Beyond Petrov-Galerkin projection by using « multi-space » priors

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Joint work with M. Diallo & P. Héas

The target problem...

Find $\boldsymbol{h}^\star \in \mathcal{H}$ such that $a(\boldsymbol{h}^\star, \boldsymbol{h}) = b(\boldsymbol{h}) \quad \forall \boldsymbol{h} \in \mathcal{H}$

where $\mathcal{H}$ is a Hilbert space ($\langle \cdot, \cdot \rangle$ and $\|\cdot\|$)

$a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a bilinear operator

$b : \mathcal{H} \to \mathbb{R}$ is a linear operator

… and its Petrov-Galerkin approximation

$$\text{Find } \hat{\boldsymbol{h}}_{\mathrm{PG}} \in V_n \text{ such that } \; a(\hat{\boldsymbol{h}}_{\mathrm{PG}}, \boldsymbol{h}) = b(\boldsymbol{h}) \quad \forall \boldsymbol{h} \in Z_n$$

where $V_n \subset \mathcal{H}$, $Z_n \subset \mathcal{H}$ are n-dimensional subspaces

The precision of Petrov-Galerkin can be quantified by an « instance optimal property »

$$\left\|\boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\mathrm{PG}}\right\| \leq C(V_n, Z_n)\,\mathrm{dist}(\boldsymbol{h}^\star, V_n),$$

Standard methods constructing V_n often return a set of subspaces and their « widths »

Standard outputs of methods constructing V_n:

$$V_0 \subset V_1 \subset \ldots \subset V_n, \qquad \dim(V_k) = k$$

such that

$$\operatorname{dist}(\boldsymbol{h}^\star, V_k) \leq \hat{\epsilon}_k, \qquad k = 0 \ldots n.$$

E.g., « reduced basis » methods

The Petrov-Galerkin projection discards most of the available information

Standard outputs of methods constructing V_n:

$$\text{✗} \subset \text{✗} \subset \ldots \subset V_n, \qquad \dim(V_k) = k$$

such that

$$\operatorname{dist}(\boldsymbol{h}^\star, \text{✗}) \le \hat{\epsilon}_k, \qquad k = 0 \ldots n.$$

E.g., « reduced basis » methods

Can we use this information to improve the projection process?

The Petrov-Galerkin projection can be reformulated as a variational problem

$$\hat{\boldsymbol{h}}_{\mathrm{PG}} = \underset{\boldsymbol{h} \in V_n}{\arg\min} \sum_{j=1}^{n} \left(b_j - \langle \boldsymbol{a}_j, \boldsymbol{h} \rangle\right)^2$$

