we discuss a particular hybrid sequence called the mixed sequence. We will provide improved discrepancy bounds for this sequence and prove a central limit theorem for the corresponding estimator. We will also provide numerical results that compare the mixed sequence with the Monte Carlo and randomized quasi-Monte Carlo methods.

Key words: Hybrid-Monte Carlo, Quasi-Monte Carlo, Option pricing

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1 Introduction

In high dimensional problems, quasi-Monte Carlo methods (QMC) start losing their effectiveness over Monte Carlo methods (MC). The dimension above which QMC is no longer competitive depends on the problem at hand. Methods such as Anova decomposition of functions, and concepts such as effective dimension (see, for instance, Moskowitz and Caflisch [1]) have been used in the past to understand the relationship between the dimension of the function and the accuracy of QMC.

In order to address the potential difficulties of QMC in high dimensions, several authors introduced "hybrid" methods that make use of low-discrepancy sequences in some elaborate way, often combining them with pseudorandom numbers. Examples of such methods are the "mixed" and "scrambled" strategies used by Spanier [2] , the mixed sequence used by Ökten [3,4], the "renumbering" and "continuation" methods used by Moskowitz [5], and similar numbering techniques used by Coulibaly and Lécot [6], Morokoff and Caflisch [7], and Lécot and Tuffin [8]. The authors of these studies report favorable numerical results when the errors obtained from these hybrid methods are compared with the MC and QMC errors.

In this paper, we will discuss in detail methods that have been named as the mixed method, padding with MC, and padding with randomized QMC

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(RQMC) [9]. Consider the problem of estimating

$$I = \int_{[0,1]^s} f(x)dx \tag{1}$$

using sums of the form

$$\hat{I} = \frac{1}{N}\sum_{k=1}^{N} f(x^{(k)}) \tag{2}$$

where $x^{(k)}$ are s-dimensional vectors chosen appropriately. If the dimension s is large, and if it is possible to identify a smaller subset of d important variables $\{i_1, ..., i_d\}$, then one has the following options:

(1) Sample $\{i_1, ..., i_d\}$ using a d-dimensional QMC sequence, and for the rest of the variables use an $(s-d)$-dimensional MC (pseudorandom) sequence (called the mixed method, or padding QMC by MC);

(2) Sample $\{i_1, ..., i_d\}$ using a d-dimensional RQMC sequence, and for the rest of the variables use an $(s-d)$-dimensional MC (pseudorandom) sequence (called the randomized mixed method, or padding RQMC by MC).

Let $x^{(k)} = (q^{(k)}, X^{(k)})$ be an s-dimensional sequence obtained by concatenating the vectors $q^{(k)}$ and $X^{(k)}$. Here $(q^{(k)})_{k\geq 1}$, is a d-dimensional QMC sequence, and $X^{(k)}, k \geq 1$, are independent random variables with the uniform distribution on $(0,1)^{s-d}$. We will call $x^{(k)}$ a mixed sequence. The underlying sequences used in both of the strategies mentioned above are mixed sequences. The first strategy, in computing (2), uses a single mixed sequence to obtain the estimate $\hat{I}$, whereas the second strategy uses independent replications of a mixed sequence, where each replication involves an independent selection of an RQMC sequence, and random vectors $X^{(k)}, k \geq 1$. In our definition of $x^{(k)}$ we took the first d dimensions to be "important" for convenience. The results of the paper are still valid if the important d variables occurred at arbitrary locations. In Section 4, we will discuss these strategies in more detail and present a computational framework that will enable us to compare their effectiveness numerically.

In the next section, we will investigate the discrepancy of the mixed sequence, which is the underlying sequence in the strategies mentioned above. The reason we study the discrepancy is the Koksma-Hlawka inequality, which states that the error, $|I - \hat{I}|$, is bounded by the variation of f (in the sense of Hardy and Krause) multiplied by the discrepancy of the sequence, and thus smaller discrepancy suggests smaller error. The results of this section generalize the earlier results given in Ökten [3]. In Section 3, we will prove a central limit theorem for the estimator used in the mixed method. And in Section 4 we will present numerical results from computational finance.

2 An upper bound for the discrepancy of the mixed sequence

In the following $x^{(k)} = (q^{(k)}, X^{(k)})$ is the kth element of the s-dimensional mixed sequence, where $q^{(k)}$ and $X^{(k)}$ are the deterministic and stochastic components of dimension d and $s - d$. We will write the components of a vector α as $(\alpha_1, ..., \alpha_s)$.

Observe that $x^{(k)} < \alpha$ iff $q^{(k)} < \alpha'$ and $X^{(k)} < \alpha''$, where α' is the d-dimensional vector that consists of the first d components of the s-dimensional vector α, and α'' is the $(s - d)$- dimensional vector that consists of the rest of the components. Hence

$$P\{x^{(k)} < \alpha\} = 1_{[0,\alpha')}(q^{(k)}) P\{X^{(k)} < \alpha''\}.$$

The interval $[0, \alpha)$ is defined as $\prod_{k=1}^{s}[0, \alpha_k)$. Clearly, $P\{X^{(k)} < \alpha''\} = \prod_{k=d+1}^{s} \alpha_k$ which we will simply denote by p.

Let $Y \equiv Y(\alpha)$ be sample frequencies, related to the set $[0, \alpha)$:

$$Y = \frac{1}{N} \sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}).$$

We have

$$E[Y] = \frac{p}{N} \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)}) = \frac{pA}{N},$$
$$Var(Y) = \frac{1}{N^2} \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)})(p - p^2) = \frac{p(1-p)}{N^2} A,$$

where we denote the sum $\sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)})$ (a function of α' and all $q^{(k)}$) by A (or by $A_N(\alpha')$ if we need to show explicitly dependence on N and α). We assume that the sequence $\{q^{(k)}\}$ is dense in $[0, 1)^d$, i.e., for any $\alpha' \in (0, 1)^d$

$$\lim_{N \to \infty} A_N(\alpha') = \infty.$$

This is obviously true when $\{q^{(k)}\}$ is a low-discrepancy sequence.

Consider the local discrepancy random variable

$$g(\alpha) = \frac{1}{N} \sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - \prod_{k=1}^{s} \alpha_k = Y - \prod_{k=1}^{s} \alpha_k.$$

We want to study the star-discrepancies

$$D_N^*(x^{(k)}) = \sup_{\alpha \in (0,1]^s} \left| \frac{1}{N} \sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - \prod_{i=1}^{s} \alpha_i \right| = \sup_{\alpha \in (0,1]^s} |g(\alpha)|,$$

$$D_N^*(q^{(k)}) = \sup_{\alpha' \in (0,1]^d} \left| \frac{1}{N} \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)}) - \prod_{i=1}^{d} \alpha'_i \right|$$

and in particular investigate the probability

$$P\{D_N^*(x^{(k)}) < \varepsilon + D_N^*(q^{(k)})\} \tag{3}$$

where ε is a positive real number. In the rest of this section, we will simply write $\sup_\alpha$ for $\sup_{\alpha\in(0,1]^s}$ and $\sup_{\alpha'}$ for $\sup_{\alpha'\in(0,1]^d}$ for convenience.

Lemma 1

$$D_N^*(x^{(k)}) - D_N^*(q^{(k)}) \le \sup_\alpha |g(\alpha) - E[g(\alpha)]| \; . \tag{4}$$

Proof.

$$\begin{aligned}
D_N^*(x^{(k)}) &= \sup_\alpha \left| \frac{1}{N}\sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - p\prod_{k=1}^{d}\alpha_k \right| \\
&= \sup_\alpha \left| \frac{1}{N}\sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - p\frac{A}{N} + p\left(\frac{A}{N} - \prod_{k=1}^{d}\alpha_k\right) \right| \\
&\le \sup_\alpha \left| \underbrace{\frac{1}{N}\sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - p\frac{A}{N}}_{g(\alpha)-E[g(\alpha)]} \right| + \sup_\alpha p\left|\frac{A}{N} - \prod_{k=1}^{d}\alpha_k\right| \\
&\le \sup_\alpha |g(\alpha) - E[g(\alpha)]| + \underbrace{\sup_{\alpha'}\left|\frac{A}{N} - \prod_{k=1}^{d}\alpha_k\right|}_{D_N^*(q^{(k)})} \\
&\le \sup_\alpha |g(\alpha) - E[g(\alpha)]| + D_N^*(q^{(k)}).
\end{aligned}$$

■

This lemma suggests that to study (3) we need to investigate the behavior of the random variables

$$g(\alpha) - E[g(\alpha)] = \frac{1}{N}\sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - p\frac{A}{N}$$

and

$$\sup_\alpha |g(\alpha) - E[g(\alpha)]| \, .$$

From Kolmogorov's strong law of large numbers, it can be shown that for any

α

$$g(\alpha) - Eg(\alpha) = \frac{1}{N}\sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - \frac{pA}{N} \to 0 \quad (a.s.)$$

as $N \to \infty$. We will now prove a stronger result.

