On Numerical Problems in Simulations of Highly Reliable Markovian Systems

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Abstract

Simulation of highly reliable Markovian systems has been the subject of an extensive literature in recent years. Among all methods, simulation using importance sampling schemes gives the best results when the state space is large. In this paper, we highlight numerical problems that arise in rare events simulation, even when using importance sampling. The literature has up to now focused on variance reduction techniques, without any relation to the variance estimation for instance. The main contribution here is to relate the estimation of the considered parameter and of its variance to the Bounded Relative Error and Bounded Normal Approximation properties. We especially show that Bounded Normal Approximation implies that the variance is well-estimated, which implies Bounded Relative Error, implying itself that the parameter is well-estimated, but that no converse implication is true. This emphasizes the importance of Bounded Normal Approximation property, not frequently used in the literature yet.

1. Introduction

In many problems, we have to study multi-component, fault tolerant and highly reliable Markovian systems. This is the case for example in telecommunications, computer systems or space research. Many important measures have to be computed, such as for instance the reliability, the availability or the mean time to failure. In general, the complexity of the state space is such that analytic computations require too much time to be realized, so approximation methods have to be used. Among the most important are Monte Carlo methods (see [4, 6] for a general description of Monte Carlo). Especially, there is an abundant literature [2, 3, 5, 7, 8, 9, 10, 11, 12, 13] on the application of regenerative Monte Carlo methods, using importance sampling techniques because of rare events. Indeed, because system failures are rare, extremely long simulations may be required in order to obtain accurate estimates of the measures of interest when using crude simulation. Importance sampling consists in changing the underlying distributions, hence the dynamics governing the system, in order to increase the failures probabilities. To obtain unbiased estimates, we must multiply our estimator by a correction factor called the likelihood ratio. Selecting an appropriate importance sampling measure is not easy since it depends on the system being simulated. Different choices for the new sampling measure have been published in the literature: Bias1 failure biasing [9], Bias2 failure biasing [5], Distance failure biasing [3] and Balanced failure biasing [13]. Two major properties have also been studied with respect to a reliability parameter: the *bounded relative error* [11, 13] which asserts that the confidence interval half-width divided by the estimation remains bounded for large reliabilities, and more recently the *bounded normal approximation* [14] which validates the normal approximation in the Central Limit Theorem, so the coverage of the confidence interval as well.

In this paper, we study the numerical problems encountered in simulations of highly reliable Markovian systems, even when using importance sampling. Indeed, when testing the importance sampling schemes and studying their properties, we have numerically observed some unexpected results with respect to the theoretical ones. To our knowledge, no notice and no explanation of this problem have already been published, especially for the variance estimation. However, it seems interesting for a user to take care of abusive interpretations from numerical results and for a theorist to understand the reasons of the problems. These reasons are wrong estimations of the parameter being evaluated and of the variance of this estimator. We will study this point and explain the interaction between estimation of the value and of the variance with bounded relative error and bounded normal approximation properties. More specifically, we show that, somehow surprisingly, good asymptotic (that is when the rarity increases) estimations of the parameter being evaluated and of its variance might not be enough to guarantee the confidence interval coverage.

We show that there is a strong link (a total order) between the good estimations and the Bounded relative Error and Bounded normal Approximation properties in the sense that Bounded Normal Approximation implies that the variance is well-estimated, which implies Bounded Relative Error, implying itself that the parameter is well-estimated, but that no converse implication is true. This extends the results in [14] where only the relation between Bounded Relative Error and Bounded Normal Approximation was studied. Also, we propose a refinment of [14] on the necessary and sufficient condition over the paths for obtaining Bounded Normal Approximation.

The paper is organized as follows. In Section 2, we describe the model and we give a brief state of the art on the applicable simulation techniques. In Section 3, we describe a typical problem that we have encountered during a simple test, to motivate the present paper. In Section 4, we study the relation between bounded normal approximation and variance estimation. The same kind of results, but between variance estimation and bounded relative error, and bounded relative error and the estimation of the value are presented respectively in Section 5 and in Section 6. We conclude in Section 7.

2. Model and state of the art

2.1. Model

Recall as in [11] and [13] that a function f is said to be $o(\varepsilon^d)$ if $f(\varepsilon)/\varepsilon^d \to 0$ as $\varepsilon \to 0$, $f(\varepsilon) = O(\varepsilon^d)$ if $|f(\varepsilon)| \leq c_1\varepsilon^d$ for some constant $c_1 > 0$ for all ε sufficiently small, $f(\varepsilon) = \underline{O}(\varepsilon^d)$ if $|f(\varepsilon)| \geq c_2\varepsilon^d$ for some constant $c_2 > 0$ for all ε sufficiently small, and $f(\varepsilon) = \Theta(\varepsilon^d)$ if $f(\varepsilon) = \underline{O}(\varepsilon^d)$ and $f(\varepsilon) = O(\varepsilon^d)$, i.e., if there exist $0 < c_2 \leq c_1$ such that $c_2\varepsilon^d \leq f(\varepsilon) \leq c_1\varepsilon^d$ for all ε sufficiently small.