where

$$\mathrm{span}\left(\{\boldsymbol{z}_j\}_{j=1}^{n}\right) = Z_n$$

$\boldsymbol{a}_j$ is the Riesz's representer of $a(\cdot, \boldsymbol{z}_j)$

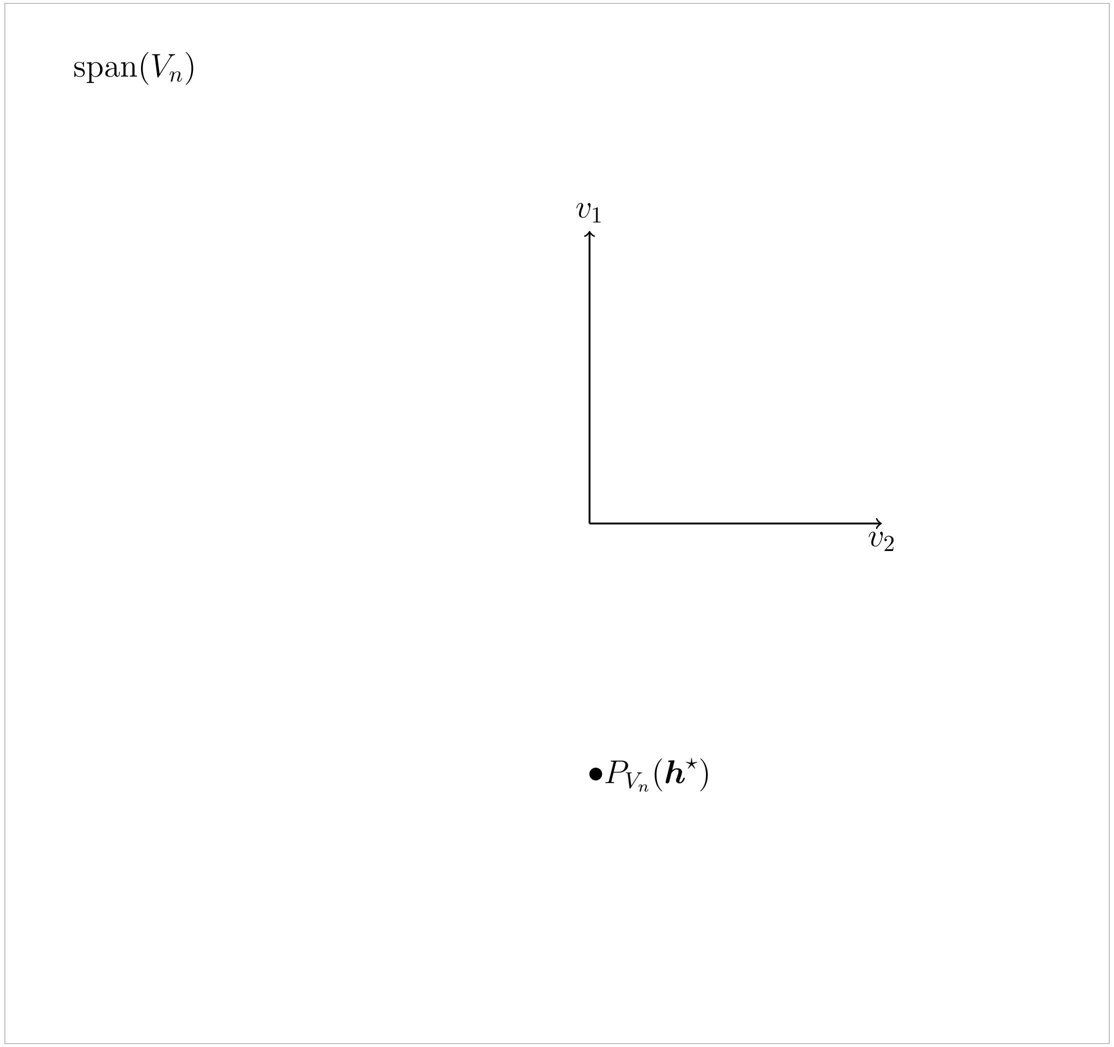
$b_j = b(\boldsymbol{z}_j)$

The proposed « multi-space » decoder adds new constraints to the variational problem

$$\hat{\boldsymbol{h}}_{\mathrm{MS}} = \operatorname*{arg\,min}_{\boldsymbol{h}\in V_n} \sum_{j=1}^{n} \left(b_j - \langle \boldsymbol{a}_j, \boldsymbol{h}\rangle\right)^2, \quad \text{subject to } \operatorname{dist}(\boldsymbol{h}, V_k) \leq \hat{\epsilon}_k, \quad k = 0 \ldots n-1.$$

See *[Binev et al., SIAM JUQ 17]* for a related work.

A graphical representation of the problem



$n = 2$

$V_k = \mathrm{span}\left(\{\boldsymbol{v}_i\}_{i=1}^{k}\right)$

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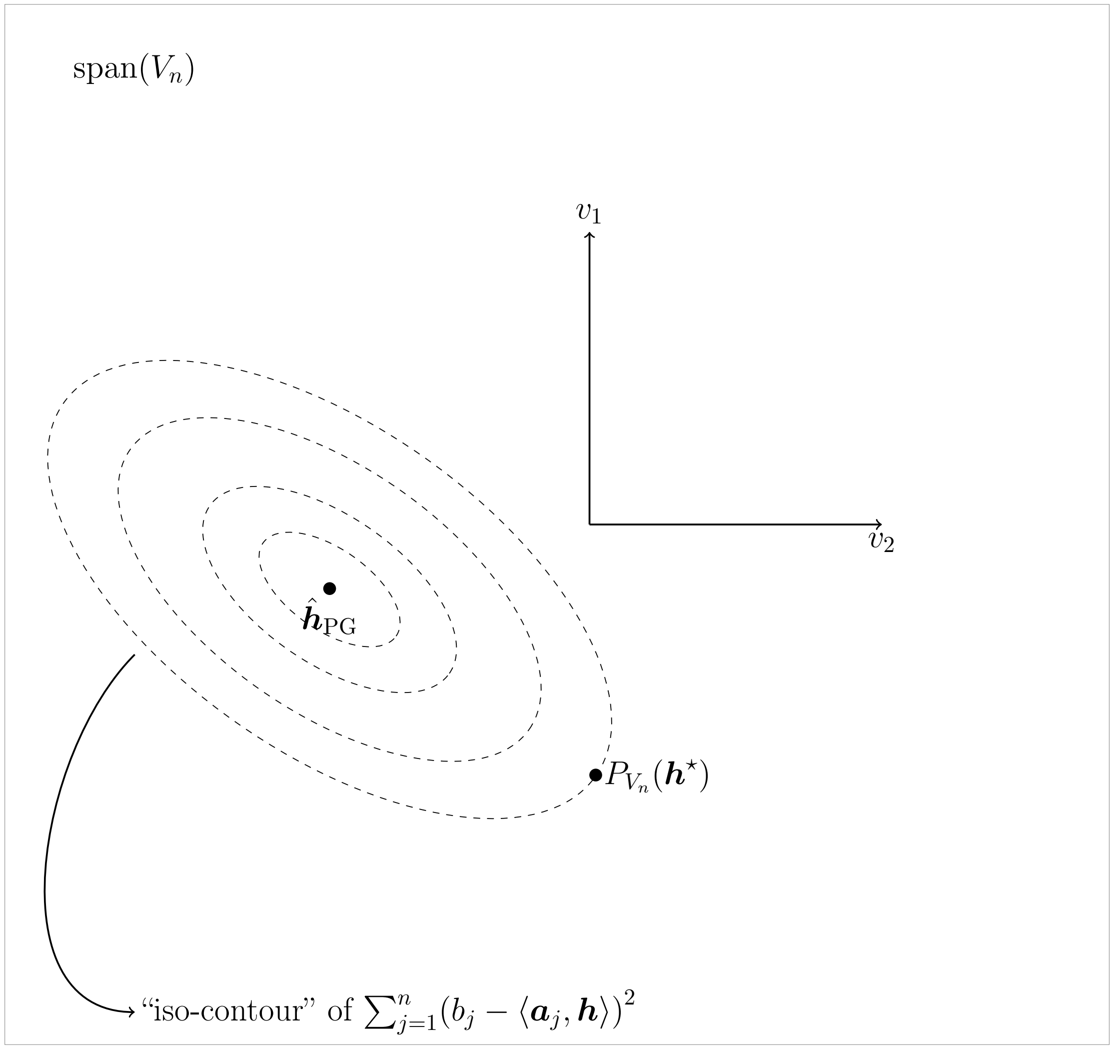

