Beyond Petrov-Galerkin projection by using « multi-space » priors

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Joint work with M. Diallo & P. Héas

The target problem...

$$\text{Find } \boldsymbol{h}^\star \in \mathcal{H} \text{ such that } \; a(\boldsymbol{h}^\star, \boldsymbol{h}) = b(\boldsymbol{h}) \quad \forall \boldsymbol{h} \in \mathcal{H}$$

where $\mathcal{H}$ is a Hilbert space ($\langle \cdot, \cdot \rangle$ and $\|\cdot\|$)

$a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a bilinear operator

$b : \mathcal{H} \to \mathbb{R}$ is a linear operator

… and its Petrov-Galerkin approximation

$$\text{Find } \hat{\boldsymbol{h}}_{\mathrm{PG}} \in V_n \text{ such that } \; a(\hat{\boldsymbol{h}}_{\mathrm{PG}}, \boldsymbol{h}) = b(\boldsymbol{h}) \quad \forall \boldsymbol{h} \in Z_n$$

where $V_n \subset \mathcal{H}$, $Z_n \subset \mathcal{H}$ are n-dimensional subspaces

The precision of Petrov-Galerkin can be quantified by an « instance optimal property »

$$\left\|\boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\mathrm{PG}}\right\| \leq C(V_n, Z_n)\,\mathrm{dist}(\boldsymbol{h}^\star, V_n),$$

Standard methods constructing V_n often return a set of subspaces and their « widths »

Standard outputs of methods constructing V_n:

$$V_0 \subset V_1 \subset \ldots \subset V_n, \qquad \dim(V_k) = k$$

such that

$$\operatorname{dist}(\boldsymbol{h}^\star, V_k) \leq \hat{\epsilon}_k, \qquad k = 0 \ldots n.$$

E.g., « reduced basis » methods

The Petrov-Galerkin projection discards most of the available information

Standard outputs of methods constructing V_n:

$$\text{✗} \subset \text{✗} \subset \ldots \subset V_n, \qquad \qquad \dim(V_k) = k$$

such that

$$\operatorname{dist}(\boldsymbol{h}^\star, \text{✗}) \leq \hat{\epsilon}_k, \qquad \qquad k = 0 \ldots n.$$

E.g., « reduced basis » methods

Can we use this information to improve the projection process?

The Petrov-Galerkin projection can be reformulated as a variational problem

$$\hat{\boldsymbol{h}}_{\mathrm{PG}} = \operatorname*{arg\,min}_{\boldsymbol{h} \in V_n} \sum_{j=1}^{n} \left(b_j - \langle \boldsymbol{a}_j, \boldsymbol{h} \rangle\right)^2$$