The description of the model we use here is more detailed in [11, 13]. We consider a system composed of C different types of components that fail and get repaired. There are n_i components of type i, so a total number $N = \sum_{i=1}^{C} n_i$ of components. The (finite) state space S is such that for each $x \in S$ we have the number of operational components of type i, $n_i(x)$ ($1 \leq i \leq C$). Let $\mathbf{1}$ be the initial state with all components up. We decompose S into two subsets U and F, where U is the set of operational states and F is the set of failed states. The sets U and F verify that if $x \in U$ and $y \in S$ with $n_i(y) \geq n_i(x)$ for all i, then $y \in U$. Failures and repairs of components are supposed to be exponentially distributed, and failure propagations, as well as repair on more than one component at a time, may occur. These event rates may be state-dependent to take into account special structures and dependences of the studied system. The model is then a continuous time Markov chain $(Y_t)_{t\geq 0}$. As in [11], a transition (x, y) from a state x to a state y is said to be a failure transition, and is denoted by $y \succ x$, if $\forall 1 \leq i \leq C$, $n_i(y) \leq n_i(x)$, with $n_k(y) < n_k(x)$ for some k. We define similarly the repair transitions (x, y), which we denote by $y \prec x$. The whole set of possible transitions is denoted by Γ. As it is assumed that the system is composed of highly reliable components, a rarity parameter $\varepsilon > 0$ is introduced in [11, 13], such that $\varepsilon \ll 1$ and such that the failure rates tend to zero with ε. In the same way we suppose that the failure propagation probabilities also depends on ε, but not the repair rates.

Let us denote by X the canonically embedded discrete time Markov chain (DTMC) and by $\boldsymbol{P}$ its transition matrix. Transitions can also be rare for X. Indeed, it is proved in [11] that there exists an integer function $b(x, y)$ and an integer $b_0 = \min_{y:(\mathbf{1},y)\in\Gamma} b(\mathbf{1}, y)$ such that for any $(x, y) \in \Gamma$,

$$\boldsymbol{P}(x, y) = \begin{cases} \Theta(\varepsilon^{b(x,y)}) & \text{if } x \neq \mathbf{1} \\ \Theta(\varepsilon^{b(x,y)-b_0}) & \text{if } x = \mathbf{1}. \end{cases}$$

The special case for state $x = \mathbf{1}$ comes from the fact that all transitions are rare for the CTMC. Define also Φ as the corresponding measure on the sample paths of the DTMC.

We also assume that the system verifies the following properties:

1. the DTMC X is irreducible on S.
2. For every state $x \neq \mathbf{1} \in S$, there exists a state y such that $y \prec x$ and $(x, y) \in \Gamma$.
3. For each state $z \in F$, such that $(\mathbf{1}, z) \in \Gamma$, $\boldsymbol{P}(\mathbf{1}, z) = o(1)$.

We consider here the evaluation of the $MTTF$ (Mean Time To Failure), but other performance measures can be studied similarly. The $MTTF$ can be expressed by [5]

$$MTTF = \frac{E_\Phi\left[\sum_{k=0}^{\min(\tau_{\mathbf{1}},\tau_F)-1} 1/q(X_k)\right]}{E_\Phi\left[1_{(\tau_F<\tau_{\mathbf{1}})}\right]}, \qquad (1)$$

where τ_F is the hitting time of the DTMC X to set F, $\tau_{\mathbf{1}}$ the hitting time to state $\mathbf{1}$ and $1/q(X_k)$ is the expectation of the sojourn time in state X_k.

2.2. Simulation

The performance measure (1) is estimated by means of regenerative simulation [5], that is using independent regenerative cycles $(C_i)_{1\leq i\leq I}$ of the Markov chain X, where C_i is for X between the $i-1^{th}$ and i^{th} return time to $\mathbf{1}$, and applying the Central Limit Theorem to those cycles. A classical estimator of the $MTTF$ is

$$\widehat{MTTF} = \frac{\sum_{i=1}^{I} G(C_i)}{\sum_{i=1}^{I} H(C_i)}$$

where $G(C_i)$ is the sum of the expectations of sojourn times in the states up to $\min(\tau_F, \tau_{\mathbf{1}})$ in the i^{th} cycle and $H(C_i) = 1_{(\tau_F<\tau_{\mathbf{1}})}(C_i)$. Another method is to estimate independently the numerator and the denominator in (1) by using ξI ($0 < \xi < 1$) cycles for the numerator estimation and $(1-\xi)I$ for the denominator [5]. Indeed, the numerator statistical estimation in (1) is efficient with crude Monte Carlo simulation [5], so we can concentrate our attention on the evaluation of the denominator

$$\gamma = E_\Phi[1_{[\tau_F<\tau_{\mathbf{1}}]}].$$

Note also that the computation of γ is useful for the determination of many other measures [11]. We have the following result showing that $[\tau_F < \tau_{\mathbf{1}}]$ is a rare event, hence the difficulty of estimating γ using crude Monte Carlo simulation:

Theorem 1 *[13] There exists a strictly positive constant r such that $\gamma = \Theta(\varepsilon^r)$.*

To estimate γ, we use importance sampling by choosing a new matrix $\boldsymbol{P}'$ so that

$$\gamma = E_{\Phi'}[1_{[\tau_F<\tau_{\mathbf{1}}]}L]$$

where $L(x_0, \cdots x_n) = \dfrac{\Phi\{(X_0,\cdots,X_{\tau_F}) = (x_0,\cdots,x_n)\}}{\Phi'\{(X_0,\cdots,X_{\tau_F}) = (x_0,\cdots,x_n)\}}$ is defined for all path $(x_0,\cdots,x_n)$ and where Φ' is the measure corresponding to matrix $\boldsymbol{P}'$. The different choices in literature are Bias1 failure biasing (also called simple failure biasing) [9], balanced failure biasing [13], Bias2 failure biasing [5] and failure distance biasing [3].