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The shape of the iso-contours depends on $\mathbf{G} = [\langle \boldsymbol{a}_i, \boldsymbol{v}_j \rangle]_{ij}$

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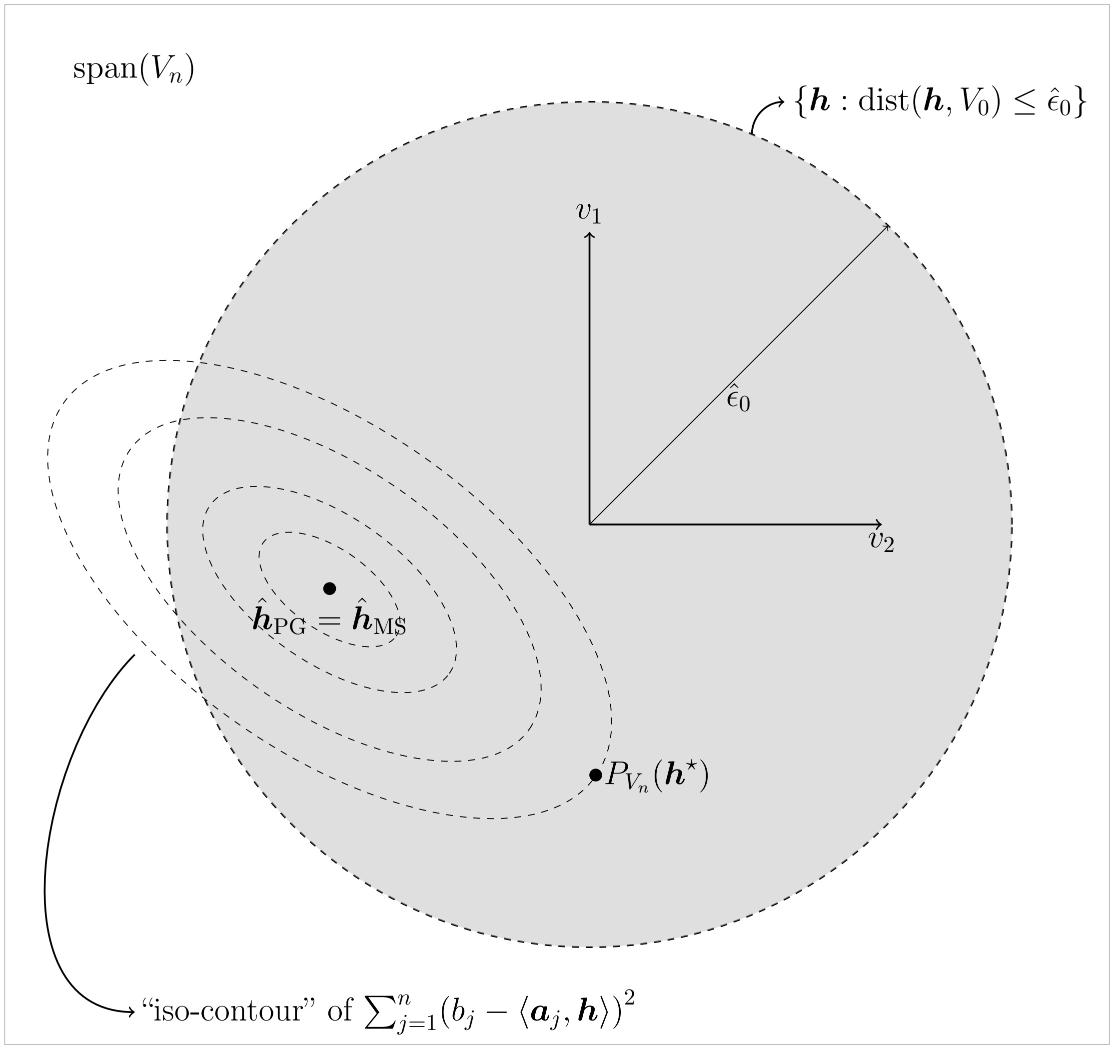

