The Progressive Second Price Mechanism in a Stochastic Environment

Patrick Maillé, Bruno Tuffin

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Abstract

We consider in this paper an auction-based pricing scheme recently introduced by Lazar and Semret to allocate bandwidth among users. This mechanism, called Progressive Second Price, was studied using tools from non-cooperative game theory, for a fixed set of players (i.e., users). We compare here the results obtained in that case with the more realistic situation when players enter the game according to a Poisson process, and leave it after an exponentially distributed sojourn time. We show that this stochastic assumption cannot be skipped since it can lead to very different results.

Keywords: Markov chains, Non-cooperative game theory, Pricing, Telecommunications.

1 Introduction

Despite the exponential growth of the available resources in communication networks, the increase of the number of users, and the new services that need more and more bandwidth, often lead to congestion problems.

It thus becomes obvious that the pricing schemes based on flat-rates (independent of users' consumption) are not suitable anymore, because they do not take into account the negative externality caused by a particular user on the others.

Many recent papers have suggested and studied new allocation and pricing mechanisms, for wireless networks where the externality is the interference [1, 15, 18], and for wired networks where the externality is expressed in terms of losses and/or delay [3, 4, 7, 8, 11, 12, 14, 21]. In both cases, the resources (bandwidth, computing power) have to be fairly distributed among users. All of those works consider an economic approach to study the different proposed mechanisms, and many authors use in particular the formalism of *non-cooperative game theory* to describe the behavior of users and resource sellers. Indeed, as the potential users of a network compete for the resource, and can be considered as behaving selfishly, the framework of non-cooperative game theory is particularly well-suited to predict the outcomes of the allocation and pricing model. Thus, the concepts of strategy and Nash equilibrium are central notions in the system.

In this paper, we focus on the recent results presented by Semret and Lazar in [10] and [16], concerning an auction mechanism, called Progressive Second Price (PSP), that could be used to allocate bandwidth among users (also called players from now). In their work, the authors present the PSP, and show its properties using tools from game theory: they exhibit an *incentive compatible* user behavior, prove that a *Nash equilibrium* is reached after a finite time, and that the corresponding allocation is *efficient* in the sense that it maximizes the social welfare.

However, one important assumption made by Lazar and Semret is that the number I of players is constant during all the game, which does not seem very realistic when considering Internet connections for example. Indeed, in the "real world", connections enter or leave the network randomly, their duration depending on the type of application. This phenomenon is likely to deteriorate the optimality of the auction, and requires further investigations. There already exists a few recent papers where the number of players vary over time [13, 19]. In particular, Stahl shows in [19] that some auction mechanisms can be efficient for a constant number of players, but that the efficiency can be lost in a stochastic environment. Nevertheless, in [13, 19] the auctions are static, meaning that players cannot modify their bids, whereas in the PSP the auction is dynamic.

In this paper, we plan to look at to what extent the properties of the PSP mechanism highlighted by Lazar and Semret still hold when players enter and leave the game according to Poisson processes. By this, we will introduce a continuous-time Markov chain that will be used to describe the auction behavior, and we will investigate the properties of the PSP in such a stochastic context.

The paper is organized as follows. In Section 2, we describe the PSP mechanism presented by Lazar and Semret in [10], and quote their main results. We introduce in Section 3 the Markov chain model used to study the game with random arrivals and departures of players, and present two different possibilities of initial bid, depending on the player's ignorance or knowledge of the bid profile at his arrival. Section 4 is dedicated to a theoretical study of that Markov chain, focusing on the existence of a stationary distribution. We then present and comment in Section 5 some numerical results obtained from simulations, since obtaining analytic results is intractable, and we finally conclude and give some directions for future work in Section 6.

2 The Progressive Second Price mechanism

We present here the PSP mechanism, that was first described by Lazar and Semret in [10], for the allocation of variable-size shares of a resource among several users. An extension of the scheme to a network context was studied later by Semret *et al* [17], but we concentrate here on the case of a single resource, basically the bandwidth of a single communication link.

Denote by $\mathcal{I} = \{1, ..., I\}$ the set of players, and by Q the total amount of available resource. Players submit bids, and then the auctioneer (i.e., the network) shares the resource and computes the price to be charged to each player based on the received requests.

Bids are chosen to be 2-dimensional: each player $i \in \mathcal{I}$ submits a bid $s_i = (q_i, p_i) \in \mathcal{S}_i = [0, Q] \times [0, +\infty)$, where q_i is the desired quantity of resource and p_i the *unit* price player i is willing to pay for that resource. Through the paper, $s = (s_1, ..., s_I)$ will denote the bid profile, and $s_{-i} = (s_1, ..., s_{i-1}, s_{i+1}, ..., s_I)$ will be the bid profile that player i faces, so that $s = (s_i; s_{-i})$. The Progressive Second Price (PSP) allocation and pricing rule is defined as follows: player i is allocated the quantity $a_i(s)$, and is charged the price $c_i(s)$, where (if $\wedge$ designs the minimum)

$$\begin{aligned} a_i(s) &= q_i \wedge \left[Q - \sum_{p_k \geq p_i, k \neq i} q_k \right]^+ && (1) \\ c_i(s) &= \sum_{j \neq i} p_j \left[a_j(s_{-i}) - a_j(s_i; s_{-i}) \right]. && (2) \end{aligned}$$

The motivation for that choice for the pricing rule is that c_i covers the "social opportunity cost" of player i's presence, i.e. the declared willingness to pay of players who are excluded by player i's bid (see Figure 1), which is often called the second price. Also, from (1), the players are sorted by their unit price, those bidding the highest get the bandwidth they request, and so on until no more resource is available.

It can be remarked that the allocation rule given by Equation (1) punishes players who bid at the same price and ask more than the available quantity at that price, which may result in a non-optimal allocation since the bandwidth is not fully allocated. This was argued by Tuffin in [20], where a slightly modified allocation rule is presented. We describe here this revisited scheme, which will be used in the rest of the paper. Define

$$Q_i(y, s_{-i}) = \left[Q - \sum_{k \neq i : p_k > y} q_k \right]^+ \qquad (3)$$

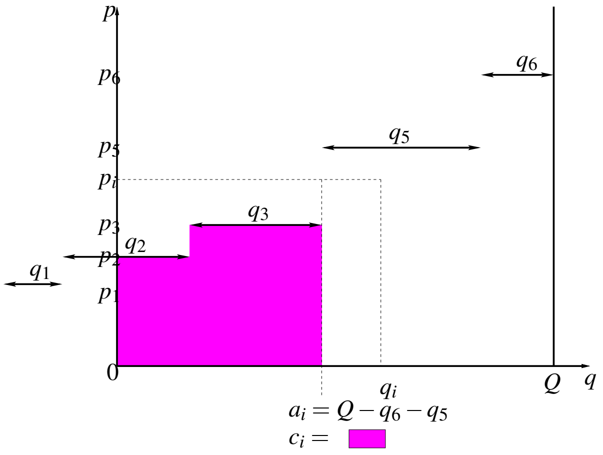

Figure 1: Allocation and price to pay for player i

as the remaining bandwidth at a given unit price. The allocation

$$a_i(s) = q_i \wedge \frac{q_i}{\sum_{k:p_k=p_i} q_k} Q_i(p_i, s_{-i}) \tag{4}$$

allows players who bid the same price p_i to totally share the available quantity at price p_i. The price charged to each player is then still computed using Equation (2). It is shown in [20] that the results that Lazar and Semret obtained are still exact for the modified allocation rule.

In order to model the players' behavior, we also need to represent their perception/valuation of the service they can get. Player i's preferences are modeled by a quasi-linear utility function of the form

$$u_i(s) = \theta_i(a_i(s)) - c_i(s), \tag{5}$$

where θ_i is player i's *valuation function*, which depends on the quantity of resource he receives.

Since players are assumed to act selfishly, so that they will choose their bid in order to maximize their individual utility function.[1] The allocation game works as follows: each player can make a bid each T units of time, knowing perfectly the bids made by the other players s_{-i}. Moreover, it is assumed that two players never bid at exactly the same time. As a player's bid is a reply to s_{-i}, Lazar and Semret define the set of player i's ϵ-best replies (for an $\epsilon > 0$) as

$$\mathcal{S}_i^\epsilon(s_{-i}) = \left\{s_i \in \mathcal{S}_i : u_i(s_i; s_{-i}) \geq u_i(s_i'; s_{-i}) - \epsilon, \forall s_i' \in \mathcal{S}_i\right\},$$

meaning that $\mathcal{S}_i^\epsilon$ is the set of bids that player i can make, ensuring him a utility close to the maximum reachable from less than ϵ.

An ϵ-Nash equilibrium is then defined as a fixed point of $\mathcal{S}^\epsilon = \prod_{i\in\mathcal{I}} \mathcal{S}_i^\epsilon(s_{-i})$, where no player can increase his utility by more than ϵ by unilaterally changing his bid.

To derive theoretical results, some assumptions are made on the valuation functions [10]:

Assumption 1 *For any $i \in \mathcal{I}$,*

- $\theta_i(0) = 0$,
- θ_i *is differentiable,*
- $\theta_i' \geq 0$, *non-increasing and continuous*

[1] In [10], a budget constraint for each player can be taken into account. However, for simplicity sake, we consider in this paper the case without budget constraint.

- $\exists \gamma_i > 0, \forall z \geq 0, \theta_i'(z) > 0 \Rightarrow \forall \eta < z, \theta_i'(z) \leq \theta_i'(\eta) - \gamma_i(z-\eta)$.

Then, defining a truthful bid as a bid $s_i = (q_i, p_i)$ such that $p_i = \theta_i'(q_i)$ (that is, bidding a price equal to the marginal valuation), Lazar and Semret prove in [10] the incentive compatibility property, which describes how players could choose their bids:

Proposition 1 *Let*

$$G_i(s_{-i}) = \sup\left\{z : z \leq Q_i\left(\theta_i'(z), s_{-i}\right)\right\}. \tag{6}$$

Under Assumption 1, $\forall i \in \mathcal{I}, \forall s_{-i} \in \mathcal{S}_{-i}$ such that $Q_i(0, s_{-i}) = 0$, $\forall \epsilon > 0$,

$$t_i = (v_i, w_i) \text{ with } \left\{ \begin{array}{c} v_i = [G_i(s_{-i}) - \epsilon/\theta_i'(0)]^+ \\ w_i = \theta_i'(v_i) \end{array} \right. \tag{7}$$

is a truthful ϵ-best reply for player i to s_{-i}.

To ensure that the bandwidth is not sold at a too low level, a bid $s_0 = (Q, p_0)$ is introduced. This means that the owner of the resource will sell it at a minimum unit price p_0, which is called the reserve price. The seller can thus be seen as a player (not in $\mathcal{I}$) with a valuation function $\theta_i(q) = p_0 q$. As a consequence, the hypothesis $Q_i(0, s_{-i}) = 0$ of Proposition 1 is verified.