Lemma 2

$$\lim_{N\to\infty} \sup_{\alpha} |g(\alpha) - E[g(\alpha)]| = \lim_{N\to\infty} \sup_{\alpha'} G_N(\alpha') = 0 \ \textit{(a.s.)}$$

where

$$G_N(\alpha') := \frac{1}{N} \sup_{\alpha''} \left| \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)}) \left(1_{[0,\alpha'')}(X^{(k)}) - p\right)\right|.$$

Proof. The first equality in Lemma 2 follows from

$$\begin{aligned}
\sup_{\alpha} |g(\alpha) - E[g(\alpha)]| &= \sup_{\alpha} \left| \frac{1}{N} \sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - p\frac{A}{N} \right| \\
&= \sup_{\alpha} \frac{1}{N} \left| \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)}) 1_{[0,\alpha'')}(X^{(k)}) - Ap \right| \\
&= \sup_{\alpha} \frac{1}{N} \left| \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)}) 1_{[0,\alpha'')}(X^{(k)}) - \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)}) p \right| \\
&= \sup_{\alpha'} \sup_{\alpha''} \frac{1}{N} \left| \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)}) \left(1_{[0,\alpha'')}(X^{(k)}) - p\right)\right| .
\end{aligned}$$

Now we will prove that the limit is zero. Note that for any $\alpha' \in (0,1]^d$ we have

$$\begin{aligned}
&\lim_{N\to\infty} \sup_{\alpha''} \frac{1}{N} \left| \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)}) \left(1_{[0,\alpha'')}(X^{(k)}) - p\right)\right| \\
&= \lim_{N\to\infty} \left(\sup_{\alpha''} \frac{1}{N} \left| \sum_{k\in\chi(\alpha')} \left(1_{[0,\alpha'')}(X^{(k)}) - p\right)\right|\right) \\
&\leq \lim_{A\to\infty} \left(\sup_{\alpha''} \frac{1}{A} \left| \sum_{k\in\chi(\alpha')} \left(1_{[0,\alpha'')}(X^{(k)}) - p\right)\right|\right) = 0 \quad (a.s.) \qquad (5)
\end{aligned}$$

from Glivenko-Cantelli's theorem. Here $\chi(\alpha')$ is the subset of the index set $\{1, ..., N\}$ that consists of k for which $1_{[0,\alpha')}(q^{(k)}) = 1$, and A is the cardinality of $\chi(\alpha')$. Also note that for any $\alpha' \in (0,1]^d$, A goes to infinity together with N, provided the sequence $\{q^{(k)}\}$ is dense in $[0,1)^d$.

Let

$$R_\varepsilon = \{\alpha' | \alpha' \in (0,1]^d, \ \min_i \alpha'_i < \varepsilon/2\},$$

where ε is an arbitrary small positive real number. Notice that

$$\prod_{i=1}^d \alpha'_i < \frac{\varepsilon}{2}, \quad \forall \alpha' \in R_\varepsilon.$$

By definition of the star-discrepancy $D_N^*(q^{(k)})$, for any α'

$$\frac{A_N(\alpha')}{N} \le D_N^*(q^{(k)}) + \prod_{i=1}^d \alpha'_i,$$

therefore

$$\sup_{\alpha' \in R_\varepsilon} \frac{A_N(\alpha')}{N} < D_N^*(q^{(k)}) + \frac{\varepsilon}{2}. \tag{6}$$

Also notice that

$$\inf_{\alpha' \in R_\varepsilon^c} A_N(\alpha') = A_N(\alpha'_\varepsilon) \tag{7}$$

where $R_\varepsilon^c = (0,1]^d \setminus R_\varepsilon$ is the complement of the set R_ε, and $\alpha'_\varepsilon = \frac{1}{2}(\varepsilon, \varepsilon, \dots, \varepsilon)$ (this is a d-dimensional vector). The denseness of the sequence $\{q^{(k)}\}$ implies that $A_N(\alpha'_\varepsilon) \to \infty$ as $N \to \infty$.

From the definition of the $G_N(\alpha')$ it follows that

$$\sup_{\alpha' \in R_\varepsilon} G_N(\alpha') \le \frac{1}{N} \sup_{\alpha' \in R_\varepsilon} \sum_{k=1}^N 1_{[0,\alpha')}(q^{(k)}) = \sup_{\alpha' \in R_\varepsilon} \frac{A_N(\alpha')}{N} < D_N^*(q^{(k)}) + \frac{\varepsilon}{2},$$

last inequality following from (6).

Now, to prove the statement of the lemma, we need to connect the supremum over α' with the supremum over $\alpha' \in R_\varepsilon$. To this end, we note

$$\begin{aligned} \sup_{\alpha'} G_N(\alpha') &= \max\left\{ \sup_{\alpha' \in R_\varepsilon} G_N(\alpha'), \sup_{\alpha' \in R_\varepsilon^c} G_N(\alpha') \right\} \\ &\le \max\left\{ D_N^*(q^{(k)}) + \frac{\varepsilon}{2}, \sup_{\alpha' \in R_\varepsilon^c} G_N(\alpha') \right\}. \end{aligned} \tag{8}$$

For a uniformly distributed sequence $\{q^{(k)}\}$, $D_N^*(q^{(k)})$ tends to zero as $N \to \infty$, and we may choose n_ε large enough so that for any $N > n_\varepsilon$

$$D_N^*(q^{(k)}) < \frac{\varepsilon}{2},$$

so

$$\sup_{N>n_\varepsilon} \sup_{\alpha'} G_N(\alpha') \geq \varepsilon \quad \Leftrightarrow \quad \sup_{N>n_\varepsilon} \sup_{\alpha' \in R_\varepsilon^c} G_N(\alpha') \geq \varepsilon$$

and consequently

$$P\left\{\sup_{N>n_\varepsilon} \sup_{\alpha'} G_N(\alpha') \geq \varepsilon\right\} = P\left\{\sup_{N>n_\varepsilon} \sup_{\alpha' \in R_\varepsilon^c} G_N(\alpha') \geq \varepsilon\right\}.$$

Now we investigate the probability in the right-hand side of the above equation in more detail:

$$\begin{aligned} & P\left\{\sup_{N>n_\varepsilon} \sup_{\alpha' \in R_\varepsilon^c} G_N(\alpha') \geq \varepsilon\right\} \\ & \quad = P\left\{\sup_{N>n_\varepsilon} \sup_{\alpha' \in R_\varepsilon^c} \sup_{\alpha''} \frac{1}{N}\left|\sum_{k \in \chi(\alpha')} \left(1_{[0,\alpha'')}(X^{(k)}) - p\right)\right| \geq \varepsilon\right\} \\ & \quad = P\left\{\sup_{N>n_\varepsilon} \sup_{\alpha' \in R_\varepsilon^c} \sup_{\alpha''} \frac{1}{N}\left|\sum_{j=1}^{A_N(\alpha')} \left(1_{[0,\alpha'')}(Z^{(j)}) - p\right)\right| \geq \varepsilon\right\} \end{aligned} \tag{9}$$

where $Z^{(j)}$ are independent random vectors uniformly distributed on $[0,1)^{s-d}$. Note that the only term in the above summation that depends on α' is the number of summands. Recall that $\chi(\alpha')$ is the subset of the index set $\{1, ..., N\}$ that consists of k for which $1_{[0,\alpha')}(q^{(k)}) = 1$, and $A_N(\alpha')$ is the cardinality of $\chi(\alpha')$. The random variables $X^{(k)}$ are from an i.i.d. sequence, so it does not matter which ones are selected by $k \in \chi(\alpha')$. To emphasize this point we introduced a new index j and replaced $X^{(k)}$ by $Z^{(j)}$ in the last expression.