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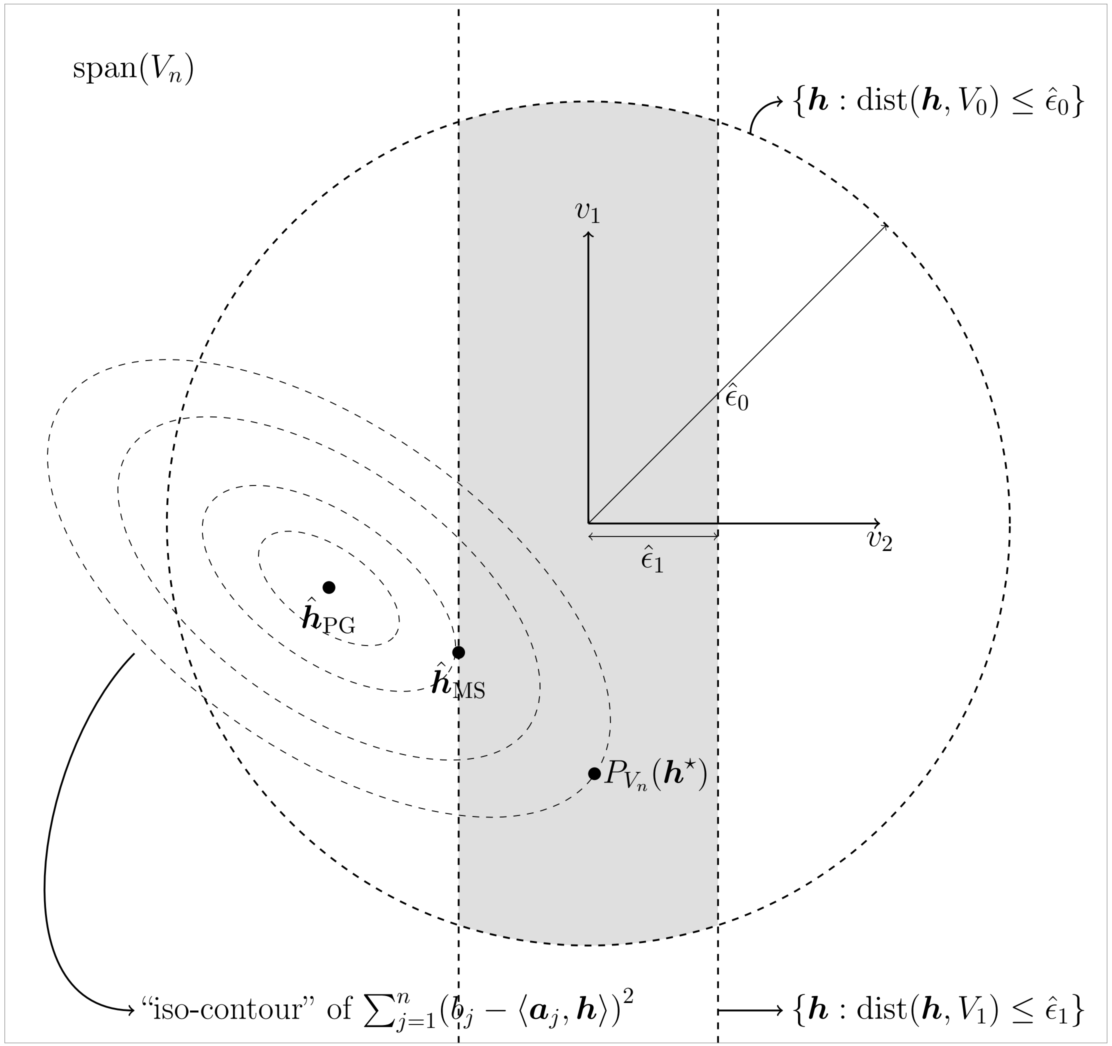


$n = 2$

$V_k = \mathrm{span}\left(\{\boldsymbol{v}_i\}_{i=1}^{k}\right)$

The shape of the iso-contours depends on $\mathbf{G} = [\langle \boldsymbol{a}_i, \boldsymbol{v}_j \rangle]_{ij}$

The feasibility region depends on $\{V_k\}_{k=1}^{n}$ and $\{\hat{\epsilon}_k\}_{k=1}^{n}$

Can we give some guarantee on the performance of the « multi-space » decoder?