The authors suggest that players submit bids according to Equation (7), with the constraint that there are never more than one player sending a bid at the same time,[2] if and only if it increases their individual utility by at least ϵ. This is justified in [10] by considering a "bid fee" of ϵ. Then it can be shown (see [16], Proposition 4) that under Assumption 1, the allocation game converges after a finite number of bids to a truthful 2ϵ-Nash equilibrium.[3]

In order to obtain results about the efficiency of the PSP mechanism, some additional conditions are imposed on the valuation functions:

Assumption 2 $\exists \kappa > 0, \forall i \in \mathcal{I}$,

- $\forall z, z', z > z' \geq 0, \theta_i'(z) - \theta_i'(z') > -\kappa(z - z')$,
- $\theta_i'(0) < +\infty$

The following result can then be shown (see [16]):

Proposition 2 *Consider an ϵ-Nash equilibrium $s^* \in \mathcal{S}^\epsilon$, and the corresponding allocation $a^* = (a_0(s^*), a_1(s^*), ..., a_I(s^*))$. Define $\underline{a}^* := \min_{i \in \mathcal{I}\cup\{0\}, a_i^* > 0} a_i^*$ and assume that Assumptions 1 and 2 hold. If $\underline{a}^* > \sqrt{\epsilon/\kappa}$, then*

$$\max_{a \in \mathcal{A}} \left(\sum_{i \in \mathcal{I}\cup\{0\}} \theta_i(a_i) \right) - \sum_{i \in \mathcal{I}\cup\{0\}} \theta_i(a_i^*) = O(\sqrt{\epsilon\kappa}) \tag{8}$$

where $\mathcal{A} = \{a \in [0, Q]^{I+1} : \sum_i a_i \leq Q\}$.

3 The Markov model for auctions, arrivals and departures

In this section, we look at the previous auction mechanism behavior in a stochastic environment. We thus make some assumptions on the way players enter the game, submit bids and leave, driving to a Markov process.

[2] This condition is met for example when players choose to bid asynchronously every T units of time.

[3] The convergence of the bids is proved in [16] when players act as we describe here, and the equilibrium that is reached is necessarily a 2ϵ-Nash equilibrium

3.1 The hypotheses

Assumption 3 *There exists a finite number T of different valuation functions. Thus a player i is characterized by his* type *(or class) $t_i \in \{1, ..., T\}$, meaning that his valuation function is $\theta_i = \theta_{(t_i)}$. New players enter the game according to a Poisson process with rate λ, and the type of a new player is chosen according to the discrete probability distribution $\mathbb{P}_t$. It is supposed that players' types are independent random variables, so that players of type t_i arrive according to a Poisson process with rate $\lambda_i = \lambda \mathbb{P}_t(t_i)$.*

Each type can correspond to a different kind of application (telephony, video, file transfer...) with different bandwidth requirements.

Assumption 4 *Each type-t_i player sojourn time in the game is an exponentially-distributed random variable, with parameter μ_i, independent of all other processes, and in particular independent of the cumulative bandwidth the player is allocated during his sojourn.*

Different mean sojourn times are necessary for different applications, since watching a video generally needs a much longer connection time than loading a web page for instance.

Assumption 5 *Each type-t_i player in the game has the opportunity to submit a new bid at different times. Inter-bid times are assumed to follow an exponential distribution with parameter ν_{t_i}, independent of all other random variables.*

Assumption 6 *The set of possible bid quantities is discrete with a discretization step α: the bids that can be submitted are of the form $s_i = (q_i, p_i)$, where $q_i \in [0, Q]_d := [0, Q] \cap \alpha\mathbb{N}$.*

Assumption 6 is introduced in order to obtain a Markov chain with a discrete state space. This is done for two reasons: first it is easier to analyze than Markov chains with continuous state space, and second in practice a continuum of prices is unlikely to happen.

Because of this discretization of the possible quantities that can be asked and because of the finite number of different valuation functions, the situation where several players bid the same unit price is likely to happen. To make sure that the bandwidth is optimally distributed, we are going to use the allocation rule (4), rather than (1), following the arguments of [20].

As another consequence, each player chooses his bid as follows.

Assumption 7 *When submitting a new bid, a previously present player chooses to maximize his immediate utility function, using truthful bids. Formally, a player i will submit the bid*

$$\hat{s}_i(s_{-i}) = (\hat{q}_i, \hat{p}_i = \theta_i'(\hat{q}_i)), \text{ with } \hat{q}_i = \arg\max_{q \in [0,Q]_d} u_i\left((q_i, \theta_i'(q_i)); s_{-i}\right).$$

It can be verified (see an example on Figure 2) that there are only two candidates: $\hat{q}_i = \bar{q}_i := \sup\left\{q \in [0, Q]_d : q \leq Q - \sum_{j \neq i: p_j \geq \theta_i'(q)} q_i\right\}$ and $\hat{q}_i = \bar{q}_i + \alpha$.

Unlike Lazar and Semret, we do not introduce a "bid fee" ϵ here, so players can submit bids as soon as it improves their utility. Actually, since the bid set is discretized, a bid change will correspond to a utility improvement greater than a certain threshold.

We then distinguish two different cases, each of which corresponding to one of the following knowledge assumptions when entering the game.

Assumption 8 *When a player i enters the game, he is not aware of the other bids, and submits the (initial) bid $(q_i, \theta_i'(q_i))$, where q_i is chosen randomly according to a uniform distribution on the possible quantities in a bid, independent of all other random variables.*

Given that the player totally ignores the bid profile, he chooses its bid randomly. Afterwards, he will be informed of the profile and will be able to change his bid according to Assumption 7. This situation could happen depending on the degree of signaling procedures that are implemented in the network. However, we will also consider the case where arriving players submit their best bid:

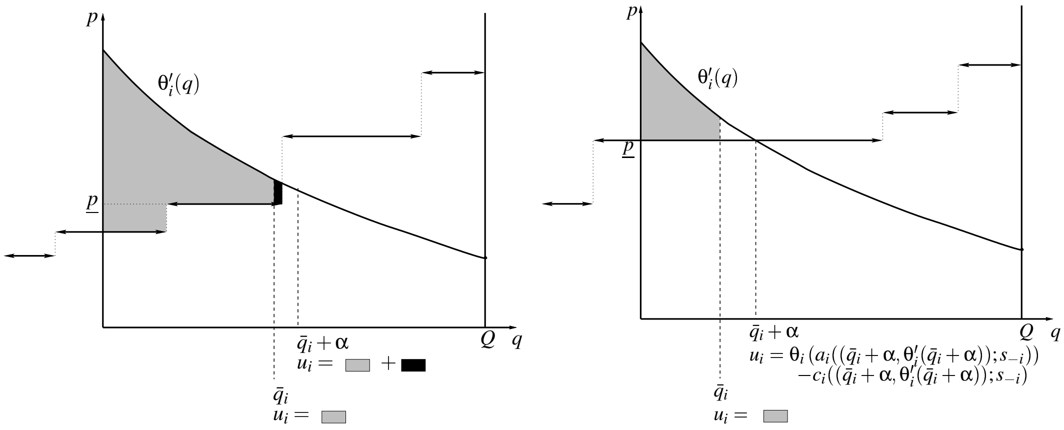

Figure 2: The two candidates for the best bid: on the left, the utility associated with the bid $\bar{q}_i$ (area in grey) is smaller than that associated with the bid $\bar{q}_i + \alpha$ (the difference is the area in black). This is the case because $\theta_i'(\bar{q}_i + \alpha)$ is strictly greater than the price $\underline{p} := \max\{p_j : p_j < \theta_i'(\bar{q}_i)\}$. The right-hand figure illustrates the case when $\theta_i'(\bar{q}_i + \alpha) = \underline{p}$; player i must then compare the utility he would obtain by bidding $\bar{q}_i$ (in grey), and that he would get by bidding $\bar{q}_i + \alpha$, considering the allocation rule (4).

Assumption 9 *When a player i enters the game, he is informed of the bid profile, and immediately submits his best bid like described in Assumption 7.*

Both situations (as expressed by Assumptions 8 and 9) will be compared numerically.

We now need to define the state space of the random process.

3.2 State space of the Markov chain

When Assumptions 3 to 7 hold, as well as one among Assumptions 8 or 9 is chosen, the random process X described above is a continuous-time discrete-space Markov chain defined over the state space

$$\mathcal{U} = \bigcup_{m \in \mathbb{N}} \left(\{1, ..., T\} \times [0, Q]_d\right)^m,$$

where a state x is represented

$$x = ((t_1, q_1), ..., (t_i, q_i), ..., (t_m, q_m)) \tag{9}$$

if there are m players in the game. Indeed, a state of the process is then the set of pairs (t_i, q_i) for all players i in the game at that moment. t_i is player i's type, characterizing his valuation function and thus his future bids, and q_i is his asked quantity, which is sufficient to describe the current bid (knowing the type t_i), since bids are assumed to be truthful.

Moreover, we decide to keep the players in their order of entry in the game, so that the arrival of a new type-t_{new} player submitting bid q_{new} will imply a transition from state x to state $x' = ((t_1, q_1), ..., (t_m, q_m), (t_{\text{new}}, q_{\text{new}}))$.

3.3 Transition rates

Denote by $x \to x'$ a transition from state x to state x' (were $x, x' \in \mathcal{U}$). There are three possible types of transition for the Markov chain X we study:

- **Arrival of a new player**

$$x = ((t_1, q_1), ..., (t_m, q_m)) \to x' = ((t_1, q_1), ..., (t_m, q_m), (t_{m+1}, q_{m+1})) \tag{10}$$

  occurring with rate

  - $\gamma(x, x') = \frac{1}{\lfloor Q/\alpha \rfloor + 1} \lambda \mathbb{P}_t(t_{m+1})\ \forall t_{m+1} \in \{1, \cdots, T\}$ for the uniform initial bid case (following Assumption 8), and
  - $\gamma(x, x') = \begin{cases} \lambda \mathbb{P}_t(t_{m+1}) & \text{if } q_{m+1} = \hat{q}_{m+1} \text{ (see Assumption 7) and } \forall t_{m+1} \in \{1, \cdots, T\} \\ 0 & \text{otherwise,} \end{cases}$
    for the initial best-reply case (Assumption 9).

- **Departure of a player**

$$\begin{aligned} x &= ((t_1, q_1), ..., (t_{i-1}, q_{i-1}), (t_i, q_i), (t_{i+1}, q_{i+1}), ..., (t_m, q_m)) \\ \to x' &= ((t_1, q_1), ..., (t_{i-1}, q_{i-1}), (t_{i+1}, q_{i+1}), ..., (t_m, q_m)) \end{aligned} \tag{11}$$

  occurring with rate $\gamma(x, x') = \mu_i$.