Since $A_N(\alpha') \le N$, the above probability is less than or equal to

$$\le P\left\{\sup_{N>n_\varepsilon}\sup_{\alpha'\in R_\varepsilon^c}\sup_{\alpha''}\frac{1}{A_N(\alpha')}\left|\sum_{j=1}^{A_N(\alpha')}\left(1_{[0,\alpha'')}(Z^{(j)})-p\right)\right|\ge\varepsilon\right\}$$
$$\le P\left\{\sup_{\alpha'\in R_\varepsilon^c}\sup_{k\ge A_{n_\varepsilon}(\alpha')}\sup_{\alpha''}\frac{1}{k}\left|\sum_{j=1}^{k}\left(1_{[0,\alpha'')}(Z^{(j)})-p\right)\right|\ge\varepsilon\right\}\ ; \tag{10}$$

the last inequality follows since if $N > n_\varepsilon$ then $k = A_N(\alpha') \ge A_{n_\varepsilon}(\alpha')$. The supremum over $\alpha' \in R_\varepsilon^c$ and $k \ge A_{n_\varepsilon}(\alpha')$ is equivalent to the supremum over $k \ge \inf_{\alpha'\in R_\varepsilon^c} A_{n_\varepsilon}(\alpha')$, and from (7) $\inf_{\alpha'\in R_\varepsilon^c} A_{n_\varepsilon}(\alpha') = A_{n_\varepsilon}(\alpha'_\varepsilon)$, where $\alpha'_\varepsilon = \frac{1}{2}(\varepsilon,\varepsilon,\ldots,\varepsilon)$. Therefore the probability simplifies to

$$= P\left\{\sup_{k\ge A_{n_\varepsilon}(\alpha'_\varepsilon)}\sup_{\alpha''}\frac{1}{k}\left|\sum_{j=1}^{k}\left(1_{[0,\alpha'')}(Z^{(j)})-p\right)\right|\ge\varepsilon\right\}.$$

From Glivenko-Cantelli's theorem, the above probability converges to zero as $A_{n_\varepsilon}(\alpha'_\varepsilon) \to \infty$ or $n_\varepsilon \to \infty$.

We have shown

$$\lim_{n_\varepsilon\to\infty} P\left\{\sup_{N>n_\varepsilon}\sup_{\alpha'} G_N(\alpha') \ge \varepsilon\right\} = \lim_{n_\varepsilon\to\infty} P\left\{\sup_{N>n_\varepsilon}\sup_{\alpha'\in R_\varepsilon^c} G_N(\alpha') \ge \varepsilon\right\} = 0$$

for any $\varepsilon > 0$, which is equivalent to the statement

$$\lim_{N\to\infty}\sup_{\alpha'} G_N(\alpha') = 0 \quad \text{(a.s.)}$$

that we wanted to prove. ■

The following lemma is from McDiarmid [10].

Lemma 3 (McDiarmid) *Let $X_1,...,X_N$ be independent random variables, with X_i taking values in a set S_i for each i. Suppose that the measurable function $f : \prod S_i \to \mathbb{R}$ satisfies $|f(x) - f(x')| \le c_i$ whenever the vectors x and x' differ only in the ith coordinate. Let $\mathbf{X}$ be the random variable $f(X_1,...,X_N)$.*

Then for any $\varepsilon > 0$,

$$P(|\mathbf{X} - E(\mathbf{X})| \geq \varepsilon) \leq 2e^{-2\varepsilon^2 / \sum_{i=1}^{N} c_i^2}.$$

We need this lemma to find a bound for $\sup_\alpha |g(\alpha) - E[g(\alpha)]|$.

Lemma 4

$$P(|\sup_\alpha |g(\alpha) - E[g(\alpha)]| - E(\sup_\alpha |g(\alpha) - E[g(\alpha)]|)| < \varepsilon) \geq 1 - 2e^{-2N\varepsilon^2}. \quad (11)$$

Proof. Let

$$h_N(\mathbf{x}) := h_N(1_{[0,\alpha)}(x^{(1)}), ..., 1_{[0,\alpha)}(x^{(N)})) = \sup_\alpha |h_N(\alpha, \mathbf{x})|$$

where

$$h_N(\alpha, \mathbf{x}) = g(\alpha) - E[g(\alpha)] = \frac{1}{N} \sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - \frac{p_\alpha}{N} A_x$$

and $x^{(k)} = (q^{(k)}, X^{(k)}), k = 1, ..., N$ is a mixed sequence. In the above expression, we modified our previous notation as $p_\alpha := p = \prod_{k=d+1}^{s} \alpha_k$ and $A_x = A = \sum_{k=1}^{N} 1_{[0,\alpha')}(q^{(k)})$, to emphasize the dependencies on their subscripts, which will be essential in this proof. Now consider another mixed sequence $z^{(k)} = (r^{(k)}, Z^{(k)})$ and associated random variables $1_{[0,\alpha)}(z^{(1)}), ..., 1_{[0,\alpha)}(z^{(N)})$ where $1_{[0,\alpha)}(z^{(k)}) = 1_{[0,\alpha)}(x^{(k)})$ for all k except for $k = i$. We want to find a bound on $|h_N(\mathbf{x}) - h_N(\mathbf{z})|$, which will help us apply the McDiarmid's Lemma to h_N. Note that in applying this lemma, we will take $x^{(k)} = (q^{(k)}, X^{(k)})$ as the random variable denoted by X_k in the statement of Lemma 3.

Keeping in view the elementary property of the sup function

$$\sup_\alpha |h_N(\alpha, \mathbf{x})| \leq \sup_\alpha |h_N(\alpha, \mathbf{x}) - h_N(\alpha, \mathbf{z})| + \sup_\alpha |h_N(\alpha, \mathbf{z})|,$$

we have

$$
\begin{aligned}
|h_N(\mathbf{x}) - h_N(\mathbf{z})| &= \left|\sup_\alpha |h_N(\alpha,\mathbf{x})| - \sup_\alpha |h_N(\alpha,\mathbf{z})|\right| \\
&\le \sup_\alpha \left| \frac{1}{N}\sum_{k=1}^{N} 1_{[0,\alpha)}(x^{(k)}) - \frac{p_\alpha}{N}A_x - \frac{1}{N}\sum_{k=1}^{N} 1_{[0,\alpha)}(z^{(k)}) + \frac{p_\alpha}{N}A_z \right| \\
&= \sup_\alpha \left| \frac{1_{[0,\alpha)}(x^{(i)}) - 1_{[0,\alpha)}(z^{(i)})}{N} + \frac{p_\alpha}{N}\left(1_{[0,\alpha')}(r^{(i)}) - 1_{[0,\alpha')}(q^{(i)})\right) \right| \\
&\le \frac{1}{N},
\end{aligned}
$$

where we took into account that the differences $1_{[0,\alpha)}(x^{(i)}) - 1_{[0,\alpha)}(z^{(i)})$ and $1_{[0,\alpha')}(r^{(i)}) - 1_{[0,\alpha')}(q^{(i)})$ either have opposite signs or are zeros. Then, the constants in McDiarmid's Lemma are

$$
c_i = 1/N \text{ and } \sum_{i=1}^{N} c_i^2 = \sum_{i=1}^{N} 1/N^2 = 1/N
$$

and thus from the same Lemma

$$
P(|h_N - Eh_N| \ge \varepsilon) \le 2e^{-2\varepsilon^2 N}
$$

or

$$
P(|\sup_\alpha |g(\alpha) - E[g(\alpha)]| - E(\sup_\alpha |g(\alpha) - E[g(\alpha)]|)| < \varepsilon) \ge 1 - 2e^{-2\varepsilon^2 N}
$$

for any $\varepsilon > 0$. ■

We can now state and prove our main theorem.

Theorem 5 *Let $x^{(k)} = (q^{(k)}, X^{(k)})$ be an s-dimensional mixed sequence, where $q^{(k)}$ is a d-dimensional low-discrepancy sequence, and $X^{(k)}$ is a random variable with the uniform distribution on $(0,1)^{s-d}$. Then for any $\varepsilon > 0$*

$$
P(D_N^*(x^{(k)}) - D_N^*(q^{(k)}) < \varepsilon) \ge 1 - 2e^{-2\varepsilon^2 N},
$$

for sufficiently large N.