We describe here briefly Bias1 failure biasing which will be used in our numerical examples. The choice of $\boldsymbol{P}'$ is the following: from a state x in $\{\mathbf{1}\} \cup F$, the probability transitions are not changed, i.e. $\boldsymbol{P}'(x,.) = \boldsymbol{P}(x,.)$. From another state, a probability ρ_0 is assigned to the whole set of failure transitions, and a probability $1-\rho_0$ is assigned to the whole set of repair transitions. In each of these two subsets, the individual probabilities are taken proportionally to the original ones. In numerical illustrations, we will take $\rho_0 = 0.8$. The advantage of this technique is that the failure set probabilities are no more $O(\varepsilon)$, meaning that observing a failure is not a rare event anymore.

2.3. Bounded Relative Error and Bounded Normal Approximation

In this Section, we describe/recall two important properties to be verified by the γ estimator.

Let us first introduce the following definition, which asserts that the confidence interval width will not be large with respect to γ for large reliabilities:

Definition 1 *[13] Define $\sigma^2_{\Phi'}$ as the variance of the random variable $1_{[\tau_F<\tau_{\mathbf{1}}]}L$ under probability measure Φ' (which has mean γ) and z_δ as the $1-\delta/2$ quantile of the standard normal distribution (i.e., mean 0 and variance 1). Then the relative error for a sample size I is defined by*

$$RE = z_\delta \frac{\sqrt{\sigma^2_{\Phi'}/I}}{\gamma}.$$

We say that we have a bounded relative error if RE remains bounded as $\varepsilon \to 0$.

Let Δ_m be the set of paths defined by

$$\begin{aligned}\Delta_m = \quad & \{(x_0,\cdots,x_n):\ n \geq 1, x_0 = \mathbf{1}, x_n \in F, \\ & x_i \notin \{\mathbf{1}, F\} \text{ for } 1 \leq i \leq n-1, (x_i, x_i+1) \in \Gamma \\ & \text{and } \Phi\{(X_0,\cdots,X_{\tau_F}) = (x_0,\cdots,x_n)\} = \Theta(\varepsilon^m)\}.\end{aligned}$$

The following result is a necessary and sufficient condition on the importance sampling measure to have a bounded relative error:

Theorem 2 *[11] Consider $\mathcal{H}$ the set of importance sampling measures Φ' corresponding to a transition matrix P' such that for any $(x,y) \in \Gamma$, if $P(x,y) = \Theta(\varepsilon^d)$, then $P'(x,y) = \underline{O}(\varepsilon^d)$. Let $\Phi' \in \mathcal{H}$. Then we have a bounded relative error if and only if for all $(x_0,\cdots,x_n) \in \Delta_m$, $r \leq m \leq 2r-1$,*

$$\Phi'\{(X_0,\cdots X_{\tau_F}) = (x_0,\cdots,x_n)\} = \underline{O}(\varepsilon^{2m-2r}).$$

In the same way, using the Berry-Esseen Theorem, or one of its variants from the Student statistic [1], we can bound the normal approximation for a given sample size, and then be sure that the confidence level of the confidence interval is controlled. The Berry-Esseen bound is the following: let $\mathcal{N}(x)$ be the standard normal distribution and, for a random variable X, let $\rho = E[|X-E(X)|^3]$, $\sigma^2 = E[(X-E[X])^2]$. Consider $X_1,\cdots,X_I$ I i.i.d. copies of X, define $\bar{X}_I = I^{-1}\sum_{i=1}^I X_i$, $\hat{\sigma}_I^2 = I^{-1}\sum_{i=1}^I (X_i - \bar{X}_I)^2$ and let F_I be the distribution of the centered and normalized sum $(X_1+\cdots+X_I)/(\hat{\sigma}_I\sqrt{I}) - E[X]\sqrt{I}/\hat{\sigma}_I$. Then there exists an absolute constant $c > 0$ such that, for each x and I

$$|F_I(x) - \mathcal{N}(x)| \leq \frac{c\rho}{\sigma^3\sqrt{I}}.$$

From this bound and the discussion in [14] on a necessary condition to be verified, we can define the Bounded Normal Approximation property.

Definition 2 *[14] If*

$$\rho_{\Phi'} = E_{\Phi'}\left[\left|1_{[\tau_F<\tau_{\mathbf{1}}]}L - E_{\Phi'}[1_{[\tau_F<\tau_{\mathbf{1}}]}L]\right|^3\right]$$

denotes the third-order absolute moment and $\sigma_{\Phi'}$ the standard deviation of the random variable $1_{[\tau_F<\tau_{\mathbf{1}}]}L$ under probability measure Φ', we say that we have a bounded normal approximation if $\rho_{\Phi'}/\sigma^3_{\Phi'}$ is bounded when $\varepsilon \to 0$.