where $\operatorname{span}\left(\{\boldsymbol{z}_j\}_{j=1}^{n}\right) = Z_n$

$\boldsymbol{a}_j$ is the Riesz's representer of $a(\cdot, \boldsymbol{z}_j)$

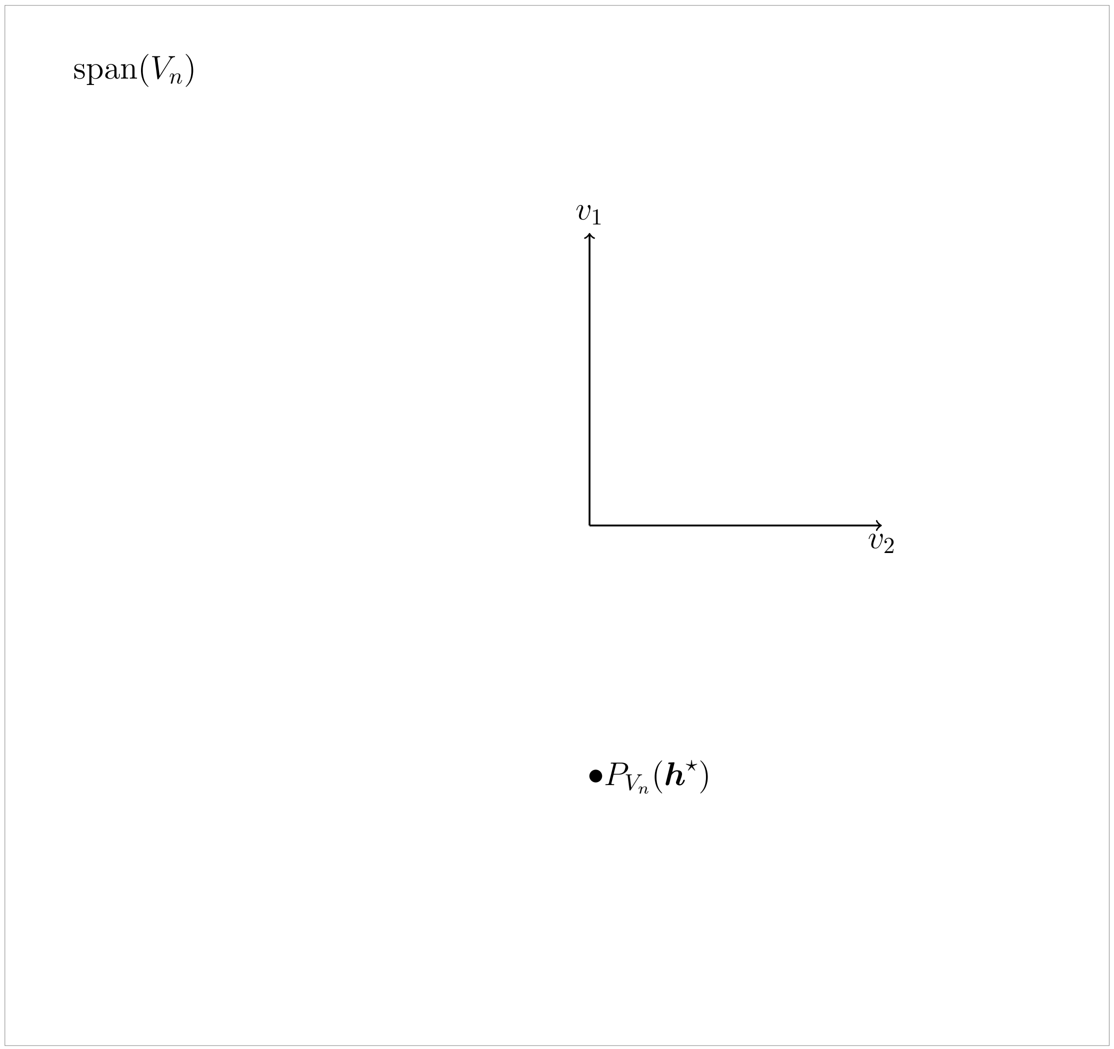
$b_j = b(\boldsymbol{z}_j)$

The proposed « multi-space » decoder adds new constraints to the variational problem

$$\hat{\boldsymbol{h}}_{\mathrm{MS}} = \arg\min_{\boldsymbol{h}\in V_n} \sum_{j=1}^{n} (b_j - \langle \boldsymbol{a}_j, \boldsymbol{h}\rangle)^2,$$
$$\text{subject to } \operatorname{dist}(\boldsymbol{h}, V_k) \leq \hat{\epsilon}_k, \quad k = 0 \ldots n-1.$$

See *[Binev et al., SIAM JUQ 17]* for a related work.