Proof. Let $\varepsilon > 0$. From Lemma 2 and the dominated convergence theorem, $E[h_N] = E[\sup_\alpha |g(\alpha) - E[g(\alpha)]|] \to 0$ as $N \to \infty$. Choose N sufficiently large so that $E[h_N] < \varepsilon/2$. Then

$$|h_N - E[h_N]| < \varepsilon/2 \Rightarrow h_N < \varepsilon$$

and from Lemma 1

$$h_N < \varepsilon \Rightarrow D_N^*(x^{(k)}) - D_N^*(q^{(k)}) < \varepsilon.$$

Therefore

$$P(|h_N - E[h_N]| < \varepsilon/2) \le P(D_N^*(x^{(k)}) - D_N^*(q^{(k)}) < \varepsilon)$$

and using the bound of Lemma 4 we conclude

$$P(D_N^*(x^{(k)}) - D_N^*(q^{(k)}) < \varepsilon) \ge 1 - 2e^{-2\varepsilon^2 N}.$$

■

Corollary 6 *Put* $\varepsilon := (\varepsilon_N) = (N^{-a/2}), \quad 0 < a < 1$, *in the above theorem, and let* $\{q^{(k)}\}_{k=1}^{\infty}$ *be a low-discrepancy sequence with* $D_N^*(q^{(k)}) \le c_d \frac{(\log N)^d}{N} + O\left(\frac{(\log N)^{d-1}}{N}\right)$. *Then, for sufficiently large* N, *the discrepancy of the mixed sequence satisfies*

$$D_N^*(x^{(k)}) < \frac{1}{N^{a/2}} + c_d \frac{(\log N)^d}{N} + O\left(\frac{(\log N)^{d-1}}{N}\right), \tag{12}$$

with probability greater than or equal to

$$1 - 2e^{-2N^{1-a}}. \tag{13}$$

The best values for $c_d, 2 \le d \le 20$, are calculated by Kritzer (see Table 3 of [11]), for Niederreiter-Xing sequences. These values improve the ones published earlier by Niederreiter in [12]. Omitting the lower order terms, let $A_1 = c_s N^{-1}(\log N)^s$ be the upper bound for the discrepancy of the s-dimensional Niederreiter-Xing sequence, and $A_2 = N^{-a/2} + c_d N^{-1}(\log N)^d$ be the probabilistic upper bound (12) for the corresponding mixed (s, d) sequence. In Table 1, we compute A_1 and A_2 using two-digit rounding arithmetic when $N = 10^7, a = 0.8, d = s/2$, and $s = 4, 6, ..., 20$. The lower bound (13) for the probability is equal to one for these parameters. We see factors of improvement as high as 10^4. Please note that the bound A_2 and its corresponding probability is valid when N is sufficiently large. In this paper, we do not investigate how large N should be for these bounds to be valid, and present these numerical results only for a rough understanding of the magnitudes involved.

Table 1

Bounds for the discrepancy

s	A_1	A_2
4	2.9×10^{-4}	1.6×10^{-3}
6	2.1×10^{-3}	1.8×10^{-3}
8	7.2×10^{-2}	1.9×10^{-3}
10	5.2×10^{-1}	2.4×10^{-3}
12	2.5	3.7×10^{-3}
14	2.7×10	1.5×10^{-2}
16	7.2×10	7.4×10^{-2}
18	1.5×10^{2}	1.2×10^{-1}
20	2.3×10^{3}	5.2×10^{-1}

3 A central limit theorem for the mixed method

The problem we are interested in is the estimation of the integral of a bounded function over the s-dimensional hypercube

$$I = \int_{[0,1]^s} f(x)dx,$$

using the estimator

$$\theta_m = \frac{1}{N}\sum_{k=1}^{N} f(x^{(k)})$$

where $\{x^{(k)}\}_{k=1}^{\infty}$ is the $s-$dimensional mixed sequence

$$x^{(k)} = (q_1^{(k)}, \cdots, q_d^{(k)}, X_{d+1}^{(k)}, \cdots,, X_s^{(k)}).$$

Define the random variables

$$Y_k = f\left(q_1^{(k)}, \cdots, q_d^{(k)}, X_{d+1}^{(k)}, \cdots,, X_s^{(k)}\right),$$

let $\mu_k = E[Y_k]$ and $\sigma_k^2 = Var(Y_k)$ and

$$s_N^2 = Var(\theta_m)N^2 = \sigma_1^2 + ... + \sigma_N^2.$$

We will next prove a central limit theorem stating that, (1) The estimator θ_m is asymptotically normally distributed; (2) Its asymptotic variance is theoretically known; (3) The estimator has a smaller variance than the MC method asymptotically.

Theorem 7 *Assume that f is bounded over $[0,1]^s$ and the functions*

$$g(x_1, ..., x_d) = \int_{[0,1]^{s-d}} \left(f(x_1, ..., x_d, X_{d+1}, ..., X_s)\right)^2 dX_{d+1}...dX_s$$
$$h(x_1, ..., x_d) = \left(\int_{[0,1]^{s-d}} f(x_1, ..., x_d, X_{d+1}, ..., X_s)dX_{d+1}...dX_s\right)^2$$

are Riemann integrable. Then

(1) The distribution of the normalized sum

$$\frac{\sum_{k=1}^{N} Y_k - \sum_{k=1}^{N} \mu_k}{s_N}$$

tends to the standard normal distribution.

(2) We have

$$s_N^2/N \to L = \int_{[0,1]^s} f(x)^2 dx - \int_{[0,1]^d} \left(\int_{[0,1]^{s-d}} f(y,x) dx \right)^2 dy;$$

(3) The mixed strategy always yields a reduction in the standard MC variance, with the reduction given by

$$\frac{\int_{[0,1]^s} f(x)^2 dx - \int_{[0,1]^d} \left(\int_{[0,1]^{s-d}} f(y,x) dx \right)^2 dy}{\int_{[0,1]^s} f(x)^2 dx - \left(\int_{[0,1]^s} f(x) dx \right)^2} \leq 1.$$

Proof. The variance of Y_k is

$$\sigma_k^2 = \int_{[0,1]^{s-d}} (f(q_1^{(k)}, \cdots, q_d^{(k)}, X_{d+1}, \cdots, X_s))^2 dX_{d+1} \cdots dX_s -$$
$$\left(\int_{[0,1]^{s-d}} f(q_1^{(k)}, \cdots, q_d^{(k)}, X_{d+1}, \cdots, X_s) dX_{d+1} \cdots dX_s \right)^2$$

Since f is bounded, Y_n are also bounded and, from a standard result (see Feller [13]), it suffices to show that $s_N \to \infty$ when $N \to \infty$ to verify the Lindeberg condition that ensures a central limit theorem for independent but non-identical random variables. But, from the theory of uniform distribution of sequences (see e.g. Corollary 1.1 and Exercise 6.3 in Chapter 1 of [14]), since g and h are Riemann integrable, we have

$$\frac{1}{N} \sum_{k=1}^{N} g(q_1^{(k)}, \cdots, q_d^{(k)}) \to \int_{[0,1]^d} f(x)^2 dx$$

and

$$\frac{1}{N} \sum_{k=1}^{N} h(q_1^{(k)}, \cdots, q_d^{(k)}) \to \int_{[0,1]^d} h(y) dy = \int_{[0,1]^d} \left(\int_{[0,1]^{s-d}} f(y,x) dx \right)^2 dy,$$

proving Claim 2. The Lindeberg condition is satisfied and we get the central limit theorem of Claim 1. For the last claim, we note that $s_N^2/N \to \int_{[0,1]^s} f(x)^2dx - \int_{[0,1]^d}\left(\int_{[0,1]^{s-d}} f(y,x)dx\right)^2 dy$ as $N \to \infty$ whereas $\sigma^2 = \int_{[0,1]^s} f(x)^2dx - \left(\int_{[0,1]^s} f(x)dx\right)^2$ is the variance of $f(X)$ for X uniformly distributed over $(0,1)^s$. The fact that we always get a variance reduction comes from

$$\int_{[0,1]^d}\left(\int_{[0,1]^{s-d}} f(y,x)dx\right)^2 dy \geq \left(\int_{[0,1]^d}\int_{[0,1]^{s-d}} f(y,x)dxdy\right)^2$$

(special case of the Cauchy-Schwarz inequality). ■

Remark 8 *It is important to note that the theorem is valid as long as the deterministic sequence used in the definition of the estimator θ_m is uniformly distributed modulo one. In particular, if we choose the sequence to be a low-discrepancy sequence, its faster convergence rate if f and g are of bounded variation (see [12]) will help reduce the bias of the estimator, and increase the convergence rate of the variance to its asymptotic value. Both of these observations follow from the Koksma-Hlawka inequality [12].*