- **New bid from an already present player** (the bid can be unchanged if it is already optimal for the concerned player)

$$\begin{aligned} x &= ((t_1, q_1), ..., (t_i, q_i), ..., (t_m, q_m)) \\ \to x' &= ((t_1, q_1), ..., (t_i, \hat{q}_i), ..., (t_m, q_m)) \end{aligned} \tag{12}$$

  with $\hat{q}_i$ described in Assumption 7. This transition occurs with rate $\gamma(x, x') = \nu_i$.

3.4 State-space organization and infinitesimal generator

We indicate here how the states can be sorted, in order to write down the infinitesimal generator matrix for the Markov chain. We decide to organize the states with the condition that the total number of players increases. For example, if we define $C = \lfloor Q/\alpha \rfloor$, the states can be ordered as follows:

$$\underbrace{(\emptyset)}_{\text{0 player}}, \underbrace{(1,0), (1,\alpha), ..., (T, C\alpha)}_{\text{1 player: } T(C+1) \text{ states}}, \underbrace{((1,0);(1,0)), ((1,0);(1,\alpha)), ..., ((T,C\alpha);(T,C\alpha))}_{\text{2 players: } T^2(C+1)^2 \text{ states}}, ...$$

where $\emptyset$ means that there are no players in the game.

Thus the state space can be separated in blocks corresponding to the number of players in the game: the m^{th} block contains the $(T(C+1))^m$ possible states with m players. The only transitions that can occur from a block are to the next one (an arrival of a player), to the previous one (a departure), or to a state in the same block (a player submitting a new bid). Consequently, the infinitesimal generator matrix $\mathbf{Q}$ of the Markov chain X is block-tridiagonal, with an increasing block size, of the form:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{B}_0 & \mathbf{A}_0 & \mathbf{0} & & & & \dots & \dots \\ \mathbf{D}_1 & \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{0} & & & \dots & \dots \\ \mathbf{0} & \mathbf{D}_2 & \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{0} & \dots & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \dots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{D}_i & \mathbf{B}_i & \mathbf{A}_i & \mathbf{0} & \dots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{13}$$

where

- $\mathbf{A}_m$ is the $(T(C+1))^m \times (T(C+1))^{m+1}$-matrix containing the transition rates for arrivals of new players (the values in this matrix depend on which of Assumptions 8 or 9 is chosen).

- $\mathbf{B}_m$ is the $(T(C+1))^m \times (T(C+1))^m$-matrix containing the transition rates that correspond to new bids of already present players.
- $\mathbf{D}_m, m \geq 1$, is the $(T(C+1))^m \times (T(C+1))^{m-1}$-matrix that contains the transition rates for departures of players.

4 Theoretical results for the Markov chain

In this section, we investigate the properties of the previous Markov chain. From the characteristics of some states that can be deduced from Assumptions 3 and 4, we are going to infer the existence of a stationary distribution.

A first result is the following and is a direct consequence of Assumptions 3 and 4:

Lemma 1 *The processes $n_1, ..., n_T$, where n_i gives the number of type-t_i players in the game, are independent (continuous time) Markov chains corresponding to the number of customers in $M/M/\infty$ queues. The chain n_i corresponds to a queue with an arrival rate $\lambda_i = \lambda \mathbb{P}_t(i)$ and a service rate μ_i.*

In particular, all those Markov chains are irreducible and positive recurrent chains, whose stationary distribution is perfectly known [9]. The independence of those processes ensures that the process $n = (n_1, ..., n_T)$ is also an irreducible positive recurrent Markov chain, whose stationary distribution is the product of the stationary distributions of processes $n_1, ..., n_T$. As a result, the state $\emptyset$, corresponding to the case when there are no players in the game, is a recurrent positive state.

We now prove the existence of a stationary distribution for the Markov chain, in the uniform case (Assumption 8). Next, we demonstrate the result for the best-reply case (Assumption 9).

4.1 Irreducibility in the uniform case

To show that the continuous time Markov chain is irreducible, it is sufficient to show that from any state $x = ((t_1, q_1), ..., (t_n, q_n)) \in \mathcal{U}$, we can reach any other state $x' = ((t'_1, q'_1), ..., (t'_{n'}, q'_{n'})) \in \mathcal{U}$.

From state x, state $\emptyset$ can be reached with a non-zero probability (by departure of all players), and state x' can also be reached from state $\emptyset$ with a non-zero probability (successive entries of the n' players, player i's type being t'_i and his bid q'_i), thanks to Assumption 8. Consequently, x' can be reached from x which proves that the Markov chain is irreducible in the uniform case.

4.2 Existence of a stationary distribution in the uniform case

Theorem 1 *The Markov chain $X = \{X(t)\}$ with uniform initial bid is positive recurrent, meaning that there exists a steady-state distribution.*

Proof: To prove the positivity of the Markov chain X, we use here a result from [2]:

Lemma 2 *Let $\{X(t)\}$ be a recurrent irreducible continuous-time Markov chain. Then there exists a unique steady-state measure φ (known up to a multiplicative constant). This measure verifies $\varphi(e) > 0$ for every state e and $\varphi \mathbf{Q} = 0$, and it can also be obtained by*

$$\varphi(e) = \mathbb{E}_b \left[\int_0^{R_b} \mathbf{1}_{[X(t)=e]} dt \right] \tag{14}$$

where b is a state, arbitrarily chosen, and R_b is the time of first return to state b, when the Markov chain begins at state b.

We apply this result to state $b = \emptyset$ (no player in the game), since our Markov chain we study is irreducible and recurrent (it has a recurrent state $\emptyset$). Our aim is to prove that there exists a probability distribution π, such that $\pi\mathbf{Q} = 0$, which would imply the positive recurrence of the chain X. To do so, we need to show that the measure φ is bounded.

We have seen that state $\emptyset$ is positive recurrent, which means that $\mathbb{E}_\emptyset[R_\emptyset] < +\infty$. This implies that

$$\begin{aligned}\sum_{i\in\mathcal{U}} \varphi(i) &= \sum_{i\in\mathbb{N}} \mathbb{E}_\emptyset\left[\int_0^{R_\emptyset} \mathbf{1}_{[X(t)=i]} dt\right] \\ &= \mathbb{E}_\emptyset\left[\int_0^{R_\emptyset} \sum_{i\in\mathbb{N}} \mathbf{1}_{[X(t)=i]} dt\right] \\ &= \mathbb{E}_\emptyset[R_\emptyset] < +\infty,\end{aligned}$$

where the second inequality comes from Fubini Theorem.

We can thus introduce the probability measure π such that $\pi(e) = \frac{\varphi(e)}{\sum_{i\in\mathcal{U}}\varphi(i)}$ for all state e. π is then the unique probability measure such that $\pi\mathbf{Q} = 0$. This shows that the Markov chain is positive recurrent in the uniform case, with stationary distribution π. □

4.3 Existence of a stationary distribution in the best-reply case

We simply extend here the results obtained in the uniform case. Assumption 9 does not imply the irreducibility of the Markov chain over the state space $\mathcal{U}$ since some states are unreachable from others. Nevertheless, we will consider the set of states that can be reached with a non-zero probability from state $\emptyset$. Denote by $\mathcal{U}_{br}$ this set.

Assuming that the game begins with no players, only the states in $\mathcal{U}_{br}$ need to be considered. We can then restrict the Markov chain state space to that set, which immediately yields the irreducibility (all the states in $\mathcal{U}_{br}$ can be reached from $\emptyset$ by hypothesis, and $\emptyset$ is still a recurrent positive state). The positive recurrence follows also, using the same arguments as for the uniform case, and consequently the existence of a stationary distribution is ensured. We then have the following theorem:

Theorem 2 *Assume that the initial state is $\emptyset$ and denote $\mathcal{S}_{br} \subset \mathcal{U}$ the space of states reachable from $\emptyset$. Then the Markov chain X is irreducible over $\mathcal{S}_{br}$ and positive recurrent, meaning that there exists a steady-state distribution.*

4.4 Solution method

We first notice that Markov processes with heterogeneous block-tridiagonal infinitesimal generator matrices were studied by Taylor and Bright in [5, 6]. That type of processes is called *Level-dependent quasi-birth-and-death processes.* The generator matrix Q of the Markov chain we study (given by Equation (13)) fits the theory developed by Taylor and Bright. Thus, the conditions they obtained regarding the existence of an equilibrium distribution and its computation could be applied. However, in the case we consider, the block size grows exponentially with m, and the suggested methods are intractable. For this reason, we will use simulation as the solution method.

5 Numerical results

In this section, we compare by means of simulation the results obtained with a fixed number of players with the more realistic results obtained when players are entering and leaving the competition. Simulation is used because of the intractability of other numerical analysis methods.

In the case of a fixed number of players, we assume that each user changes his bid randomly (following an exponential random variable as explained in previous sections), and not sequentially as in Lazar and Semret's original method. This allows to check the effects of the varying number of players in the game. The other difference with respect to the original method is that we use the allocation procedure (4), in order to take into account the cases of equal bids (especially because the bid space is discrete).

We consider the two different strategies described above when players enter and leave randomly the game: either their first bid is randomly and uniformly chosen, which would correspond to a situation where they do not have any indication of the profile before entering the game, or they give their best reply as their initial bid in the case when they do know the profile.

The steady-state values we are going to display (in the tables in figures) are the following:

- The average number of players, per class (or type) and overall.
- The average revenue for the network, and the part generated by each class of players.
- The average ratio of overall utility with respect to the optimal one that would be obtained in steady state (from Lazar and Semret's theory). Somehow, it measures the transient effect caused by the random entrance and departures. Note that in the case where there is no players, this ratio is taken to 1 and that we do not consider the seller's utility.
- The average cumulative value of the derivative of the valuation function for each class, i.e. the average of $\sum_{i:t_i=t}\theta_i'(a_i)$ for $t=1,...,T$.
- The average cumulative utility generated per class.
- The average bandwidth obtained by each class.
- The average ratio of quantity asked with respect to quantity obtained for each class.