Instance optimality properties

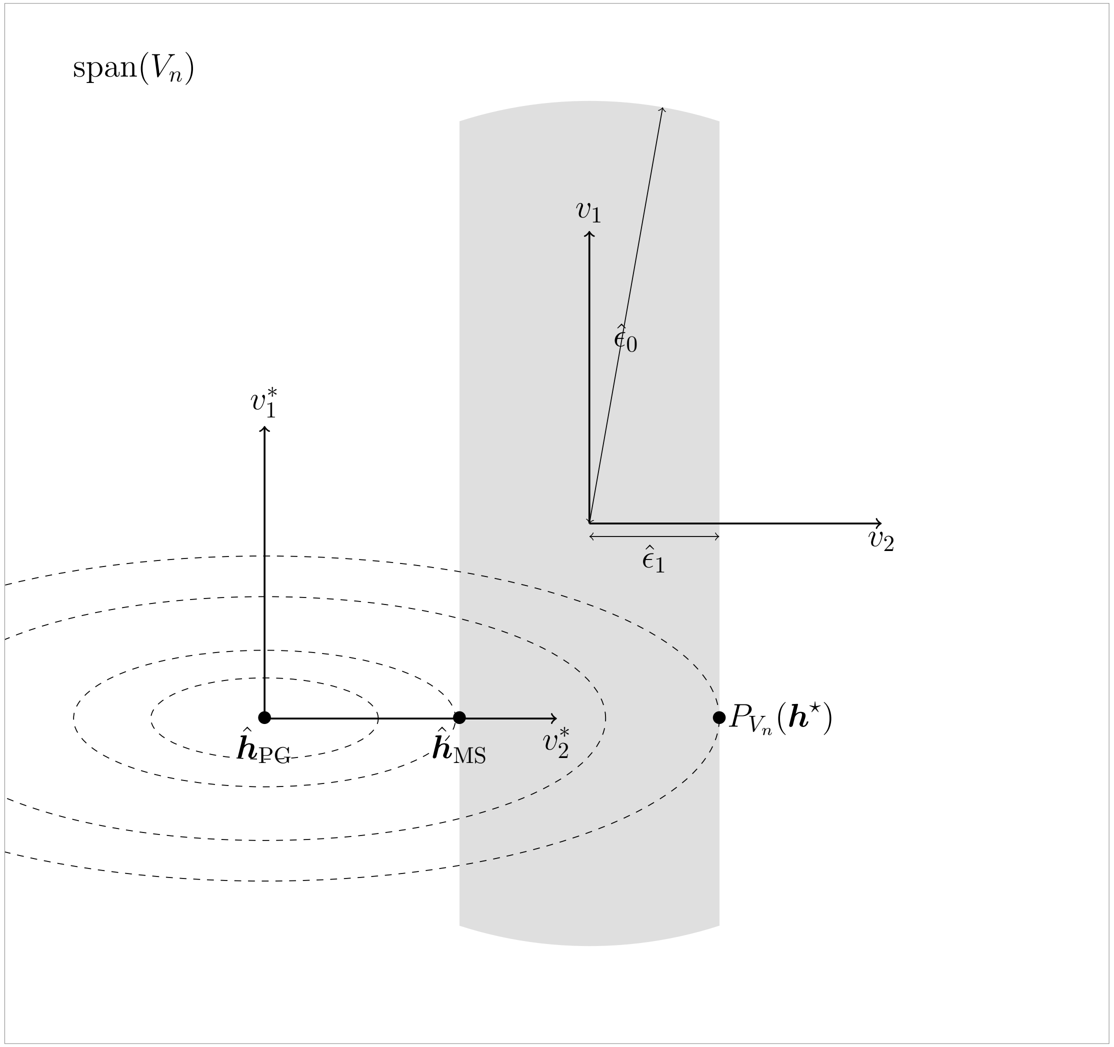
- *Petrov-Galerkin*: $\left\|\boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\mathrm{PG}}\right\| \leq C(\mathbf{G})\,\mathrm{dist}(\boldsymbol{h}^\star, V_n)$,

- *« Multi-space » decoder*:

$$\left\|\boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\mathrm{MS}}\right\| \leq \left(\sum_{j=\ell+1}^{n} \delta_j^2 + \rho\,\delta_\ell^2 + \left(\mathrm{dist}(\boldsymbol{h}^\star, V_n)\right)^2\right)^{\frac{1}{2}}$$

where ℓ and δ_j's are "easily-computable" quantities only depending on $\mathbf{G} = [\langle \boldsymbol{a}_i, \boldsymbol{v}_j\rangle]_{i,j}$, $\{\hat{\epsilon}_k\}_{k=1}^{n-1}$ and $\{\mathrm{dist}(\boldsymbol{h}^\star, V_k)\}_{k=1}^{n}$