Currently we do not know a practical and efficient way of estimating s_N. An upper bound for s_N, however, can be found using the variance of the MC estimator. Indeed, let us assume that the d-dimensional functions f, f^2 are Riemann integrable. Using this fact, and the fact that the discrepancy of the first N points of the sequence $(q_1^{(k)}, ..., q_d^{(k)}, X_{d+1}^{(k)}, ..., X_s^{(k)})_k$ tends almost surely to zero when $N \to \infty$ (from Lemmas 1 and 2), we obtain

$$\frac{1}{N}\sum_{k=1}^{N} f^2(q_1^{(k)}, \cdots, q_d^{(k)}, X_{d+1}^{(k)}, \cdots, , X_s^{(k)}) \to \int_{[0,1]^s} f^2(x)dx$$
$$\frac{1}{N}\sum_{k=1}^{N} f(q_1^{(k)}, \cdots, q_d^{(k)}, X_{d+1}^{(k)}, \cdots, , X_s^{(k)}) \to \int_{[0,1]^s} f(x)dx$$

and thus

$$\frac{1}{N}\sum_{k=1}^{N} f^2(q_1^{(k)},...,q_d^{(k)},X_{d+1}^{(k)},...,X_s^{(k)}) - \left(\frac{1}{N}\sum_{k=1}^{N} f(q_1^{(k)},...,q_d^{(k)},X_{d+1}^{(k)},...,X_s^{(k)})\right)^2 \to \sigma^2$$

almost surely as $N \to \infty$.

4 Randomization and numerical results

4.1 Randomization, estimators and efficiency

In this section we will compare the mixed method with MC and randomized mixed (Rmixed) methods numerically, when they are applied to problems from security pricing. For simplicity, we define our estimators in the context of numerical quadrature; they are extended easily to the more complicated problem from finance. To this end, consider the problem of computing

$$I = \int_{[0,1]^s} f(x)dx.$$

Let $X^{(k)}, k = 1, ...$ be a sequence of i.i.d random variables with distribution $U(0,1)^s$, $X_i^{(k)}, i = d+1, ..., s$; $k = 1, ...,$ be a sequence of i.i.d random variables with distribution $U(0,1)$, $x^{(k)} = (q_1^{(k)}, \cdots, q_d^{(k)}, X_{d+1}^{(k)}, \cdots, X_s^{(k)})$ be the kth element of an s-dimensional mixed sequence with a d-dimensional deterministic component, and let $u^{(k,i)}$ be the kth element of the ith realization of a mixed sequence whose deterministic component is the ith realization of a d-dimensional RQMC sequence, and the remaining $(s-d)$ components are sampled from $U(0,1)^{s-d}$. We then define estimators (earlier discussed in Introduction):

$$\theta = \frac{1}{NM}\sum_{k=1}^{NM} f(X^{(k)}) \text{ - MC}$$

$$\theta_{mixed} = \frac{1}{NM}\sum_{k=1}^{NM} f(x^{(k)}) \text{ - Mixed (padding QMC by MC)}$$

$$\theta_{Rmixed} = \frac{1}{M}\sum_{i=1}^{M}\left(\frac{1}{N}\sum_{k=1}^{N} f(u^{(k,i)})\right) \text{ - Randomized mixed (padding RQMC by MC)}$$

Note that θ_{mixed} is a biased estimator. We want to know how the bias and standard deviation of θ_{mixed} compare with the standard deviations of the unbiased estimators θ and θ_{Rmixed}. Here is one interpretation of the estimators θ_{mixed} and θ_{Rmixed}: θ_{mixed} goes NM "deep" in one realization of the underlying sequence, whereas θ_{Rmixed} goes N "deep" and averages over M realizations of the sequence. Also note that if we take $d = s$ in θ_{Rmixed} (no padding) we obtain the RQMC estimator. In our numerical results we will also compare the methods based on padding with the RQMC estimator.

In the numerical examples, we will consider two implementations of θ_{Rmixed}. One will use the scrambled (t, d) sequences of Owen [15], and the other will use the linear scrambling approach of Matoušek [16,17]. Both scrambling methods are applied to a $(0, d)$-sequence in base p with p smallest prime number larger than or equal to d. Our main concern is the behavior of the error for moderate sample sizes and how expensive it is to generate the estimates, and thus the existing asymptotical results on the variance of RQMC methods (see [18] and the references mentioned) are not useful to us. Instead we will compare the efficiency of these methods numerically. We define the efficiency $\varepsilon(\theta)$ of an estimator θ as

$$\varepsilon(\theta) = \left(\left(Var(\theta) + (E[\theta - I])^2\right) t\right)^{-1}$$

where t is the complexity of the computation. We will estimate $\varepsilon(\theta)$ as follows:

t will be taken as the computation time, $E[\theta - I]$ will be taken as the computed bias for the θ_{mixed} estimator (in our examples we will know the true answer so that bias can be computed), and $Var(\theta)$ will be the sample variance. For the MC and Rmixed methods, the variance is estimated like in usual MC methods from the respectively NM and M independent random variables. The variance of the mixed sequence cannot be computed directly (we can only find an upper bound as discussed in the previous section). Instead, we estimate the variance by computing the sample variance of 100 independent replications (i.e., independent uniform random coordinates between the $(d+1)$st and the sth coordinates, the first d determined by the low-discrepancy sequence).

4.2 Pricing of financial securities

Here we consider a problem from computational finance: pricing of geometric Asian options. The price of these options can be computed exactly, however, a close relative, arithmetic Asian options, do not have exact pricing formulas. In simulation, we generate a sequence of asset prices $S_0, S_1, ..., S_K$ that are subject to an Ito process $dS = \mu S dt + \sigma S dX$, where t is time, μ and σ are the drift and volatility of the underlying respectively, and $X = (X(t))_t$ is a standard Brownian motion. The payoff function is defined as $h(S_0, S_1, ..., S_K) = \max\left(G(S_0, S_1, ..., S_K) - F, 0\right)$, where $G(S_0, S_1, ..., S_K) = \left(\prod_{i=0}^{K} S_i\right)^{1/(K+1)}$ is the geometric average of the asset prices, and F is the strike price. The price of the option is the expected value $E\left[e^{-rT} h(S_0, S_1, ..., S_K)\right]$, which is estimated by simulation. In this expression r is the risk-free interest rate and T is the expiration time, i.e., the time when we observe the final price S_K. Details on geometric options, including the exact pricing formula can be found in [19].

We estimated the option price using MC, mixed, and Rmixed methods. In this problem K corresponds to the dimension of the problem (which was denoted by s in the previous sections), and in the first numerical examples K is taken to be 256. The dimension of the deterministic part of the mixed sequence is taken to be $d = 32$. The other constants are: $r = \mu = 0.1$, $\sigma = 0.1$, $T = 128$, $F = 5$ and $S_0 = 500$, leading to an exact price of 0.76561. The Brownian bridge construction [20] is first used to solve the model, so that most of the variance is concentrated in the first coordinates (even if it is not always the case, see [21]). Recall that the Brownian bridge formula assumes in its simplest implementation that K is a power of 2. From S_0, S_K is first computed, then $S_{K/2}$, $S_{K/4}$, $S_{3K/4}$, $S_{K/8}$, $S_{3K/8}$, $S_{5K/8}$, $S_{7K/8}$ and so on (see [20] for details). Figure 1 displays the results when the number of points NM increases (M is fixed at 100, we only increase N).

We plot confidence interval width (CI width), computation time, bias for the mixed method, and the efficiency in Figure 1. The first three plots give us specific information about each method, and the last plot for efficiency shows the overall effectiveness of the methods. Among other things we notice the high execution time for the Rmixed-Owen, which is expected, and the way the error for the mixed method is broken into two components as bias and CI width. Overall, Rmixed-Matoušek has the best efficiency ($d = 32$) with an average improvement factor of 4.5 in efficiency over MC. The efficiency of the mixed method is between MC and Rmixed-Owen for the first three samples, and then it gets better, giving the best efficiency for the last sample size.