In Table 1 (with numerical values described in the caption), we compare the results obtained for the three above schemes. The utility function that we consider is

$$\theta_{t_i}(q) = u_{t_i}(Q\min(Q,q) - (\min(Q,q))^2/2)$$

where u_{t_i} is a class-dependent parameter that has to be specified. The values of arrival and departure rates are chosen so that the per-class mean number of players in the random cases is the same than when the per-class number of players is constant (it can be checked in Table 1 and from traditional $M/M/\infty$ queue analysis). We can point out several results. First, looking at the utility ratio with respect to the optimal one, we can see that the result is not exactly 1 in the case of a fixed number of players. This is due to the discretization of the possible bids. Also, because of the transient effects of arrivals and departures, it is smaller in the random cases. Second, the revenue is much larger when players randomly enter and leave the game, because a bid made by an arriving player can imply a high social cost due to the alrzady present players who obtain less resource than asked. This phemomenon does not appear in the case of a constant number of players, as when the equilibrium is reached players obtain almost all the resource they ask. The high difference in the seller's revenue that results shows that the assumption of a stochastic environment can hardly be skipped to study the mecanism. Surprisingly, the uniform initial bid provides a larger revenue and utility ratio than the best reply case. There is an explanation to this phenomenon: this can be due to the fact that when a player enters, he has probability 0.5 of leaving the network before changing his bid, also there is a good chance (2/3) that a new player enters also, changing then the profile, so that the initial best-reply is not that good anymore and in average can even be worse. Third, the utilities and bandwidth allocations are also different when the number of players is random from the case where this number is fixed. Finally, one can check that the ratio of the allocated amount of bandwidth with respect to the asked one is quite low in the random case (between 0.52 and 0.68), thus players might be unhappy with their allocation. It is also interesting to note that even when the number of players is constant, the players do not get the bandwidth they ask in steady-state. This is due to the fact that they propose the same unit

Value	confidence interval (case of a uniform first bid)	confidence interval (best-reply case)	case of a constant number of players
$\bar{N}$	(1.99749e+00, 2.00220e+00)	(1.99807e+00, 2.00278e+00)	2.00000e+00
Average Revenue	(1.39219e+03, 1.39613e+03)	(8.33222e+02, 8.35983e+02)	1.15501e+02
Average utility ratio/optimum	(8.73367e-01, 8.75090e-01)	(8.36121e-01, 8.37735e-01)	9.99986e-01
$\bar{N}_1$	(9.98669e-01, 1.00077e+00)	(9.99483e-01, 1.00159e+00)	1.00000e+00
Type 1 utility (prime)	(6.91164e+01, 6.92943e+01)	(6.21871e+01, 6.23534e+01)	6.63366e+01
Type 1 utility	(2.15759e+03, 2.16261e+03)	(2.19628e+03, 2.20128e+03)	2.79973e+03
Type 1 revenue	(3.71955e+02, 3.73056e+02)	(3.05133e+02, 3.06140e+02)	6.02824e+01
Type 1 bandwidth	(3.07314e+01, 3.08021e+01)	(3.77419e+01, 3.78249e+01)	3.36634e+01
Type 1 ratio allocated/asked	(6.16461e-01, 6.16923e-01)	(5.19584e-01, 5.20024e-01)	9.90099e-01
$\bar{N}_2$	(9.99073e-01, 1.00118e+00)	(9.98836e-01, 1.00094e+00)	1.00000e+00
Type 2 utility (prime)	(1.07001e+02, 1.07280e+02)	(1.05101e+02, 1.05383e+02)	6.73267e+01
Type 2 utility	(6.16583e+03, 6.18036e+03)	(5.34291e+03, 5.35532e+03)	8.86678e+03
Type 2 revenue	(1.02014e+03, 1.02317e+03)	(5.28035e+02, 5.29897e+02)	5.52182e+01
Type 2 bandwidth	(4.63886e+01, 4.64962e+01)	(4.73145e+01, 4.74209e+01)	6.63366e+01
Type 2 ratio allocated/asked	(6.88488e-01, 6.88955e-01)	(5.72479e-01, 5.72917e-01)	9.90099e-01

Table 1: Values obtained when $T = 2$, $\lambda = 2$, $p = 0.5$, $\mu_1 = \mu_2 = 1$, $\nu_1 = 1$, $\nu_2 = 2$, $u_1 = 1$, $u_2 = 2$, $Q = 100$, $\alpha = 1$ and $p_0 = 0.5$.

Value	confidence interval (case of a uniform first bid)	confidence interval (best-reply case)	case of a constant number of players
$\bar{N}$	(1.99747e+00, 2.00218e+00)	(1.99826e+00, 2.00297e+00)	2.00000e+00
Average Revenue	(1.34032e+03, 1.34408e+03)	(1.73501e+03, 1.73963e+03)	6.45000e+02
Average utility ratio/optimum	(8.65663e-01, 8.67384e-01)	(9.31474e-01, 9.33298e-01)	9.98819e-01
$\bar{N}_1$	(9.99008e-01, 1.00111e+00)	(9.99362e-01, 1.00147e+00)	1.00000e+00
Type 1 utility (prime)	(7.16868e+01, 7.18697e+01)	(7.00260e+01, 7.02096e+01)	6.36364e+01
Type 1 utility	(2.05516e+03, 2.06000e+03)	(2.14114e+03, 2.14599e+03)	2.97521e+03
Type 1 revenue	(3.66256e+02, 3.67325e+02)	(5.13497e+02, 5.14899e+02)	3.96818e+02
Type 1 bandwidth	(2.81950e+01, 2.82606e+01)	(2.98906e+01, 2.99567e+01)	3.63636e+01
Type 1 ratio allocated/asked	(5.53001e-01, 5.53472e-01)	(5.82434e-01, 5.82885e-01)	9.09091e-01
$\bar{N}_2$	(9.98717e-01, 1.00082e+00)	(9.99148e-01, 1.00125e+00)	1.00000e+00
Type 2 utility (prime)	(1.09330e+02, 1.09612e+02)	(1.01201e+02, 1.01469e+02)	7.27273e+01
Type 2 utility	(6.29609e+03, 6.31107e+03)	(6.82312e+03, 6.83912e+03)	8.67769e+03
Type 2 revenue	(9.73981e+02, 9.76844e+02)	(1.22142e+03, 1.22483e+03)	2.48182e+02
Type 2 bandwidth	(4.51886e+01, 4.52942e+01)	(4.92965e+01, 4.94089e+01)	6.36364e+01
Type 2 ratio allocated/asked	(6.59441e-01, 6.59924e-01)	(6.88869e-01, 6.89327e-01)	9.09091e-01

Table 2: Values obtained when $T = 2$, $\lambda = 2$, $p = 0.5$, $\mu_1 = \mu_2 = 1$, $\nu_1 = 1$, $\nu_2 = 2$, $u_1 = 1$, $u_2 = 2$, $Q = 100$, $\alpha = 10$ and $p_0 = 0.5$.

price and require in overall more than the available bandwidth so that it is shared proportionally among them (this shows the interest of totally sharing the bandwidth as suggested in [20]).

Table 2 looks at the same example, but with a larger discretization step. It shows that the best reply case gives better results here (in terms of revenue). It illustrates the difficulty of choosing between both mechanisms. Here too, the players do not get the quantity they ask in steady-state in the fixed number of players' case, and the proportion they get is worse, due to the larger discretization step.

In all the following figures, we consider the first example (with $T = 2$, $\lambda = 2$, $p = 0.5$, $\mu_1 = \mu_2 = 1$, $\nu_1 = 1$, $\nu_2 = 2$, $u_1 = 1$, $u_2 = 2$, $Q = 100$, $\alpha = 1$ and $p_0 = 0.5$) as the basic result and we observe how the performance measures react when one parameter is varying.

Figures 3 and 4 consider (for the uniform and best reply case respectively) the case where the probability that an arriving new player is of class-1 varies from 0.1 to 0.9. At the same time, we vary the departure rates in order to make sure that the mean number of players for each class is one (so that the results can still be compared with those obtained with a fixed number of players in Table 1). The ratio of allocated bandwidth with respect to the asked one is increasing for class-1 and decreasing for class-2. This could be a consequence of the random effect for the uniform case: there is a high probability that a class-1 player enters the game, with a high bid price, after a new bid is made by a class-2 player, and so the new player is allocated the resource asked by the class-2 player. Indeed, as the probability increases, class-2 players (i.e., those who will pay more) stay longer in the network, optimizing their bids more easily, but the very frequent arrivals and departures of class-1 players perturbate the game: in the uniform case, class-1 players get less and less resource (their initial bid is often under class-2 players' bids), whereas in the best-response case, their allocation grows. The same kind of ideas can be used for understanding the utility curves, as well as the for the revenues.

Figures 5 and 6 illustrate the evolution when the total arrival rate increases. Both initial bid cases show similar behaviors. The revenue (and of course the average number of players) increases with the arrival rate. Concerning the obtained average bandwidth and the average utilities, the curves show that the more demanding (class-2) players have increasing values whereas class-1 players hit a maximum, meaning that they suffer more of congestion: when there are too many players in the game, the resource is only allocated to players who value it most, i.e. class-2 players here. Also, for the allocated/asked, both classes reach a maximum, still because of congestion, but the optimal point is at a larger value for class-2. Finally, the ratio revenue/optimum is decreasing because the number of arrivals increases between two changes of bid rate, making it harder to approach the optimal point.

In Figures 7 and 8, only the proportion of class-1 and class-2 players varies. The results are similar for both initial bid cases. The revenue decreases with the number of class-2 players (as they value more the bandwidth), and the average per-class bandwidth and total per-class utility also vary with the per-class average number of players. The single notable difference between the uniform and best reply cases is that, for the best reply case, the ratio of the revenue with respect to the optimal one has a minimum. The fact that it increases after a while is probably due to the idea that most of the overall utility is generated by class-2 players (who value more the resource), less numerous for large values of p and that this ratio is set to one when no customers are in the game. Also, for the best reply case, one can note that the minimum value occurs about when the revenue generated by class-1 begins to exceed class-2's.

Figures 9 and 10 show how the game reacts to an increasing reserve price. The revenue increases with the reserve price, as well as the bids (given by the derivative of utilities), at least up to a given level where the price is somehow considered too expensive. This level is of course higher for class-2 than for class-1. On another hand, we can observe different behaviors between the uniform and best reply cases: the average ratios revenue with respect to the optimum and allocated bandwidth with respect to the asked one decrease in the first case, but rather increase in the second case. This might be due to the fact that the initial bid in the uniform case has a high probability of being smaller than the reserve price.