Particularization of the instance optimality bound to some examples



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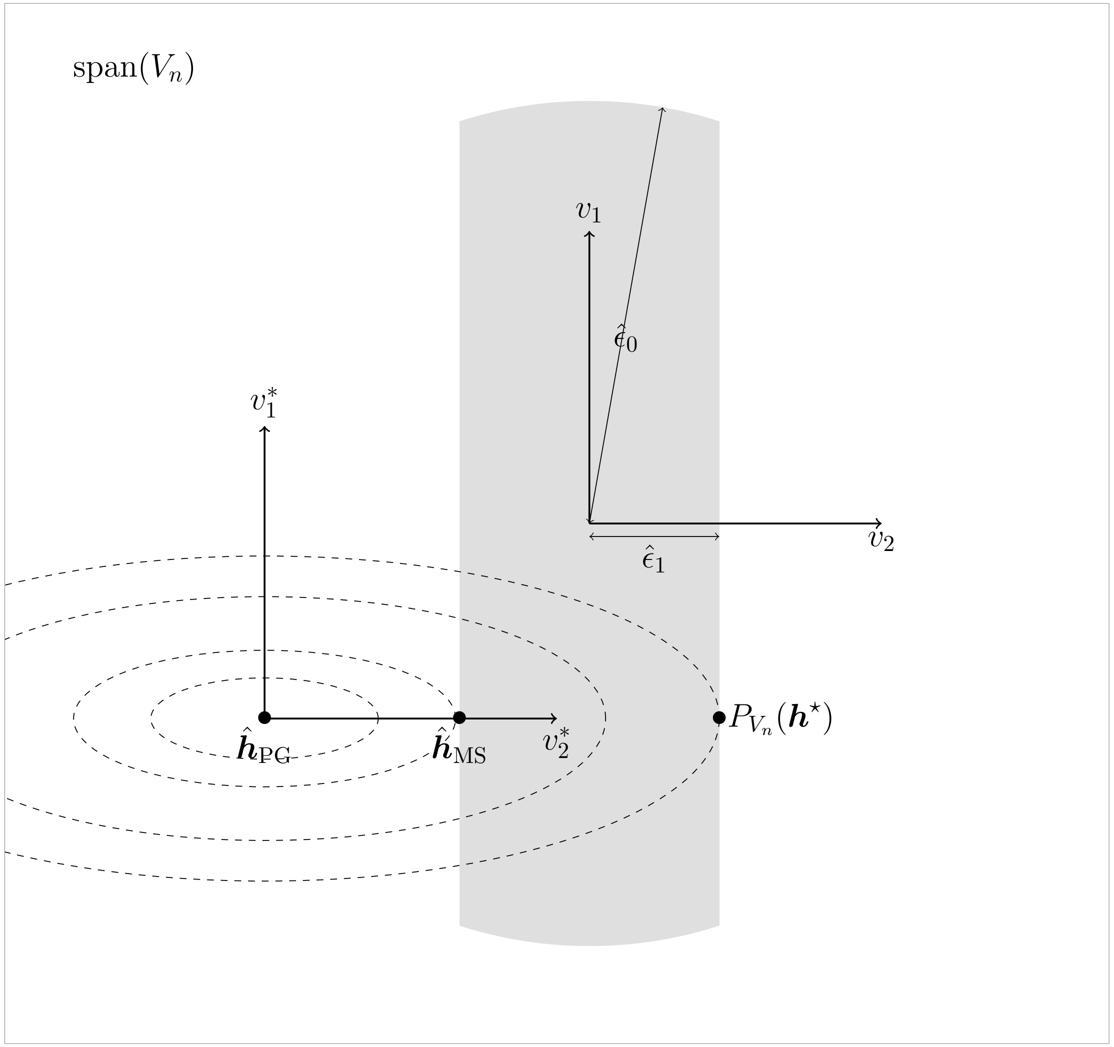


$$\hat\epsilon_j = \begin{cases} 1 & j = 0 \dots n-3, \\ \epsilon^{\frac{1}{2}} & j = n-2, n-1, \\ \epsilon & j = n, \end{cases}$$

$$\mathrm{dist}(\boldsymbol{h}^\star, V_k) = \hat\epsilon_j$$

$$\sigma_j = \text{sing. values of } \mathbf{G}$$
$$= \begin{cases} 1 & j = 1 \dots n-3, \\ \epsilon^{\frac{1}{2}} & j = n-2, n-1, \\ \epsilon & j = n. \end{cases}$$

$$\left\| \boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\mathrm{PG}} \right\| \leq 1$$
$$\left\| \boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\mathrm{MS}} \right\| \leq 3\epsilon^{\frac{1}{2}}$$

Quid of the computational complexity?

PG projection can been carried out efficiently with a complexity $\mathcal{O}(n^2)$ per iteration

$$\hat{\boldsymbol{h}}_{\mathrm{PG}} = \underset{\boldsymbol{h}\in V_n}{\arg\min} \sum_{j=1}^{n} \left(b_j - \langle \boldsymbol{a}_j, \boldsymbol{h} \rangle\right)^2$$

« Least square » problem: can be solved efficiently via gradient-based methods with a complexity $\mathcal{O}(n^2)$ per iteration.