A graphical representation of the problem



$n = 2$

$V_k = \text{span}\left(\{\boldsymbol{v}_i\}_{i=1}^{k}\right)$

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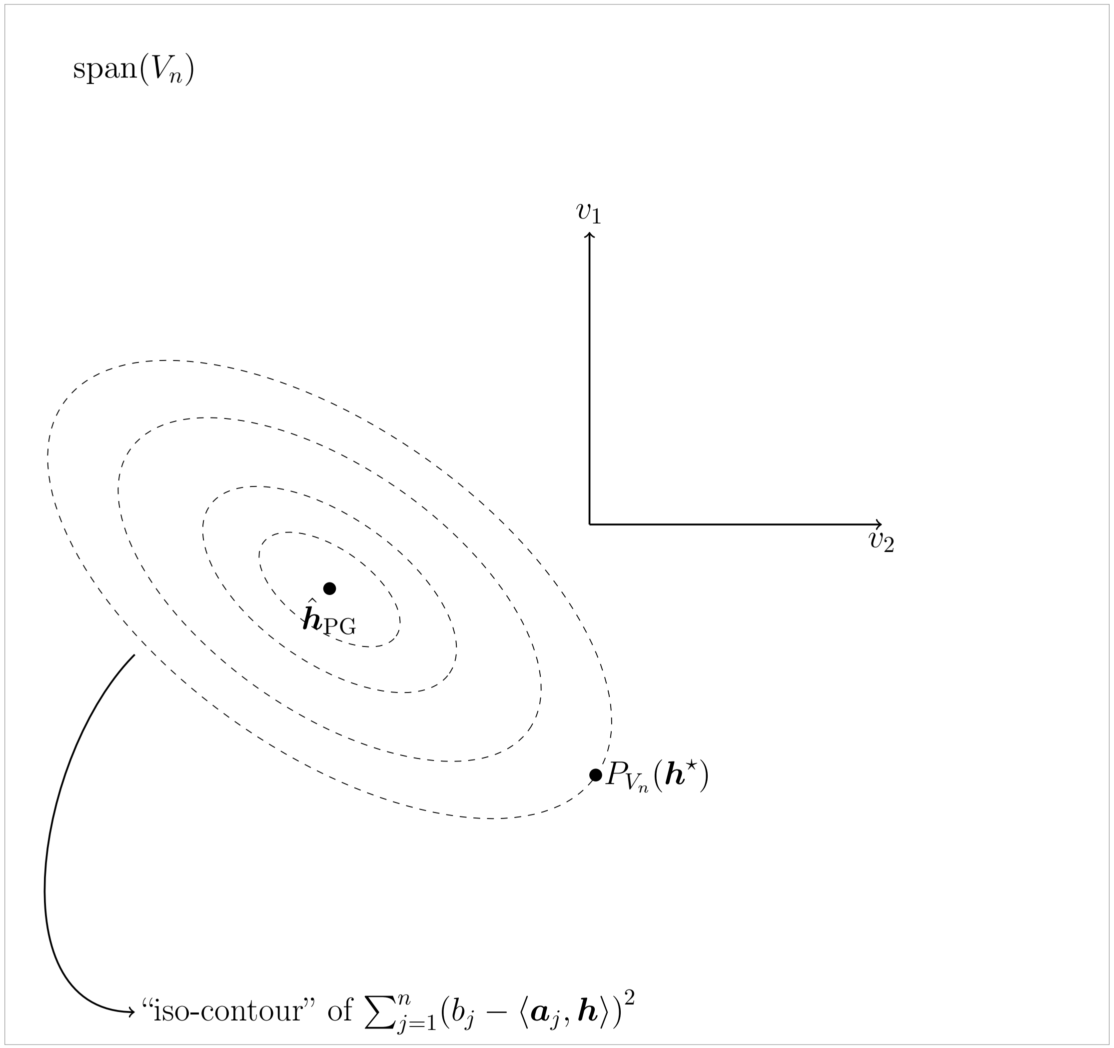

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The shape of the iso-contours depends on $\mathbf{G} = [\langle \boldsymbol{a}_i, \boldsymbol{v}_j \rangle]_{ij}$