Let

$$\Delta = \bigcup_{m=r}^{\infty} \Delta_m,$$

and for Φ' an importance sampling measure, define as in [14]

$$\begin{aligned}\Delta_{m,k} = \{ \quad & (x_0, \cdots, x_n) \in \Delta \ : \\ & \Phi\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\} = \Theta(\varepsilon^m) \text{ and} \\ & \Phi'\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\} = \Theta(\varepsilon^k)\},\end{aligned}$$

$$\Delta'_t = \bigcup_{m,k\,:\,m-k=t} \Delta_{m,k},$$

and s the integer such that $\sigma^2_{\Phi'} = \Theta(\varepsilon^s)$.

In [14], we have considered the so-called cancellation case where $s > 2r$ from cancellation of the highest order terms of γ^2 and $E_{\Phi'}[1_{[\tau_F<\tau_{\mathbf{1}}]}L^2]$ when these quantities are of the same order of magnitude. But an important remark is that if the variance $\sigma^2_{\Phi'} = E_{\Phi'}\left[\left(1_{[\tau_F<\tau_{\mathbf{1}}]}L - E_{\Phi'}[1_{[\tau_F<\tau_{\mathbf{1}}]}L]\right)^2\right]$ includes the computation over a path $(\mathbf{1}, x, \mathbf{1})$ in $\Theta(1)$ (with one failure and one repair) such that $1_{[\tau_F<\tau_{\mathbf{1}}]} = 0$, the cancellation case does not occur since

$$\begin{aligned}\sigma^2_{\Phi'} \quad > \quad & \left(1_{[\tau_F<\tau_{\mathbf{1}}]}(\mathbf{1}, x, \mathbf{1})L(\mathbf{1}, x, \mathbf{1}) - \gamma\right)^2 \Phi'(\mathbf{1}, x, \mathbf{1}) \\ & = \gamma^2\Phi'(\mathbf{1}, x, \mathbf{1}) = \Theta(\varepsilon^{2r}).\end{aligned}$$

This is especially the case of the class of measures $\mathcal{I}$ defined below. This situation was not considered in [14].

A necessary and sufficient condition to obtain a bounded normal approximation property is the following (refinement of the one in [14], from the above remark):

Theorem 3 *Let $\mathcal{I}$ be the class of measures Φ' corresponding to matrix $\boldsymbol{P}'$ defined as follows: for all $(\omega, y) \in \Gamma$, $\omega \neq \mathbf{1}$ and $y \succ \omega$,*

$$\text{if } \boldsymbol{P}(\omega, y) = \Theta(\varepsilon^d), \text{ then } \boldsymbol{P}'(\omega, y) = \underline{O}(\varepsilon^{d-1})$$

and for all (ω, y) with either $y \prec \omega$ or $y \succ \omega$ and $\omega = \mathbf{1}$,

$$\text{if } \boldsymbol{P}(\omega, y) = \Theta(\varepsilon^d), \text{ then } \boldsymbol{P}'(\omega, y) = \underline{O}(\varepsilon^d).$$

The normal approximation is bounded for a fixed number of observations and a measure $\Phi' \in \mathcal{I}$ if and only if $\forall k, m$ such that $m - k < r$, $(x_0, \cdots, x_n) \in \Delta_{m,k}$,

$$\Phi'\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\} = \underline{O}(\varepsilon^{3m/2-3s/4})$$

(i.e. $k \leq 3m/2 - 3s/4$).

The necessary and sufficient conditions of Theorems 2 and 3 will be used in next sections to prove our assertions.

Using these conditions, we have also proved in [14] that, for a measure $\Phi' \in \mathcal{I}$, the bounded normal approximation implies bounded relative error and that among all the importance sampling techniques that can be modeled by the framework presented here, only balanced failure biasing scheme verify in general the Bounded Normal Approximation and Bounded Relative Error properties.

3. A Comparison between numerical and theoretical results

One important point is that, even using importance sampling scheme, numerical results can be far from the theoretically expected ones. In this section we consider a small and instructive example allowing to understand this problem. It not only illustrates that importance sampling does not always yield correct results, but also that theoretical properties might look verified even if there are not theoretically.

The system consists of two types of component with two components of each type. The transition probabilities of the embedded discrete time Markov chain are described in Figure 1, where $< i, j >$ denotes the state with i (resp. j) operational components of type 1 (resp. 2) and where the states

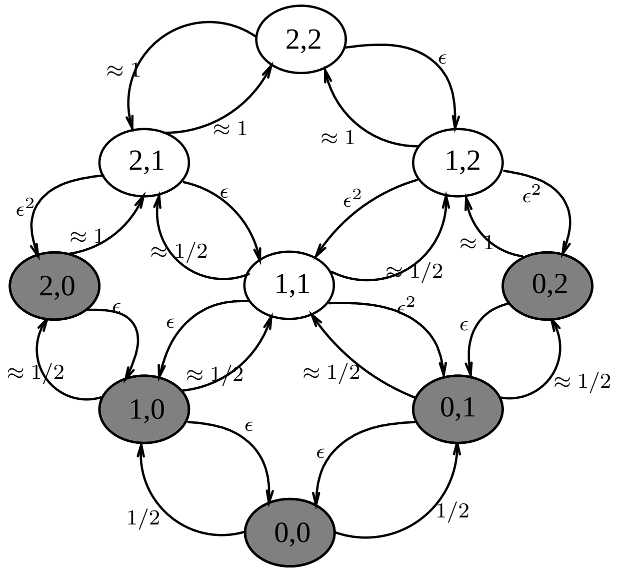

Figure 1. System I with its probability transitions.

representing the system down (i.e. the states in F) are colored in grey. More exactly, the system is up if and only if at least one component of each type is up. Moreover, the transitions probabilities of this system, using Bias1 failure biasing scheme, are described in Figure 2.