Our decoder can also be implemented with a complexity $\mathcal{O}(n^2)$ per iteration

Our problem can be rewritten as:

$$\hat{\boldsymbol{h}}_{\mathrm{MS}} = \underset{\boldsymbol{h} \in V_n}{\arg\min} \sum_{j=1}^{n} (b_j - \langle \boldsymbol{a}_j, \boldsymbol{h} \rangle)^2,$$

$$\text{subject to } \left\| P_{V_k}^{\perp}(\boldsymbol{h}) \right\| \leq \hat{\epsilon}_k, \quad k = 0 \ldots n-1.$$

We use the « Alternating Directions Method of Multipliers » to solve this convex problem with a complexity $\mathcal{O}(n^2)$ per iteration

Some results

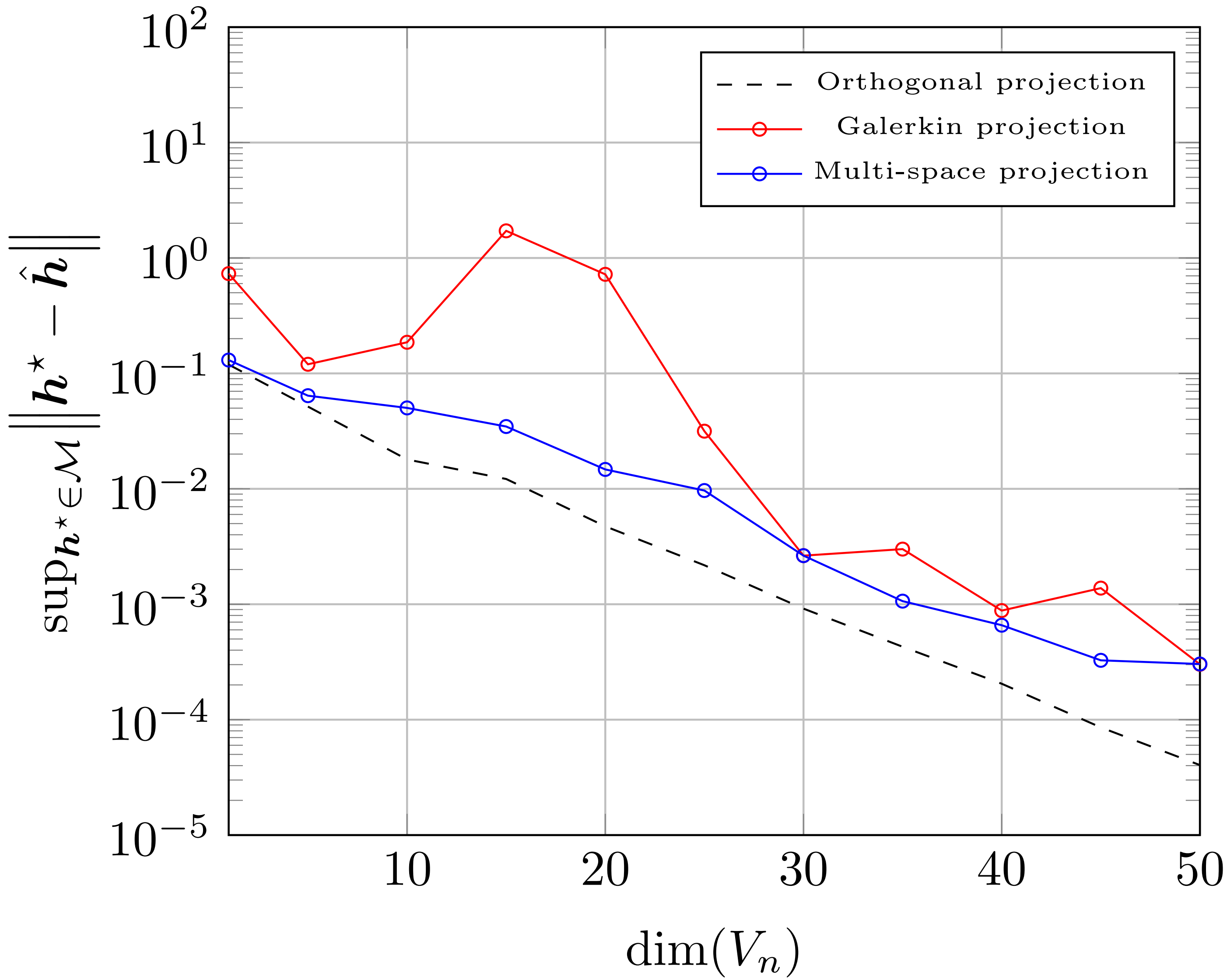
We consider a parametric mass transfert problem

$$\begin{aligned} \mu_1 \triangle \boldsymbol{h} + \mathbf{b}(\mu_2) \cdot \nabla \boldsymbol{h} &= \boldsymbol{s} \quad && \text{in } \Omega \\ \mu_1 \nabla \boldsymbol{h} \cdot \mathbf{n} &= 0 \quad && \text{in } \partial\Omega \end{aligned}$$

where

$$\begin{aligned} \mathbf{b}(\mu_2) &= [\cos(\mu_2)\,\sin(\mu_2)]^{\mathrm{T}} \\ \boldsymbol{s} &= \exp\left(-\frac{\|\mathbf{x} - \mathbf{m}\|_2^2}{2\sigma^2}\right) \\ \mu_1 &= [0.03, 0.05] \\ \mu_2 &= [0, 2\pi] \end{aligned}$$