We next try different values for d, using the Matoušek implementation. Figure 2 compares the results for the case of Rmixed-Matoušek with the above inputs but for $d = 32$, $d = 64$ and, $d = 256$ (which corresponds to the tradi-

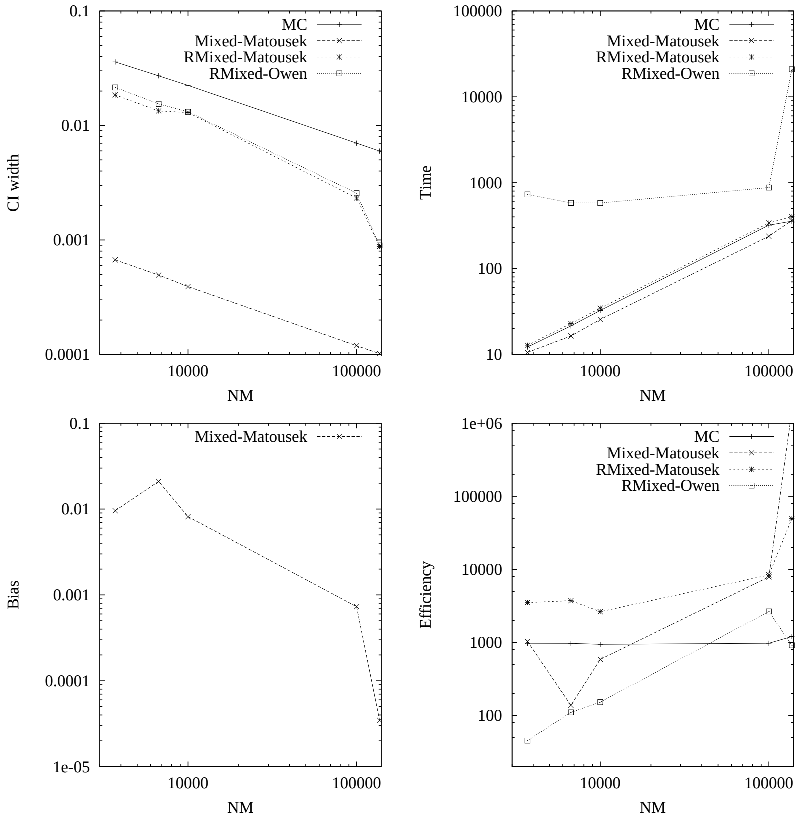

Fig. 1. Pricing an Asian option in dimension 256 using a 32-dimensional low discrepancy sequence and the Brownian bridge implementation

tional RQMC method - no padding).Note that $d = 32$ gives better efficiency than $d = 256$ (RQMC) for all except one sample size. When $N = 100,000$, the improvement is about a factor of 8.5.

How do these results change if Brownian bridge is not used? Figure 3 solves the same problem and uses the same methods as Figure 1 (except that we ignore the mixed method) without the Brownian bridge implementation. As before, Rmixed-Matoušek has the best efficiency ($d = 32$), but the improvement over MC is approximately a factor of 1.3, which is a smaller improvement than the case when Brownian bridge was employed.

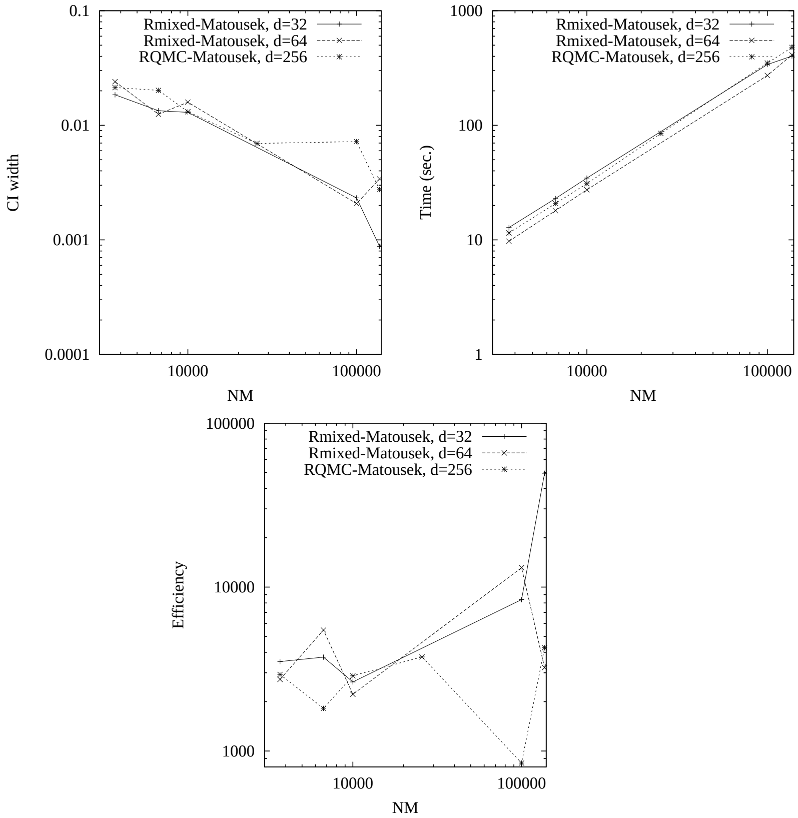

Fig. 2. Pricing an Asian option in dimension 256 using Rmixed-Matoušek scrambling and different values for d with the Brownian bridge implementation

Figure 4 compares different values for d like Figure 2, but without the Brownian bridge implementation. Comparing these two figures we make an interesting observation: When there is no Brownian bridge, the efficiency of RQMC-Matoušek is pretty bad compared to Rmixed methods for smaller sample sizes. However, for larger sample sizes, the efficiencies get closer. If Brownian bridge is used, than exactly the opposite seems to be true; efficiencies are closer for smaller samples, and farther apart for larger samples.

Comparing the plots for CI width in Figure 3 & Figure 1, and Figure 4 & Figure 2 also show that the Brownian bridge implementation lowers the variance

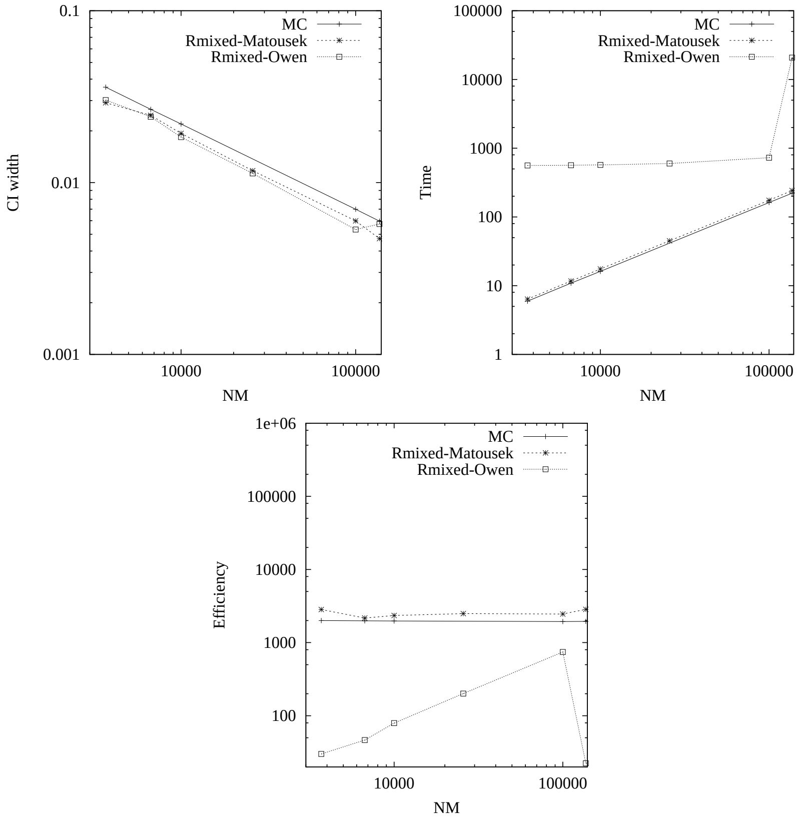

Fig. 3. Pricing an Asian option in dimension $K = 256$ and $d = 32$, without the Brownian bridge implementation

for Rmixed and RQMC methods, but not for the MC method.