For this model, using Bias1 failure biasing scheme as the

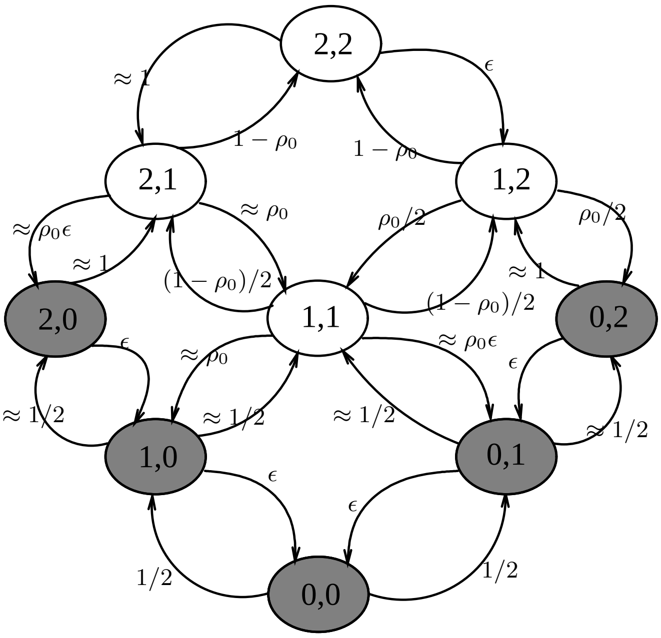

Figure 2. System I with Bias1 failure biasing probability transitions.

importance sampling measure Φ', we have

$$\gamma = 2\varepsilon^2 + o(\varepsilon^2) \text{ and } \sigma^2_{\Phi'} = \frac{1}{\rho_0}\varepsilon^3 + o(\varepsilon^3). \qquad (2)$$

(Note that, even if exact values can be computed on these examples, we only give a first order approximation since we are interested in the asymptotic results.) A first observation is then that we do not have bounded relative error. Nevertheless, if we attempt to observe this numerically, we note, see the last column of Table 1, that considering a fixed sample size I and decreasing ε, bounded relative error is observed. More exactly, the observed relative error grows as ε decays (for $\varepsilon \geq$2e-04) but suddenly drops and remains constant for $\varepsilon <$2e-04.

In the same way, consider the evaluation of γ. As $\varepsilon \to 0$, using (2), the estimation $\bar{\gamma}_I$ of γ should be approximately $2\varepsilon^2$. Table 1, columns 2 to 4, describe simultaneously $2\varepsilon^2$, $\bar{\gamma}_I$ and the confidence interval (at risk 5% and with an estimation of $\sigma^2_{\Phi'}$) as $\varepsilon \to 0$ and for a fixed sample size I using Bias1 failure biasing scheme. The estimated value is bad as $\varepsilon \to 0$: as we can observe, $\bar{\gamma}_I$ seems to be close to the expected value for $\varepsilon \geq$2e-04, and the confidence interval seems suitable too, but, between 2e-04 and 1e-04, as ε decays, the results are far from the expected one and $2\varepsilon^2$ is not included in the confidence interval.

The problem is that, even using importance sampling schemes, some paths important for γ and $\sigma^2_{\Phi'}$ evaluations

ε	$2\varepsilon^2$	$\bar{\gamma}_I$	Confidence Interval	Est. RE
10^{-2}	2e-04	2.03e-04	(1.811e-04 , 2.249e-04)	1.08e-01
10^{-3}	2e-06	2.37e-06	(1.561e-06 , 3.186e-06)	3.42e-01
$2\ 10^{-4}$	8e-08	6.48e-08	(1.579e-08 , 1.138e-07)	7.56e-01
10^{-4}	2e-08	9.95e-09	(9.801e-09 , 1.010e-08)	1.48e-02
10^{-6}	2e-12	9.95e-13	(9.798e-13 , 1.009e-12)	1.48e-02
10^{-8}	2e-16	9.95e-17	(9.798e-17 , 1.009e-16)	1.48e-02

Table 1. Asymptotic development $2\varepsilon^2$ of γ, estimation of γ, confidence interval and estimated relative error for system I using Bias1 failure biasing scheme ($\rho_0 = 0.8$) with a fixed sample size $I = 10^4$ and diverse values of ε.

are still rare, i.e. their measures are still in $O(\varepsilon)$. So, using a fixed sample size I from a pseudo-random generator, all the sample paths with probability measure in $O(\varepsilon)$ are not sampled for ε sufficiently small.

Following these observations, it seems to be interesting to understand more precisely the dependencies between, first the important properties (bounded relative error and bounded normal approximation), and secondly the observed estimation and confidence interval, as $\varepsilon \to 0$. This is done in the next sections and constitutes the main contribution of the paper.

4. Relation between $\sigma^2_{\Phi'}$ estimation and Bounded Normal Approximation

Let us start with the following definition.