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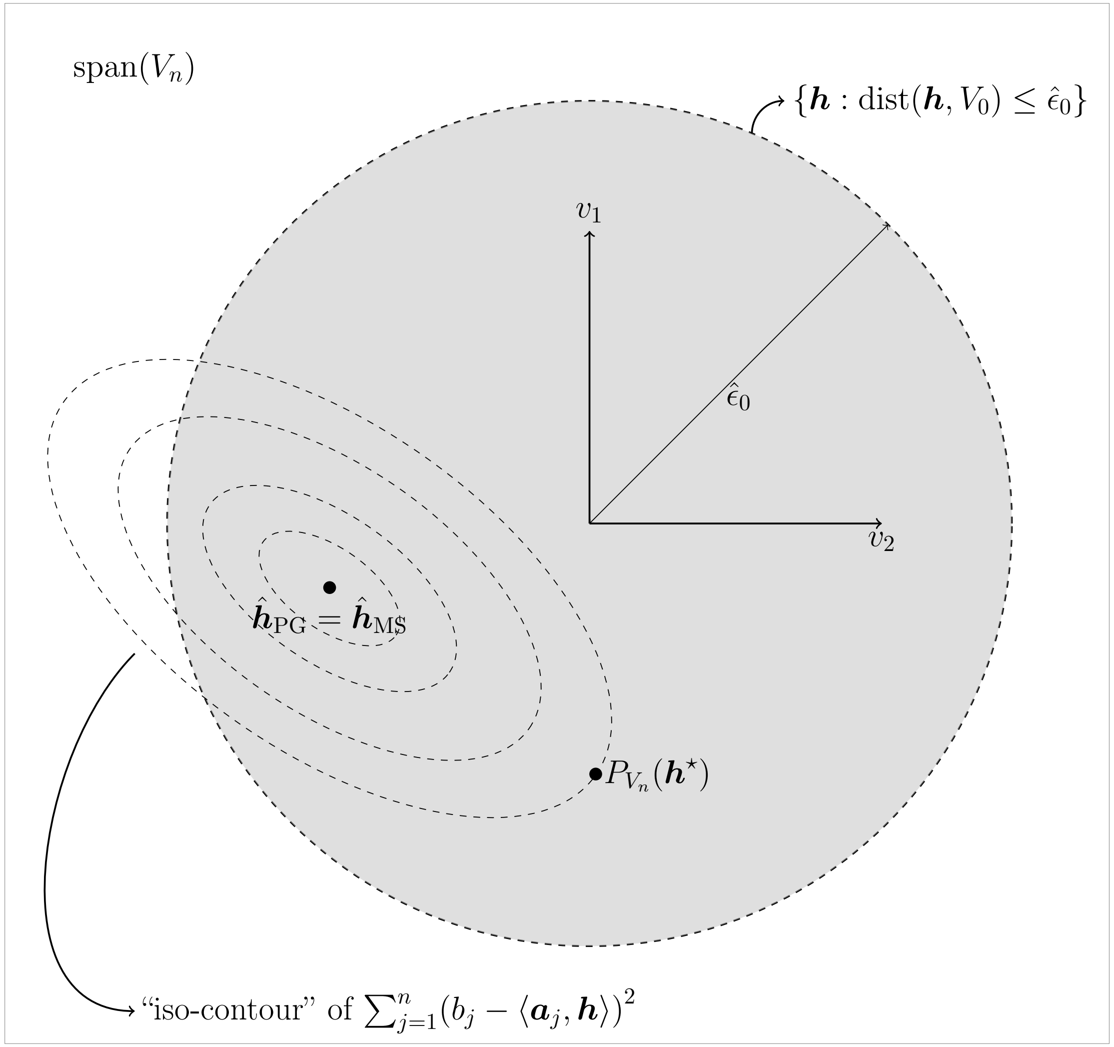

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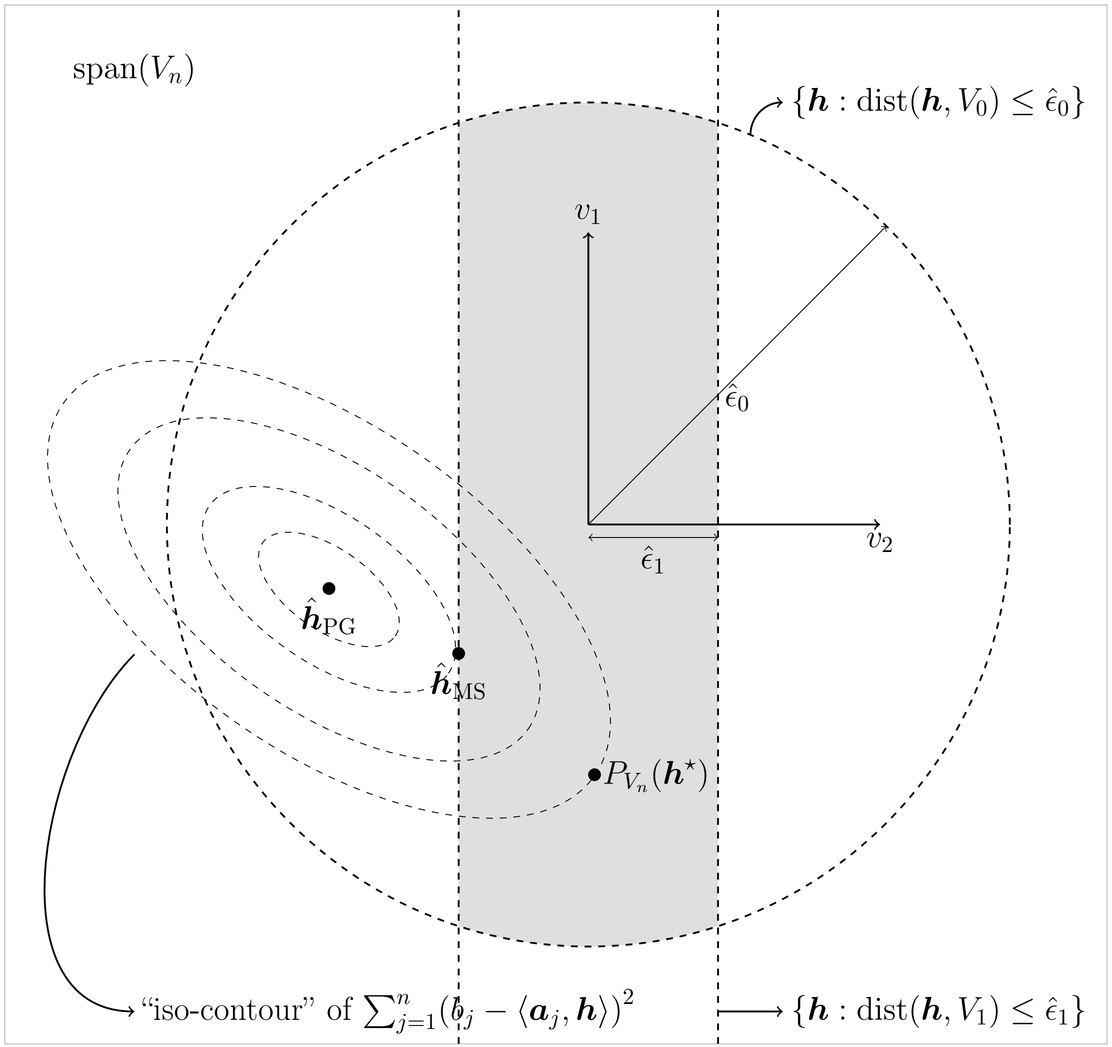