Figures 11 and 12 show the results depending on the value of the discretization step. The results are quite chaotic. We can note that a discretization step of one provides bad results (in

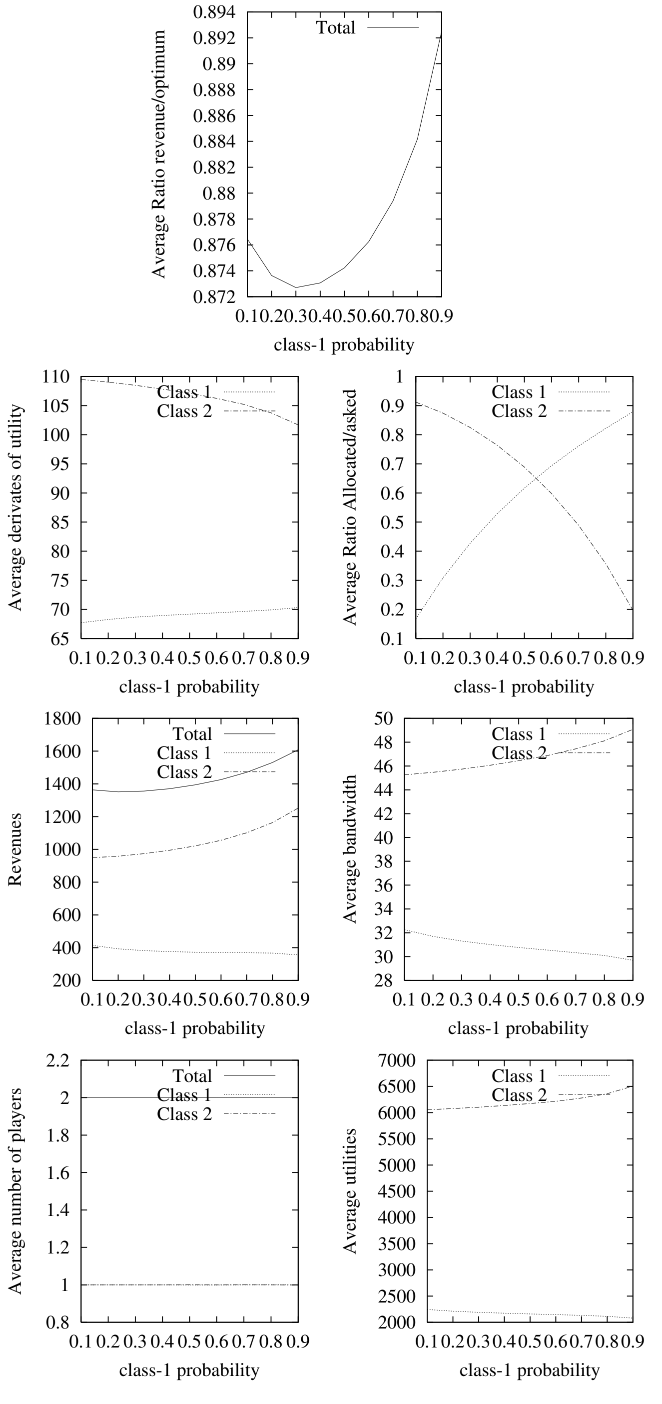

Figure 3: Evolution when the class probabilities and departure rates vary, so that the average number of players is kept constant to one (uniform case).

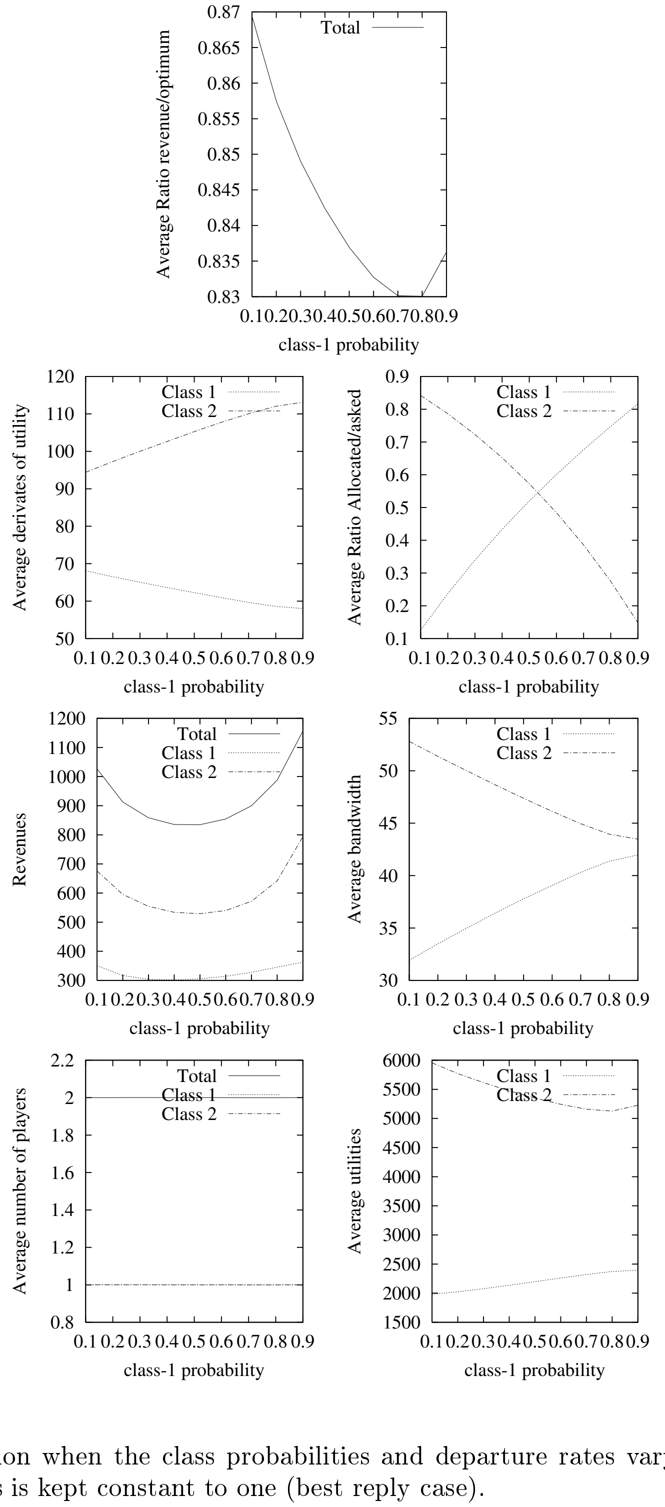

Figure 4: Evolution when the class probabilities and departure rates vary, so that the average number of players is kept constant to one (best reply case).

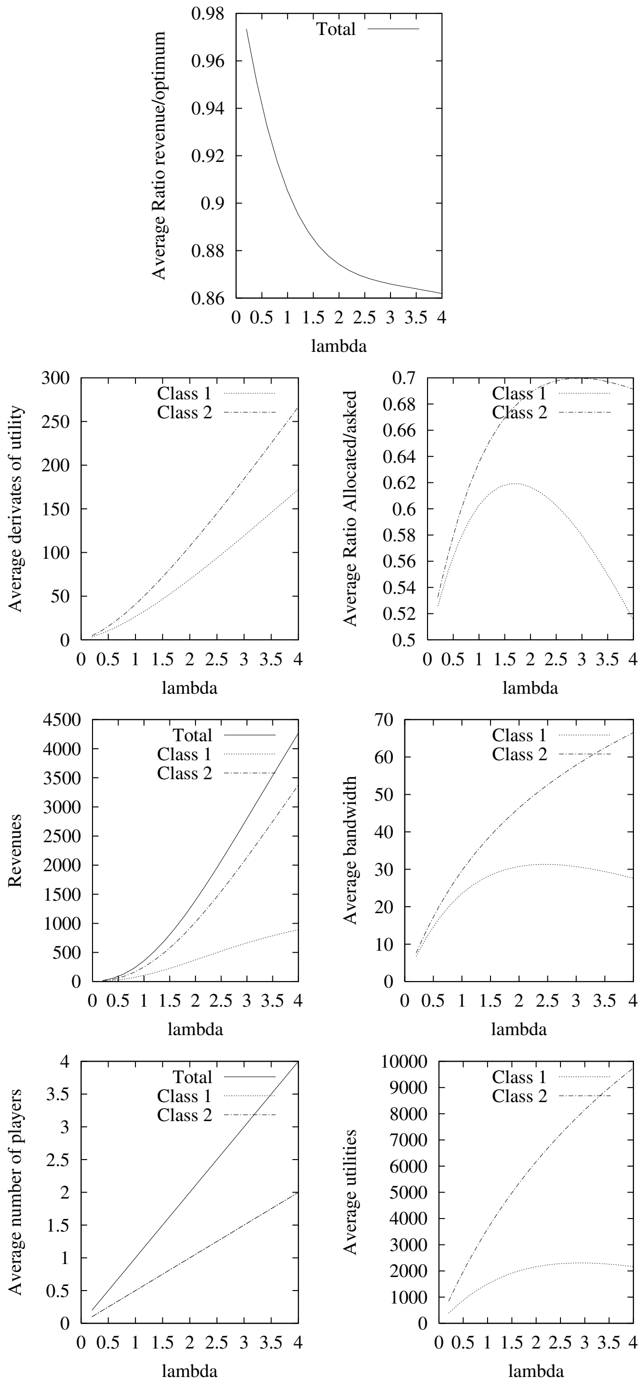

Figure 5: Evolution when the total arrival rate λ varies (uniform case)

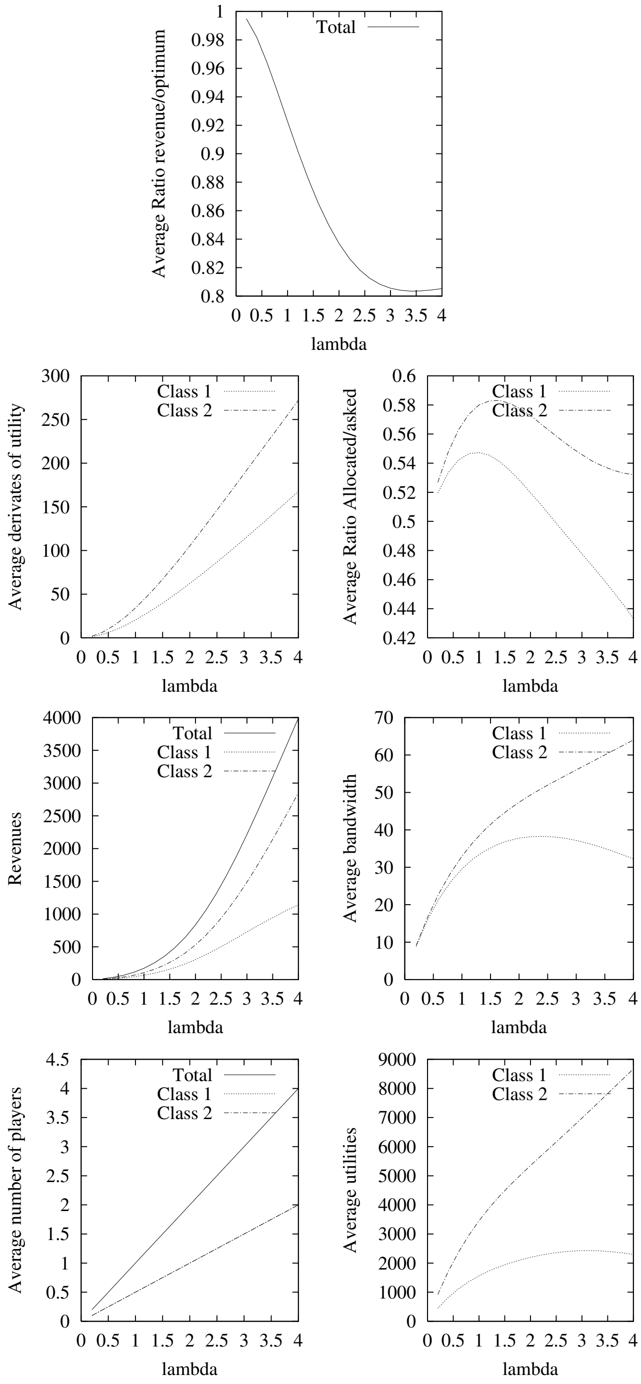

Figure 6: Evolution when the total arrival rate λ varies (best reply case)