Definition 3 *Let f be a function defined over Δ (we set $f = 0$ elsewhere) and $t \geq 0$ be such that*

$$E_\Phi[f(X_0, \cdots, X_{\tau_F})] = \Theta(\varepsilon^t).$$

We will say that $E_\Phi[f(X_0, \cdots, X_{\tau_F})]$ is well estimated as $\varepsilon \to 0$ under probability measure Φ' if for all $(x_0, \cdots x_n) \in \Delta$ such that $f(x_0, \cdots x_n)\Phi\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\} = \Theta(\varepsilon^t)$, then

$$\Phi'\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\} = \Theta(1).$$

This definition means that all the paths in Δ important for the evaluation of $E_\Phi[f(X_0, \cdots, X_{\tau_F})]$ as $\varepsilon \to 0$ are not rare events under probability measure Φ'. This definition can also be summarized saying that, using a (fixed) pseudo-random sequence and sample size I, and defining by $\overline{f_I L(\varepsilon)}$ the estimated value of $E_{\Phi'}[f(X_0, \cdots, X_{\tau_F})L]$ under probability measure Φ',

$$\begin{aligned} E_{\Phi'}\left[\lim_{\varepsilon \to 0} \overline{f_I L(\varepsilon)}\right] &= \lim_{\varepsilon \to 0} E_{\Phi'}[f(X_0, \cdots X_{\tau_F})L] \\ &= \lim_{\varepsilon \to 0} E_\Phi[f(X_0, \cdots X_{\tau_F})]. \end{aligned}$$

Let us wonder if there is a relation between $\sigma^2_{\Phi'}$ estimation and Bounded Normal Approximation property.

Proposition 1 *Let $\Phi' \in \mathcal{I}$ (see Theorem 3). Bounded Normal Approximation property implies that $\sigma^2_{\Phi'}$ is well estimated.*

Proof: Bounded Normal Approximation property implies Bounded Relative Error property, so $s = 2r$.

Let m, k and $(x_0, \cdots, x_n) \in \Delta_{m,k}$ be such that

$$\frac{\Phi^2\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\}}{\Phi'\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\}} = \Theta(\varepsilon^s) = \Theta(\varepsilon^{2r}).$$

If $m - k \geq r$ then $\Phi'\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\} = \Theta(1)$ (i.e. $k = 0$), since $2r = (2m - k) = 2(m - k) + k$.

If $m-k < r$, consider the necessary and sufficient condition of Theorem 3. Then $s = 2m-k$ and $k \leq 3m/2-3s/4$, which means that $2m - s \leq 3/4(2m - s)$. As $s = 2r$ (we have Bounded Relative Error) and $m \geq r$ by definition of r, then $2m = 2r = s$, that is $k = 0$. ∎

Proposition 2 *The converse of Proposition 1 is false, i.e. there exists a system and a measure $\Phi' \in \mathcal{I}$ such that $\sigma^2_{\Phi'}$ is well estimated but such that Bounded Normal Approximation property is not verified.*

Proof: consider the example of Figure 3, using Bias1 failure biasing as described in Figure 4. The states where the system is down are still colored in grey.

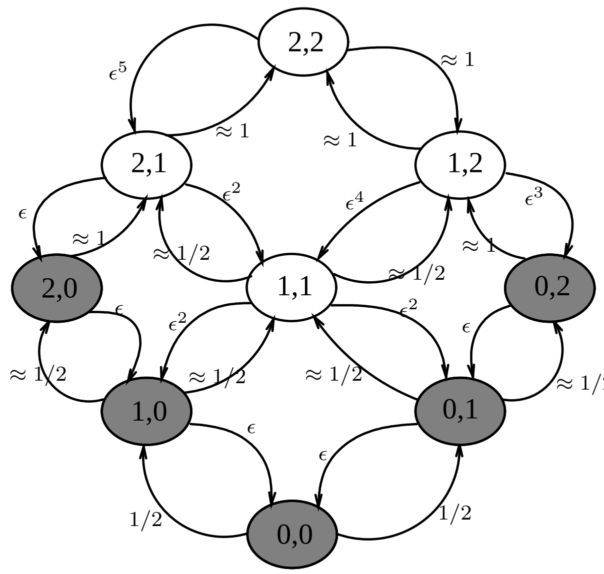

Figure 3. System II with its probability transitions.

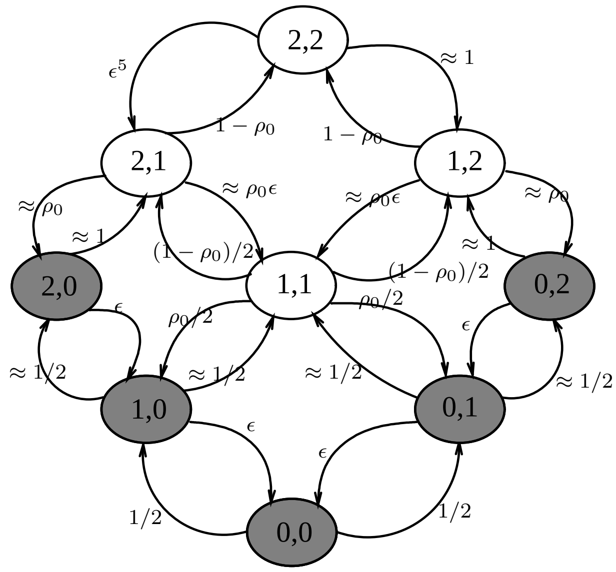

Figure 4. System II with Bias1 failure biasing probability transitions.