$n = 2$

$V_k = \mathrm{span}\left(\{\boldsymbol{v}_i\}_{i=1}^{k}\right)$

The shape of the iso-contours depends on $\mathbf{G} = [\langle \boldsymbol{a}_i, \boldsymbol{v}_j \rangle]_{ij}$

The feasibility region depends on $\{V_k\}_{k=1}^{n}$ and $\{\hat{\epsilon}_k\}_{k=1}^{n}$

Can we give some guarantee on the performance of the « multi-space » decoder?

Instance optimality properties

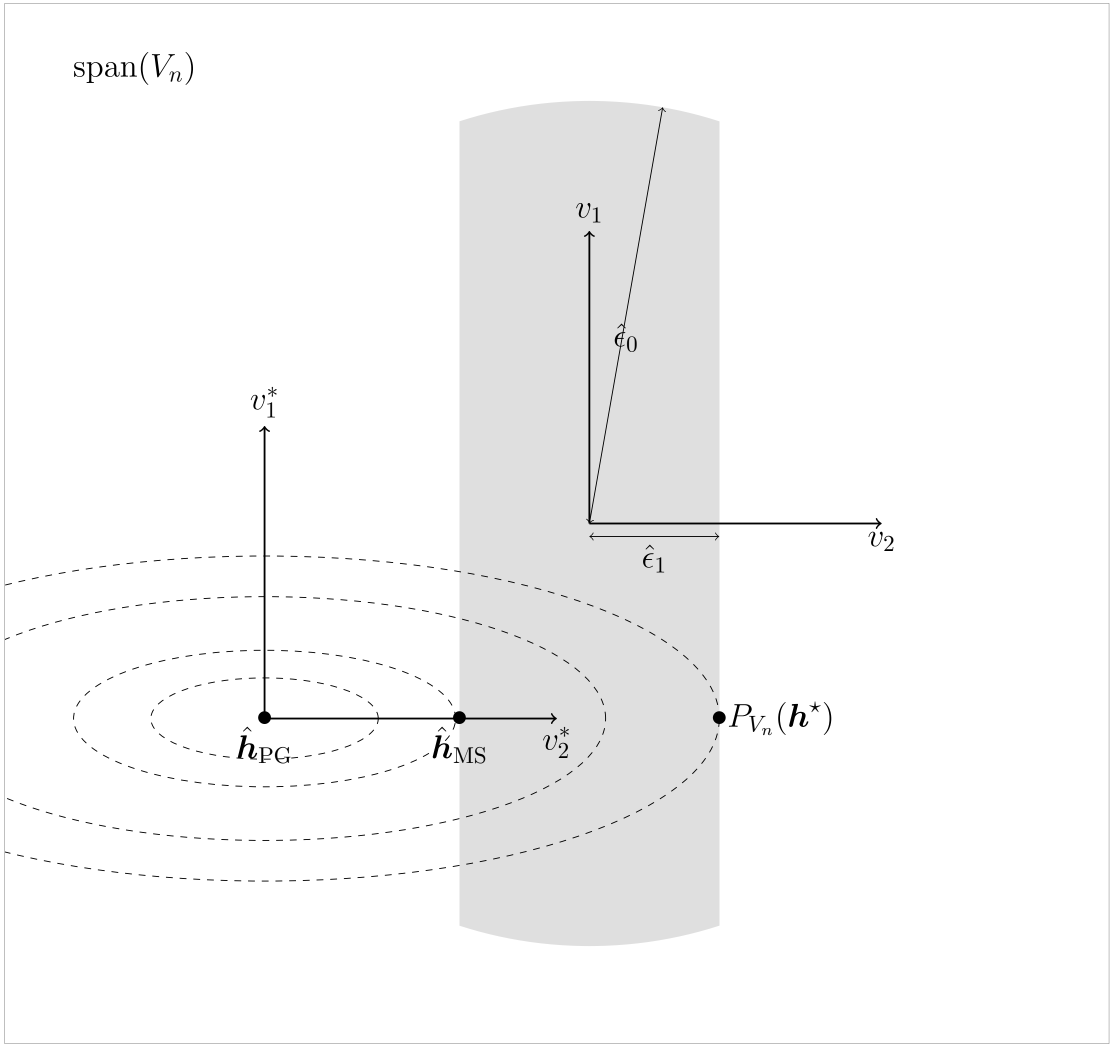
- *Petrov-Galerkin*: $\left\| \boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\mathrm{PG}} \right\| \leq C(\mathbf{G}) \, \mathrm{dist}(\boldsymbol{h}^\star, V_n)$,