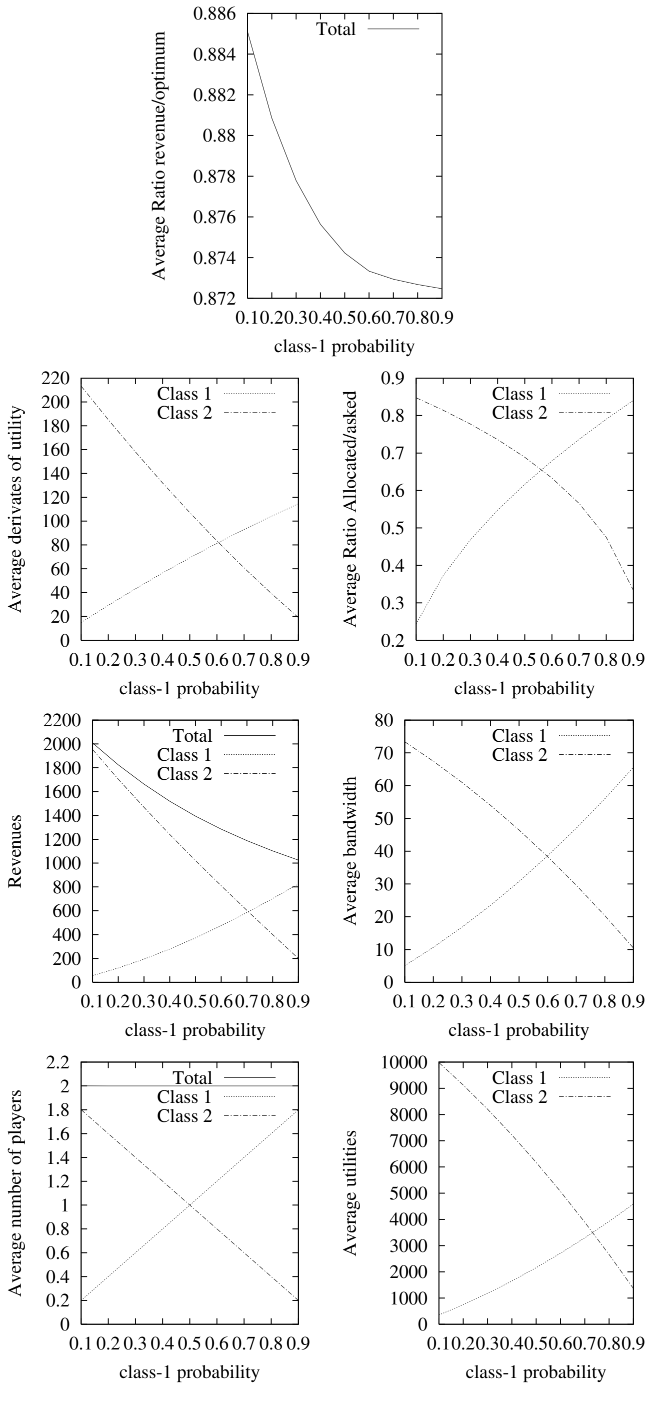

Figure 7: Evolution when the class-1 probability varies (uniform case)

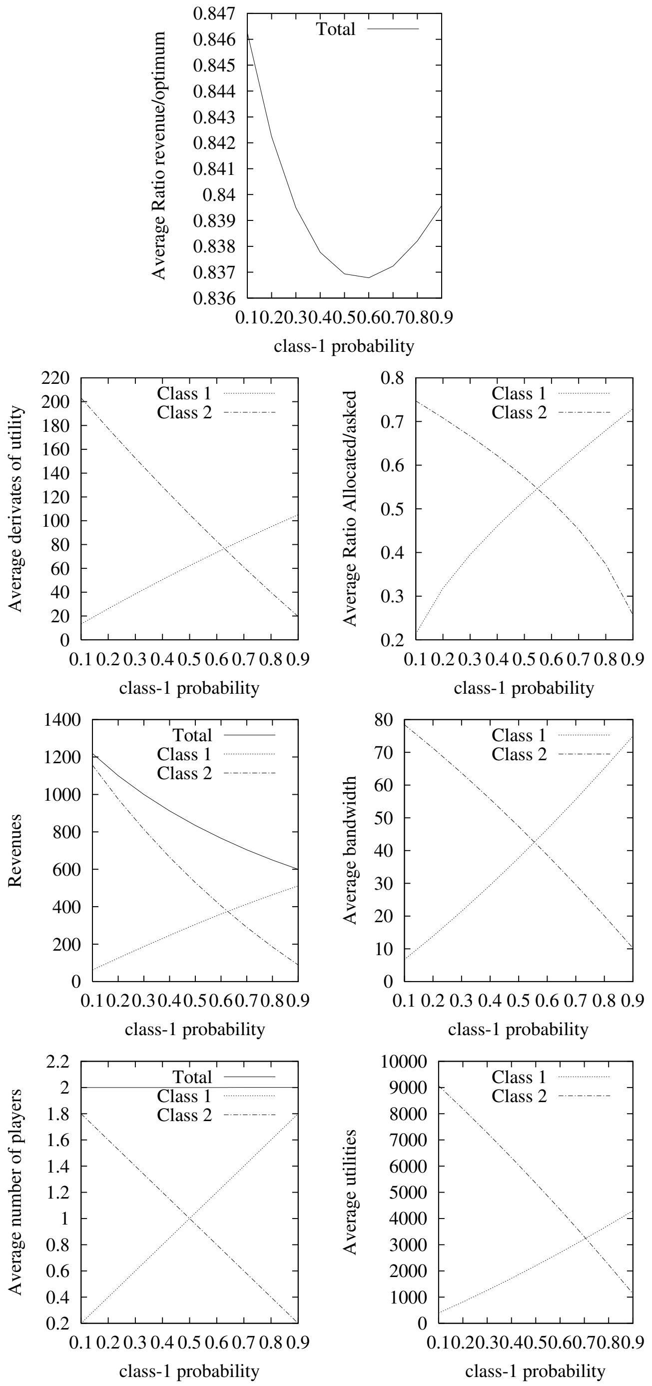

Figure 8: Evolution when the class-1 probability varies (best reply case)

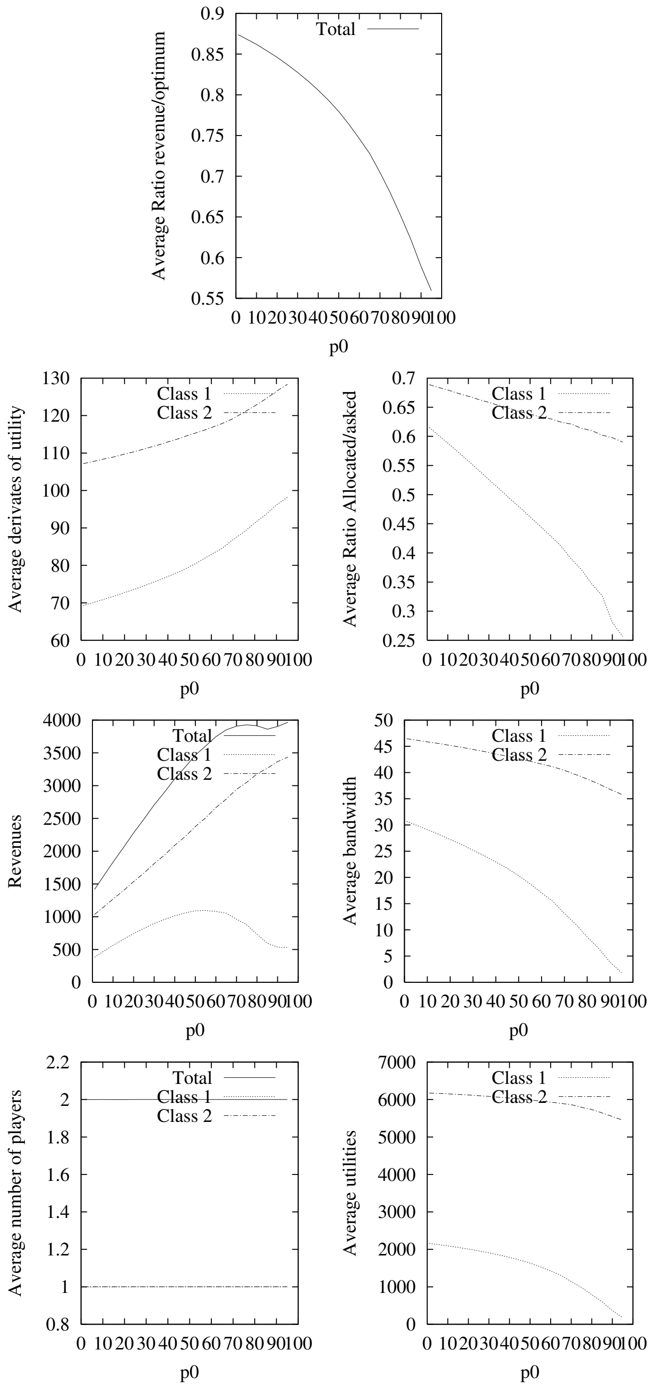

Figure 9: Evolution when the reserve price p_0 varies (uniform case)