For this model, as it can be easily seen in Figure 3, $r = 3$ and Δ_3 is constituted of only one path: $(< 2, 2 >, < 1, 2 >, < 0, 2 >)$. Moreover $s = 6$ and the sole path such that $\frac{\Phi^2\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\}}{\Phi'\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\}} = \Theta(\varepsilon^6)$ is the path in Δ_3 for which Figure 4 shows that it is well estimated.

However, the path $(< 2, 2 >, < 2, 1 >, < 2, 0 >)$ is in $\Delta_{m,k}$ with $m = 6$ and $k = 5$. Then $5 = k > 3m/2 - 3s/4 = 4.5$, so the necessary and sufficient condition of Theorem 3 is not verified. ∎

A first idea would be to think that if γ and $\sigma^2_{\Phi'}$ are well estimated, then the normal approximation is good enough (i.e. bad confidence interval coverage is only due to bad $\sigma^2_{\Phi'}$ estimation). Proposition 2 shows that this is not the case. Thus the Bounded Normal Approximation property is important to be verified in practice to be convinced that no theoretical or numerical error is realized.

5. Relation between $\sigma^2_{\Phi'}$ estimation and Bounded Relative Error

In this section, we prove that if $\sigma^2_{\Phi'}$ is well estimated, then the Bounded Relative Error property is verified, but that the converse is not true.

Proposition 3 *If $\sigma^2_{\Phi'}$ is well estimated, using a probability measure $\Phi' \in \mathcal{H}$, then we have the Bounded Relative Error property.*

Proof: we will prove the equivalent proposition that if Bounded Relative Error property is not verified, then $\sigma^2_{\Phi'}$ is not well estimated.

Unless we have Bounded Relative Error, there exist from Theorem 2 integers m, k and $(x_0, \cdots, x_n) \in \Delta_{m,k}$ such that $2m - k = s < 2r$. But, by definition of r, $2m \geq 2r$, so that $k > 0$. ∎

Proposition 4 *The converse of Proposition 3 is false, i.e. there exists a system and a measure $\Phi' \in \mathcal{H}$ such that Bounded Relative Error is verified but $\sigma^2_{\Phi'}$ is not well estimated.*

Proof: consider the example of Figure 5, using Bias1 failure biasing as described in Figure 6.

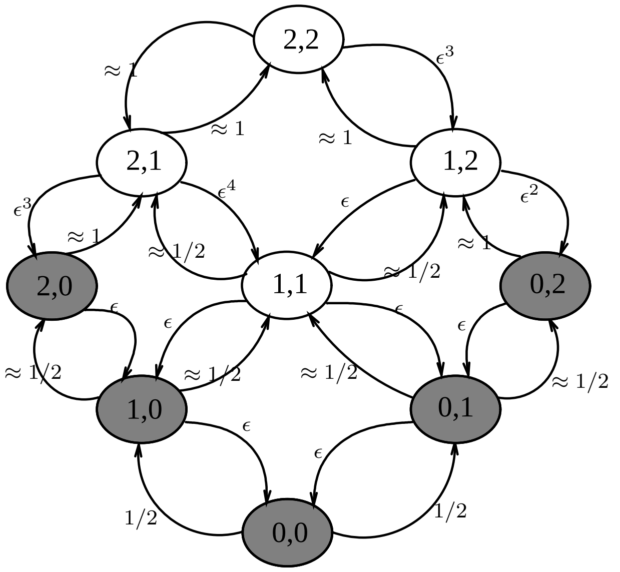

Figure 5. System III with its probability transitions.

For this model, we have $r = 3$, $s = 6$, so this system verifies the Bounded Relative Error property. Moreover, the set of paths such that

$$\frac{\Phi^2\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\}}{\Phi'\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\}} = \Theta(\varepsilon^6)$$

is constituted of two paths: $(< 2, 2 >, < 2, 1 >, < 2, 0 >)$, which is well estimated (see Figure 6), and $(< 2, 2 >, < 1, 2 >, < 0, 2 >)$ for which Φ' is in $O(\varepsilon^3)$. Thus $\sigma^2_{\Phi'}$ is not well estimated. ∎

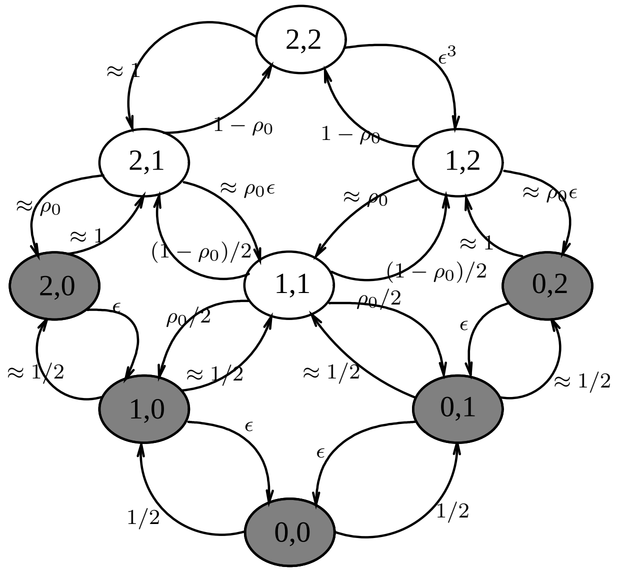

Figure 6. System III with Bias1 failure biasing probability transitions.

6. Relation between γ estimation and Bounded Relative Error

Return now to γ estimation.