- « *Multi-space* » *decoder*:

$$\left\| \boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\mathrm{MS}} \right\| \leq \left(\sum_{j=\ell+1}^{n} \delta_j^2 + \rho \, \delta_\ell^2 + (\mathrm{dist}(\boldsymbol{h}^\star, V_n))^2 \right)^{\frac{1}{2}}$$

where ℓ and δ_j's are "easily-computable" quantities only depending on $\mathbf{G} = [\langle \boldsymbol{a}_i, \boldsymbol{v}_j \rangle]_{i,j}$, $\{\hat{\epsilon}_k\}_{k=1}^{n-1}$ and $\{\mathrm{dist}(\boldsymbol{h}^\star, V_k)\}_{k=1}^{n}$

Particularization of the instance optimality bound to some examples



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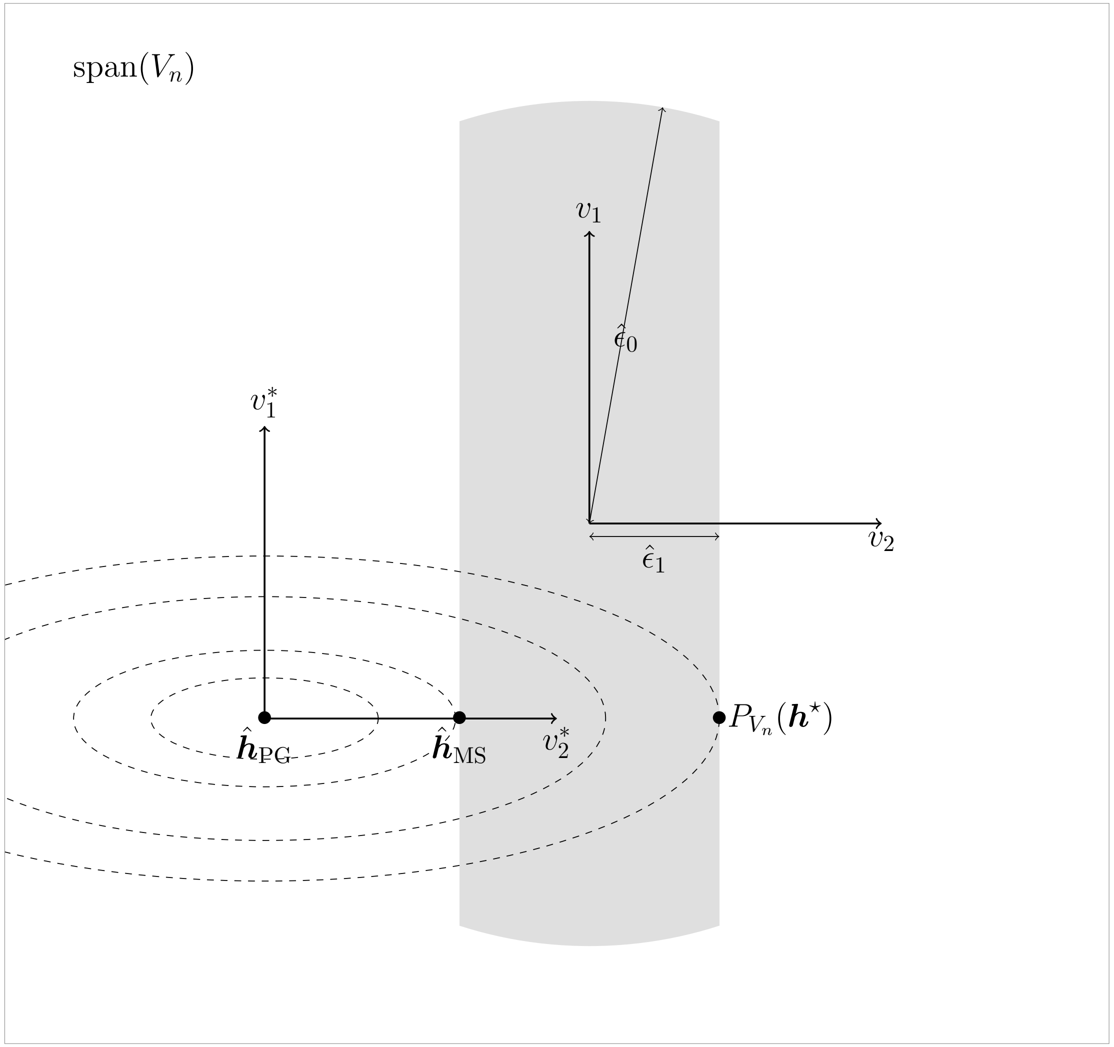