We now increase the dimension of the problem to $K = 1024$, and compare the efficiency of Rmixed-Matoušek ($d = 32$) with full scrambling, RQMC-Matoušek ($d = 1024$). Figure 5 shows that when Brownian bridge is used the Rmixed-Matoušek ($d = 32$) method has a much better efficiency than the full RQMC-Matoušek, by an average factor of 10, although there is quite a bit of variation. When Brownian bridge is not used, Rmixed-Matoušek has better efficiency for all except one sample size. We also considered large samples and simulated this problem upto $N = 10^7$. The efficiency of Rmixed-Matoušek

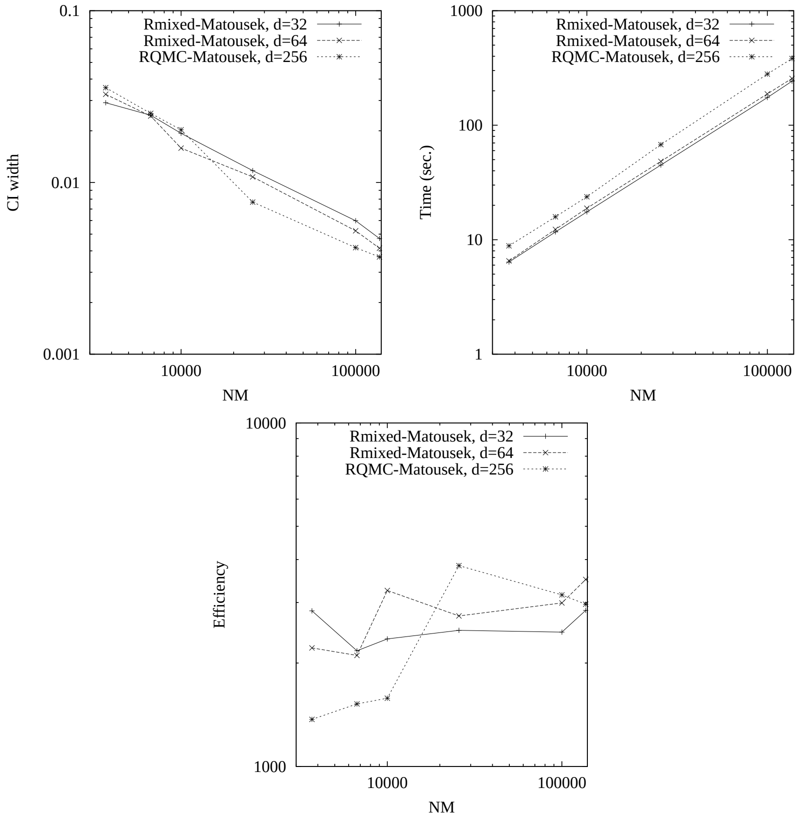

Fig. 4. Pricing of an Asian option in dimension 256 using Rmixed-Matoušek and different values for d without the Brownian bridge implementation

($d = 32$) gets even better with a wider margin than RQMC-Matoušek as sample size grows, in the case of Brownian bridge implementation. However, if Brownian bridge is not used, RQMC-Matoušek efficiency gets slightly better.

Our second example is pricing of digital options. We assume the stock price follows the geometric Brownian motion model as in the Asian option example. The payoff function is

$$h(S_1, ..., S_K) = \frac{1}{K}\sum_{i=1}^{K}(S_i - S_{i-1})_+^0 \; S_i,$$

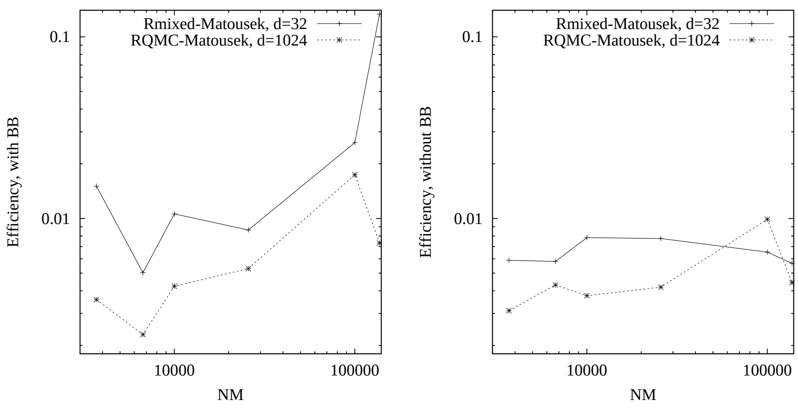

Fig. 5. Pricing of an Asian option in dimension 1024 using Rmixed-Matoušek with $d = 32$ and RQMC, $d = 1024$. The figure on the left is with the Brownian bridge implementation, and the figure on the right is without the Brownian bridge implementation

where $(x)_+^0$ is equal to 1 if $x > 0$; otherwise it is 0. These options were considered by Papageorgiou [21] who showed that the Brownian bridge implementation consistently performed worse than the standard implementation. We therefore do not consider the Brownian bridge implementation in this example.

We start with a 256 dimensional problem and compare Rmixed-Matoušek methods ($d = 32$ and $d = 64$) with the full RQMC-Matoušek implementation. Examining Figure 6, we make a similar observation we had earlier: The efficiency of RQMC-Matoušek is worse initially than the Rmixed methods, but as the sample size gets larger the efficiencies get closer.

We now investigate how the biased mixed estimator compares with the others. In Figure 7, we plot the CI width, time, bias, and efficiency when the methods MC, Mixed-Matoušek ($d = 32$), Rmixed-Matoušek ($d = 32$), and RQMC are used. Perhaps surprisingly, the mixed method gives the best efficiency for all

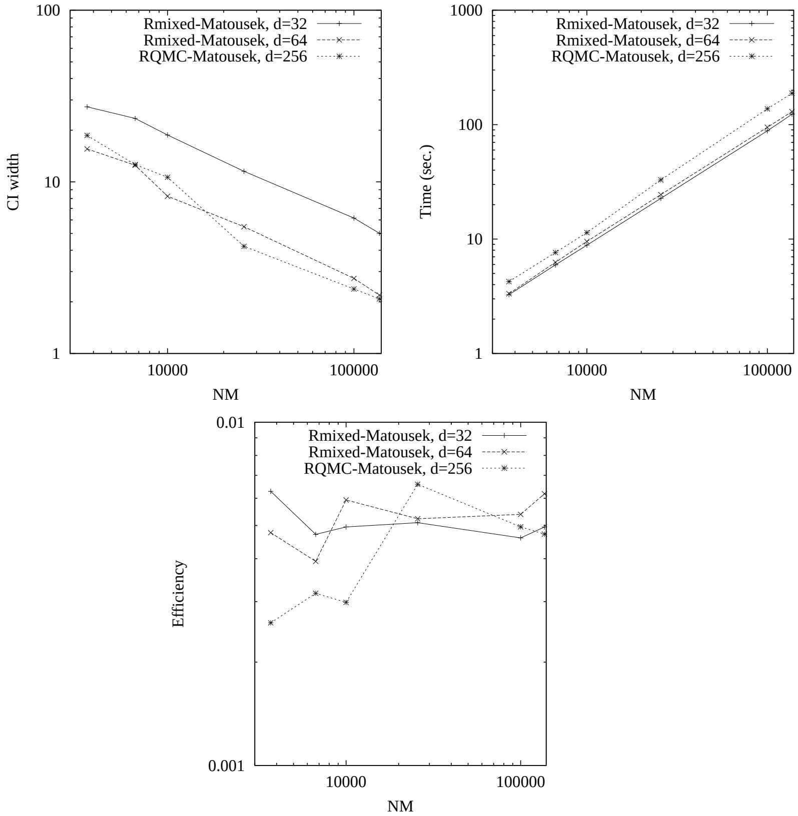

Fig. 6. Pricing of a digital option in dimension 256 using Rmixed-Matoušek with $d = 32$, $d = 64$ and RQMC-Matoušek, with $d = 256$.

except two sample sizes. Rmixed-Matoušek ($d = 32$) comes second in overall efficiency. Both methods outperform MC consistently, and RQMC efficiency gets close to the mixed and Rmixed methods for large samples.