Proposition 5 *If we have Bounded Relative Error property using a measure $\Phi' \in \mathcal{H}$ (see Theorem 2 to recall $\mathcal{H}$ definition), then γ is well estimated as $\varepsilon \to 0$ under Φ'.*

Proof: a path $(x_0, \cdots, x_n) \in \Delta$ important for the evaluation of $\gamma = E_\Phi[1_{[\tau_F < \tau_{\mathbf{1}}]}] = \Theta(\varepsilon^r)$ verifies $\Phi\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\} = \Theta(\varepsilon^r)$. It follows from the necessary and sufficient condition of Theorem 2 on probability measure Φ' that $\forall (x_0, \cdots, x_n) \in \Delta_r$,

$$\begin{aligned} \Phi'\{(X_0, \cdots, X_{\tau_F}) = (x_0, \cdots, x_n)\} &= \underline{O}(\varepsilon^{2r-2r}) \\ &= \underline{O}(1) \\ &= \Theta(1). \end{aligned}$$

Thus γ is well estimated under Φ'. ∎

Proposition 6 *The converse of Proposition 5 is false, i.e. there exists a system and a measure $\Phi' \in \mathcal{H}$ such that γ is well estimated but such that Bounded Relative Error property is not verified.*

Proof: consider the example of Figure 7, using Bias1 failure biasing scheme as described in Figure 8. The states where the system is down are still colored in grey.

As it can be easily seen in Figure 7, Δ_r (with $r = 6$ for this example) is composed of two paths: $(< 2, 2 >, <$

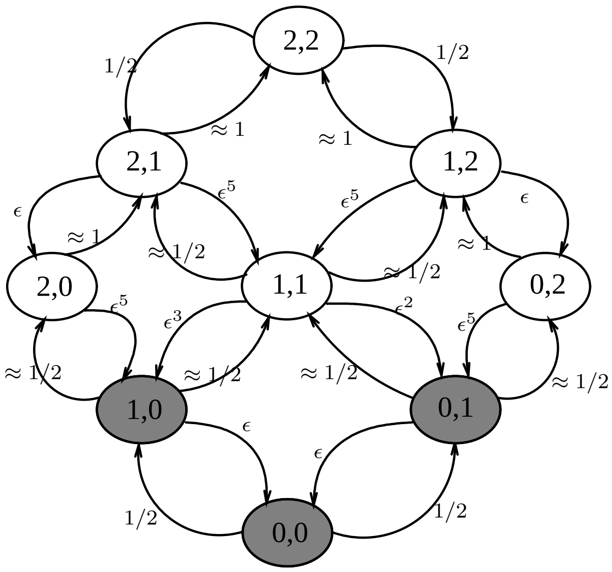

Figure 7. System IV with its probability transitions.

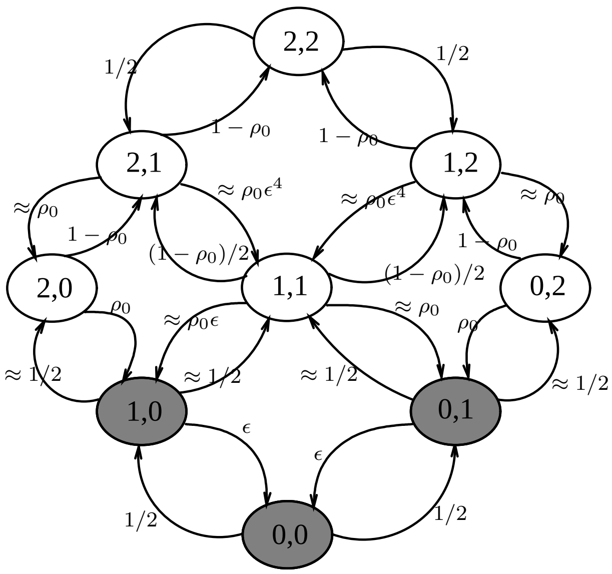

Figure 8. System IV with Bias1 failure biasing probability transitions.

$2,1>,<2,0>,<1,0>)$ and $(<2,2>,<1,2>,<0,2>,<0,1>)$. It can be observed in Figure 8 that these two paths are well estimated. However, the path $(<2,2>,<2,1>,<1,1>,<1,0>)$ is in $\Delta_{m,k}$ with $m=8$ and $k=5$. Then $5=k>2m-2r=4$ so the necessary and sufficient condition of Theorem 2 to have Bounded Relative Error is not verified. ■

Thus these results reinforce the importance of Bounded Relative Error property.

7. Conclusion

The aim of this paper was to show up the numerical problems encountered with rare event simulation, even when using importance sampling schemes, and to give some explanations to these phenomena. These problems are due to events which, although more frequent using an importance sampling scheme, are still rare and then cause bad estimations of $\sigma^2_{\Phi'}$ and γ. We have analyzed the dependences between the important estimations and the important properties which are bounded relative error and bounded normal approximation. These dependences reinforce the role of these two last properties, principally the Bounded Normal Approximation one. Indeed, the Bounded Normal Approximation property implies that $\sigma^2_{\Phi'}$, γ and the confidence interval are well estimated, whereas good $\sigma^2_{\Phi'}$ and γ estimations (surprisingly) do not imply Bounded Normal Approximation, so they do not imply a controlled confidence interval coverage. Note finally that Balanced failure biasing scheme verifies Bounded Normal Approximation [14]. As future work, we plan to develop the Bounded Normal Approximation theory to other rare event problems.

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