The multi-space decoder improves the worst-case approximation error over PG



Our contributions

- We exploit additional information in the Petrov-Galerkin projection
- We derive an instance optimal guarantee for our « multi-space » decoder
- We propose an efficient implementation for the « multi-space » decoder

Thank you for your attention.

Appendix

Main Theorem (IOP for multi-space decoder)

Theorem 1. *The solution of the "multi-space" decoder satisfies:*

$$\left\|\boldsymbol{h}^\star - \hat{\boldsymbol{h}}_{\mathrm{MS}}\right\| \leq \left(\sum_{j=\ell+1}^{n} \delta_j^2 + \rho\, \delta_\ell^2 + (\mathrm{dist}(\boldsymbol{h}^\star, V_n))^2 \right)^{\frac{1}{2}},$$

where ℓ is the largest integer such that

$$\sum_{j=\ell}^{n} \sigma_j^2 \delta_j^2 \geq 4\gamma^2 (\mathrm{dist}(\boldsymbol{h}^\star, V_n))^2,$$

and $\rho \in [0,1]$ is defined as

$$\rho \sigma_\ell^2 \delta_\ell^2 + \sum_{j=\ell+1}^{n} \sigma_j^2 \delta_j^2 = 4\gamma^2 (\mathrm{dist}(\boldsymbol{h}^\star, V_n))^2.$$

$$\gamma = \sup_{\boldsymbol{h} \in V^\perp, \|\boldsymbol{h}\|=1} \left(\sum_{j=1}^{n} \langle \boldsymbol{a}_j, \boldsymbol{h} \rangle^2 \right)^{\frac{1}{2}},$$

$\mathbf{G} = [\langle \boldsymbol{a}_i, \boldsymbol{v}_j \rangle]_{i,j}$

$\mathbf{X}$ = right singular vectors of $\mathbf{G}$,

σ_k = kth singular value of $\mathbf{G}$,

$$\delta_j = \sum_{k=1}^{n} |x_{kj}| (\hat{\epsilon}_{k-1} + \mathrm{dist}(\boldsymbol{h}^\star, V_k)).$$

ADMM (first step): problem reformulation

$$\hat{\boldsymbol{h}}_{\mathrm{MS}} = \underset{\boldsymbol{h}_n \in V_n}{\arg\min} \sum_{j=1}^{n} (b_j - \langle \boldsymbol{a}_j, \boldsymbol{h}_n \rangle)^2,$$

$$\text{subject to } \begin{cases} \left\| P^{\perp}_{V_k}(\boldsymbol{h}_k) \right\| \leq \hat{\epsilon}_k, \quad k = 0 \ldots n-1, \\ \boldsymbol{h}_0 = \boldsymbol{h}_1 = \ldots = \boldsymbol{h}_n. \end{cases}$$

We add n new variables and n new constraints.

ADMM (second step): main recursions

$$\begin{aligned}
\boldsymbol{h}_n^{(l+1)} &= \underset{\boldsymbol{h}_n \in V_n}{\arg\min} \sum_{j=1}^{n} (b_j - \langle \boldsymbol{a}_j, \boldsymbol{h}_n \rangle)^2 + \frac{\rho}{2} \sum_{k=1}^{n-1} \left\| \boldsymbol{h}_n - \boldsymbol{h}_k^{(l)} + \boldsymbol{u}_k^{(l)} \right\|^2, \\
\boldsymbol{h}_k^{(l+1)} &= \underset{\boldsymbol{h}_k}{\arg\min} \left\| \boldsymbol{h}_k - \boldsymbol{h}_n^{(l+1)} + \boldsymbol{u}_k^{(l)} \right\|^2 \text{subject to } \left\| P_{V_k}^{\perp}(\boldsymbol{h}_k) \right\| \leq \hat{\epsilon}_k, \\
\boldsymbol{u}_k^{(l+1)} &= \boldsymbol{u}_k^{(l)} + \boldsymbol{h}_n^{(l+1)} - \boldsymbol{h}_k^{(l+1)}
\end{aligned}$$

$\boldsymbol{h}_k^{(l)} \in V_n$ and $\boldsymbol{u}_k^{(l)} \in V_n \ \forall k, l$

They can thus be represented by n-dimensional vectors.