How do these results change if the dimension of the deterministic part of the mixed sequence is increased to 64? In Figure 8, we see that the efficiency of the mixed method gets even better: Now the mixed-Matoušek ($d = 64$) efficiency is better than the other methods for all sample sizes but one. The efficiency of mixed-Matoušek ($d = 64$) is about a factor of 1.3 (meaning 30%) better than MC. An approximate figure of merit is harder to come up with due to

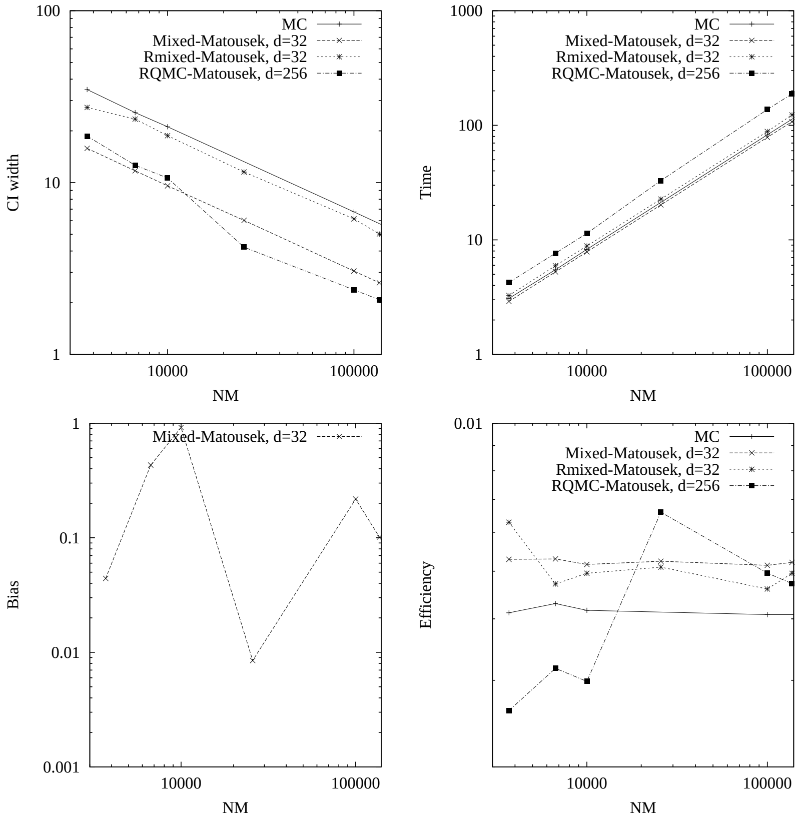

Fig. 7. Pricing of a digital option in dimension 256 using MC, Rmixed-Matoušek with $d = 32$, and RQMC-Matoušek, with $d = 256$.

high oscillations in the efficiency of RQMC-Matoušek and Rmixed-Matoušek ($d = 64$), however, especially for smaller sample sizes, the improvement is pretty noteworthy.

Finally, we look at the efficiency when the dimension is increased to $K = 1024$, and $d = 128$. The mixed-Matoušek has better efficiency than all of the other methods for all except two sample sizes. These results are consistent with the previous ones.

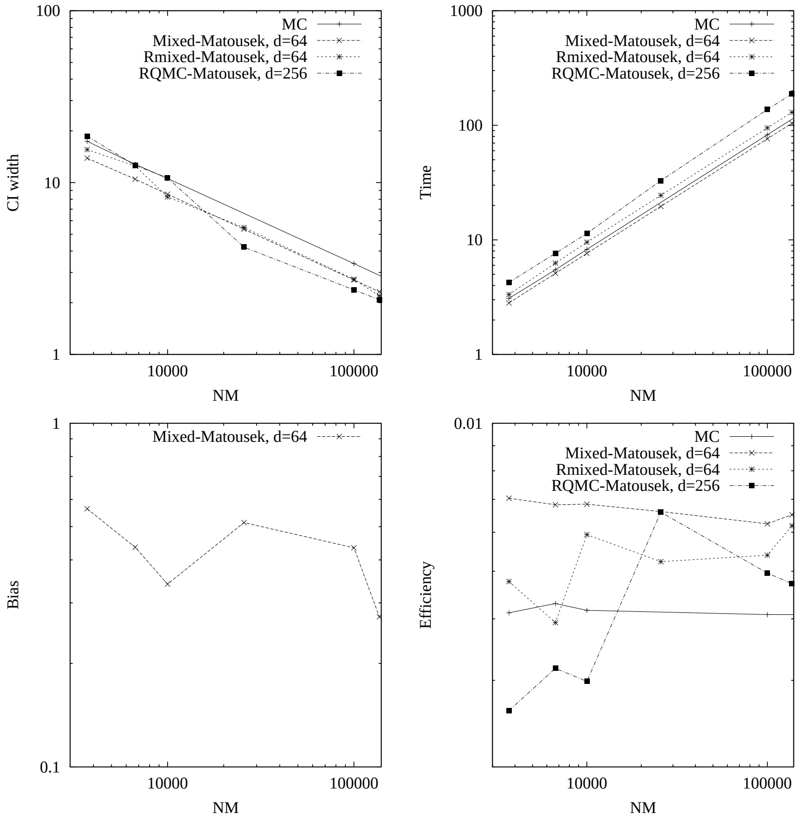

Fig. 8. Pricing of a digital option in dimension 256 using MC, Rmixed-Matoušek with $d = 64$, and RQMC-Matoušek, with $d = 256$.

5 Conclusions

In this paper, we studied the mixed method for high-dimensional integration, where the first coordinates are sampled using a QMC sequence and the remaining ones are sampled by MC. The method was known to give good experimental results, but little was known theoretically about the approximation error. We proved an upper bound for the discrepancy of the mixed sequence improving the earlier results of Ökten [3]. Next, we obtained a central limit theorem that enables the use of confidence intervals for the integral.

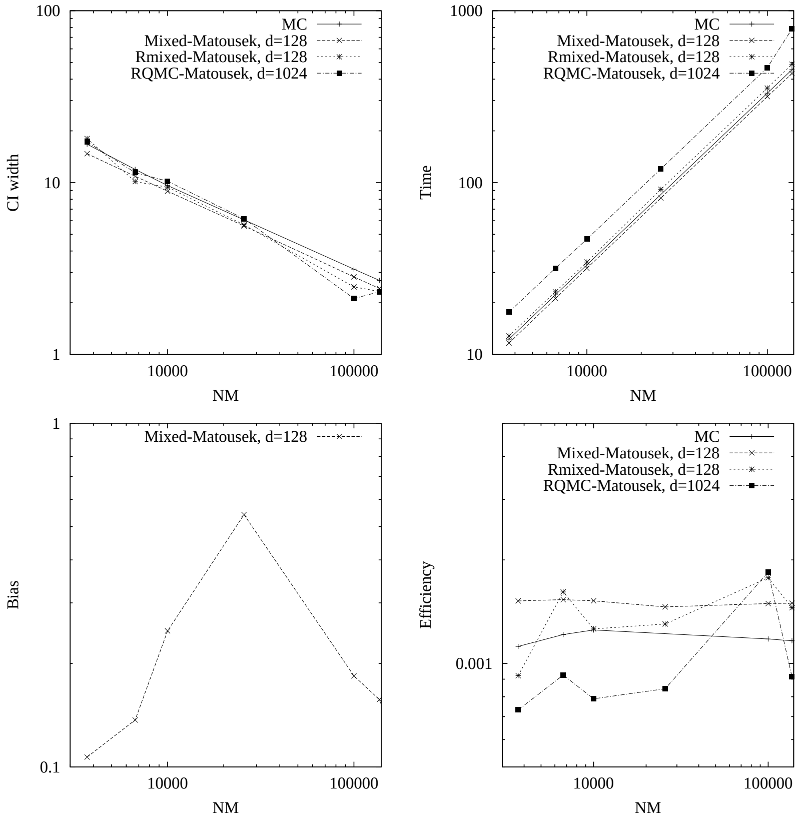

Fig. 9. Pricing of a digital option in dimension 1024 using MC, Rmixed-Matoušek with $d = 128$, and RQMC-Matoušek, with $d = 1024$.

We then discussed numerical results when the mixed method and its randomized versions were applied to problems from option pricing. Our numerical investigations suggest that the mixed method (padding QMC with MC) and its randomized version, the Rmixed method (padding RQMC with MC), can significantly improve efficiency in high dimensional problems for especially moderate sample sizes. Although we see improvements with and without the Brownian bridge implementation, the use of Brownian bridge magnified the factors of improvement in the Asian option example. We also observed that the biased mixed method has the potential of outperforming its randomized version as well as the full RQMC strategy in terms of efficiency. This happens when the bias is small compared to the variance, and there is significant gain in computation time.

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