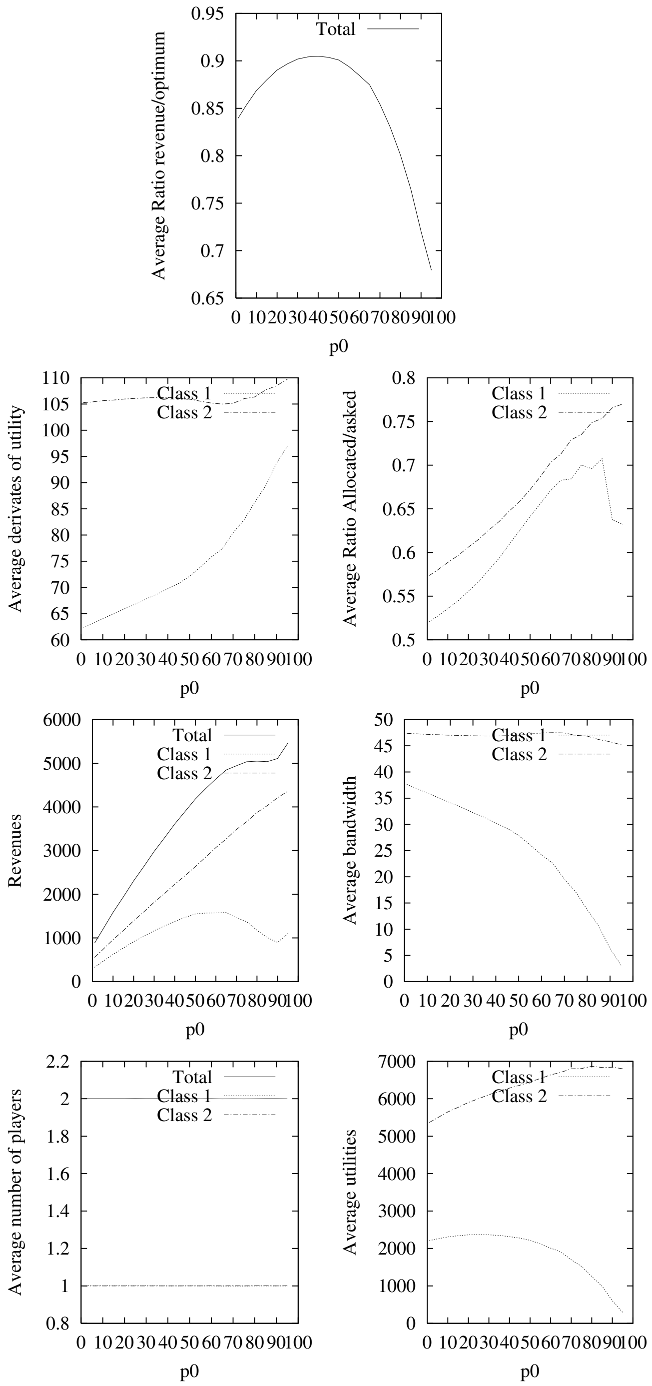

Figure 10: Evolution when the reserve price p_0 varies (best reply case)

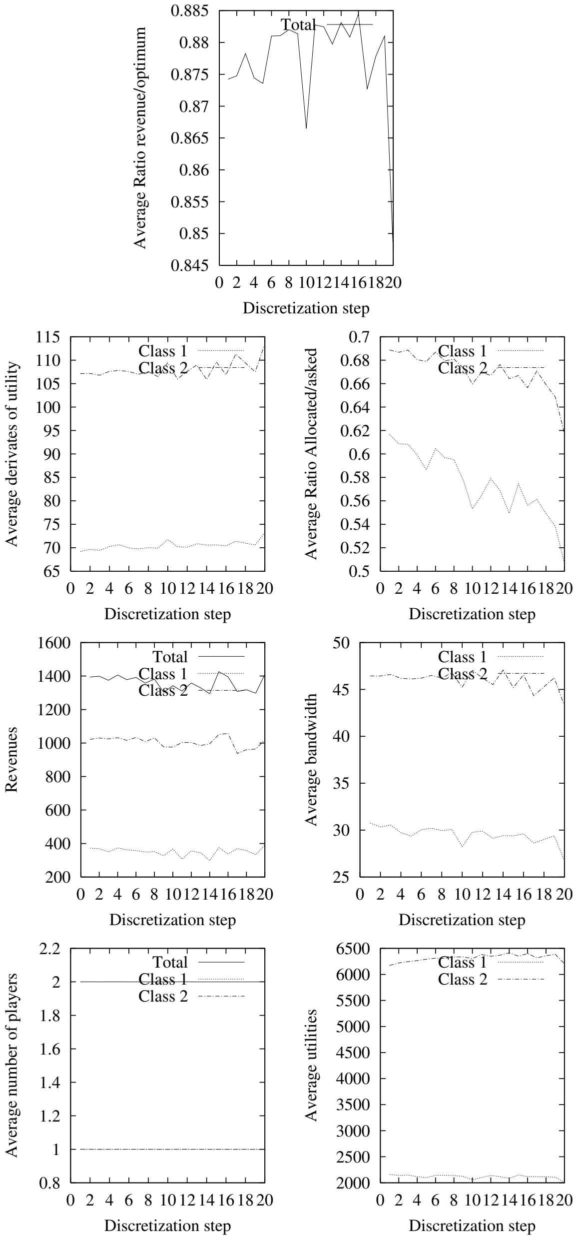

Figure 11: Evolution when the discretization step varies (uniform case)

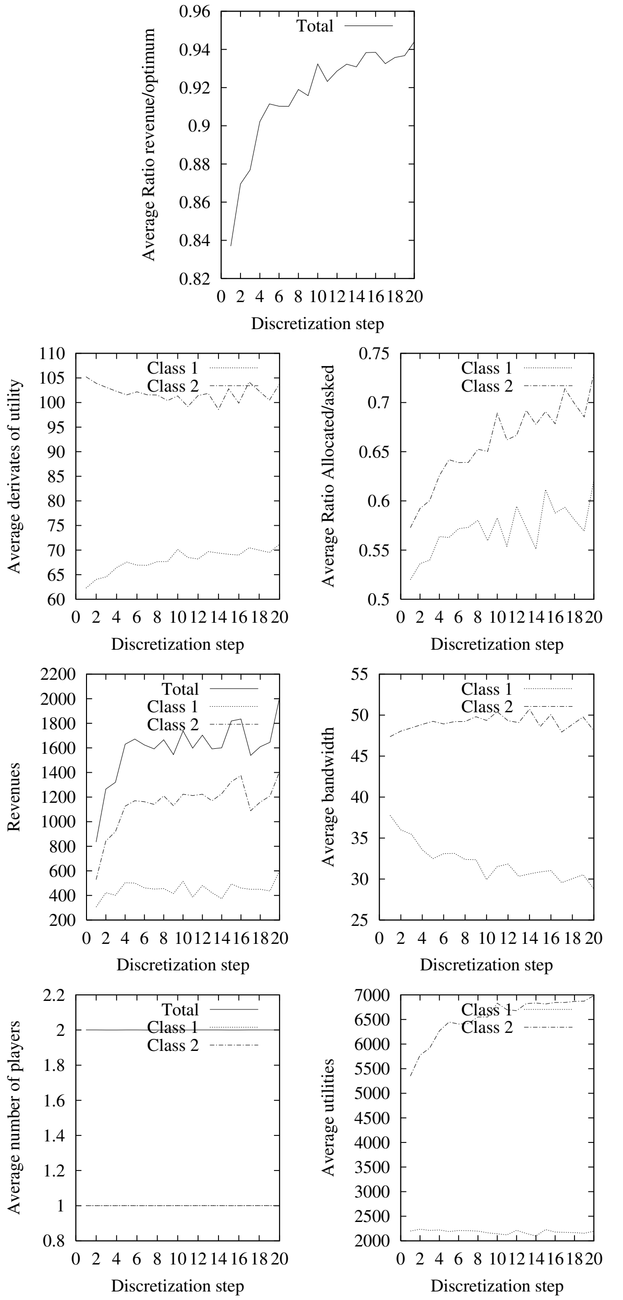

Figure 12: Evolution when the discretization step varies (best reply case)

terms of revenue for the best reply initial bid, with respect to the uniform one) as observed in Table 1.

Figures 13 and 14 show the evolution of the performance measures by simultaneously varying both utility parameters. It can be verified that the average utilities increase with the parameters as well as the network revenue since players value more the allocated bandwidth. One interesting result is (still) with the average ratio of revenue with respect to the optimal one where it can be observed that, at a given level, the ratio decreases with the class-1 utility parameter, but increases with class-2's. The reason is that the class-1 change-of-bid rate is exactly the departure rate and two times less that the arrival rate, meaning that the network state changes often between two change of bids, making the optimal value unlikely to happen, whereas it is less true for class-2, for which the change-of-bid rate is two times class-1's.

Figures 15 and 16 show the results when changing the change-of-bid rates ν_i $(i = 1, 2)$. In both cases, one class-performance increases with its change-of-bid rate (as it can optimize its utility), but it also decreases if the rate of the other class increases (as then it adapts proportionally less its bid than the other class). The case of the average ratio of revenue with respect to the optimal one, which seems to decrease with an increasing class-1 rate, might be due to the fact that most of the revenue is generated by class-2 and that class-1 changes decrease its revenue (and in general a player leaves or enters before that the optimal profile is obtained).

6 Conclusions and perspectives

In this paper, we have confronted an allocation and pricing scheme which is efficient in a static context to a more realistic case, where players can enter and leave the game. We have used Markov chains theory to model the process, and to prove the existence of a stationary distribution. We have then studied the behavior of the mechanism, using simulation, and highlighted the loss of efficiency of the PSP mechanism in a stochastic environment, especially if the time interval between two changes in the set of present players is not big enough with respect to the time of convergence of the auction process. Also, surprisingly, as observed numerically, not providing the bid profile before the initial bid could result in a higher revenue for the seller.

We now address some points that can be the subject of future work. During our study, we have assumed that the sojourn time of each player is exponentially distributed, and independent of the resource obtained so far. This assumption can be justified for applications like telephony or video, where the sojourn time is the duration of a conversation or a program. It would also be interesting to consider that the users have a certain amount of data to transmit (like for file transfer). In that case, their sojourn time in the game would be related to the cumulative bandwidth they have obtained. This situation can be investigated, but it would imply the loss of the Markov property.

Some more work could also be done on the strategic aspect of the game: we assume in this paper that players follow the strategy described in [10], i.e. maximize their immediate utility. However, in game theory, the problem of the strategy choice may rely on a very complex optimization calculus, where time discounting and predictions of future bid profiles would be taken into account. Though those considerations are beyond the framework currently chosen, but deserve some investigation.