$$\hat{\epsilon}_j = \begin{cases} 1 & j = 0 \dots n-3, \\ \epsilon^{\frac{1}{2}} & j = n-2, n-1, \\ \epsilon & j = n, \end{cases}$$

$$\text{dist}(\boldsymbol{h}^\star, V_k) = \hat{\epsilon}_j$$

$$\sigma_j = \text{sing. values of } \mathbf{G}$$

$$= \begin{cases} 1 & j = 1 \dots n-3, \\ \epsilon^{\frac{1}{2}} & j = n-2, n-1, \\ \epsilon & j = n. \end{cases}$$

$$\left\| \boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\text{PG}} \right\| \leq 1$$

$$\left\| \boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\text{MS}} \right\| \leq 3\epsilon^{\frac{1}{2}}$$

Quid of the computational complexity?

PG projection can been carried out efficiently with a complexity $\mathcal{O}(n^2)$ per iteration

$$\hat{\boldsymbol{h}}_{\mathrm{PG}} = \arg\min_{\boldsymbol{h}\in V_n} \sum_{j=1}^{n} (b_j - \langle \boldsymbol{a}_j, \boldsymbol{h} \rangle)^2$$

« Least square » problem: can be solved efficiently via gradient-based methods with a complexity $\mathcal{O}(n^2)$ per iteration.

Our decoder can also be implemented with a complexity $\mathcal{O}(n^2)$ per iteration

Our problem can be rewritten as:

$$\hat{\boldsymbol{h}}_{\mathrm{MS}} = \underset{\boldsymbol{h}\in V_n}{\arg\min} \sum_{j=1}^{n} (b_j - \langle \boldsymbol{a}_j, \boldsymbol{h} \rangle)^2,$$

$$\text{subject to } \left\| P^{\perp}_{V_k}(\boldsymbol{h}) \right\| \leq \hat{\epsilon}_k, \quad k = 0 \dots n-1.$$

We use the « Alternating Directions Method of Multipliers » to solve this convex problem with a complexity $\mathcal{O}(n^2)$ per iteration

Some results

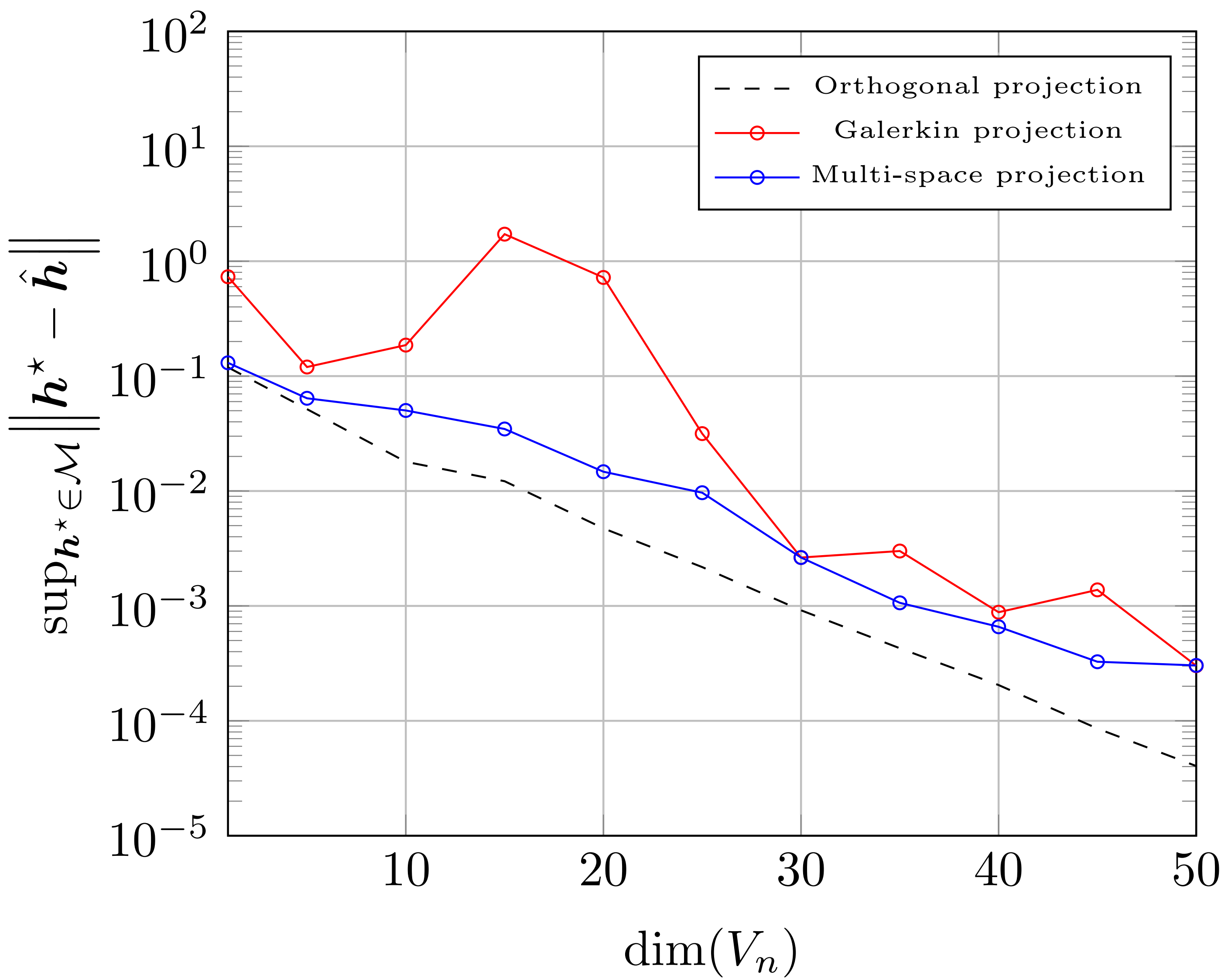
We consider a parametric mass transfert problem

$$\begin{aligned} \mu_1 \triangle \boldsymbol{h} + \mathbf{b}(\mu_2) \cdot \nabla \boldsymbol{h} &= \boldsymbol{s} \quad && \text{in } \Omega \\ \mu_1 \nabla \boldsymbol{h} \cdot \mathbf{n} &= 0 \quad && \text{in } \partial\Omega \end{aligned}$$

where

$$\begin{aligned} \mathbf{b}(\mu_2) &= [\cos(\mu_2)\,\sin(\mu_2)]^{\mathrm{T}} \\ \boldsymbol{s} &= \exp\left(-\frac{\|\mathbf{x}-\mathbf{m}\|_2^2}{2\sigma^2}\right) \\ \mu_1 &= [0.03, 0.05] \\ \mu_2 &= [0, 2\pi] \end{aligned}$$

The multi-space decoder improves the worst-case approximation error over PG



Our contributions

- We exploit additional information in the Petrov-Galerkin projection
- We derive an instance optimal guarantee for our « multi-space » decoder
- We propose an efficient implementation for the « multi-space » decoder

Thank you for your attention.