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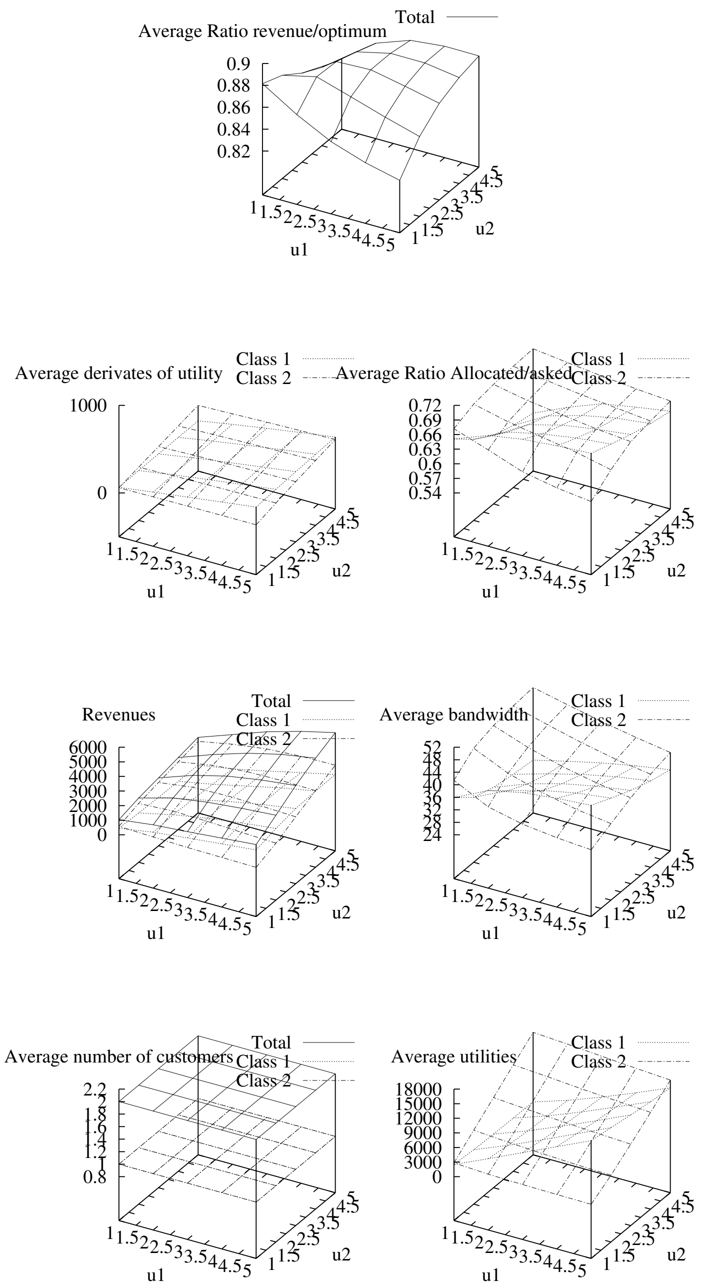

Figure 13: Evolution when the utility parameters vary (uniform case)

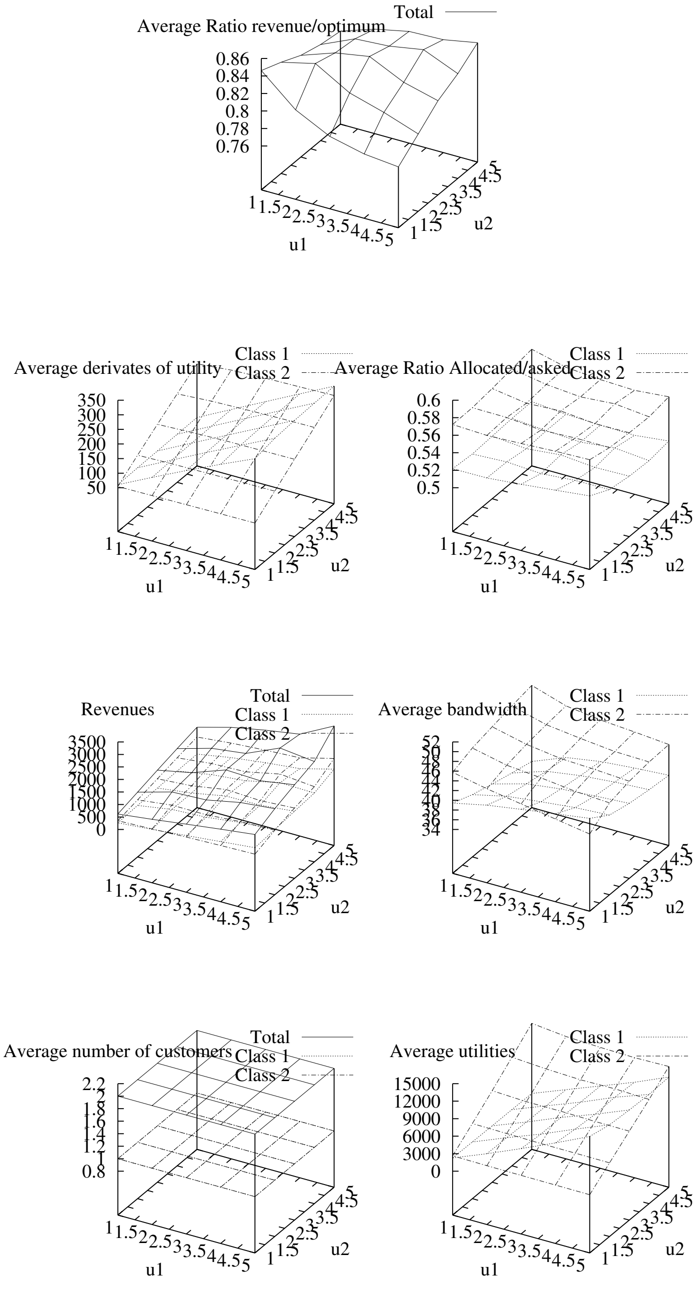

Figure 14: Evolution when the utility parameters vary (best reply case)

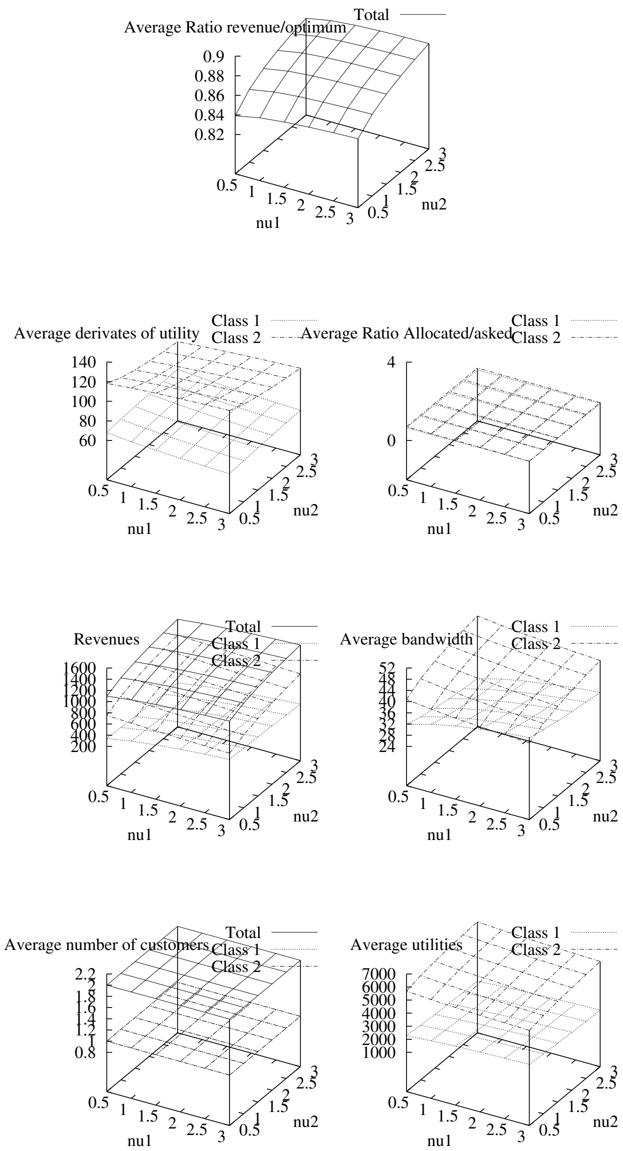

Figure 15: Evolution when the change-of-bid rates vary (uniform case)

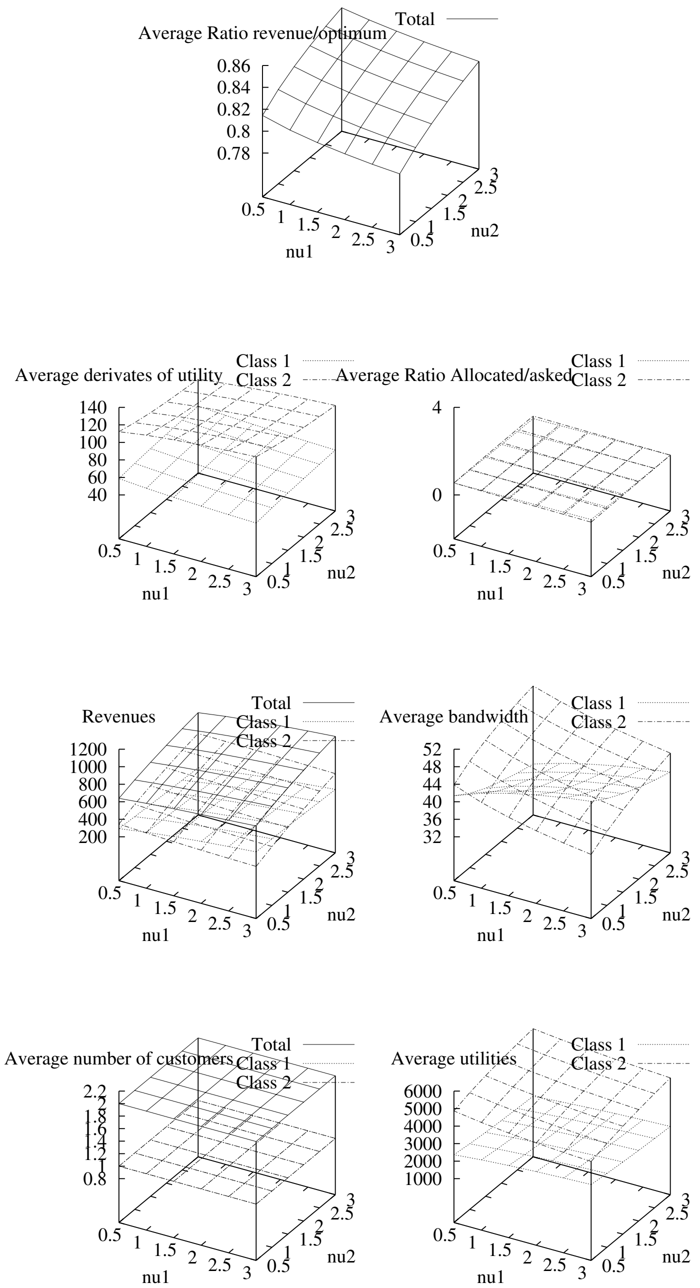

Figure 16: Evolution when the change-of-bid rates vary (best reply case)

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