Model Reduction from Partial Observations

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Our contribution takes place in the context of model reduction for parametric partial differential equation:

$$\mathrm{PDE}(u, \mu) = 0, \tag{1}$$

where u belongs to an Hilbert space $\mathcal{H}$ (with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$) and $\mu \in \mathcal{P}$ is a parameter. When the solution $u(\mu)$ of (1) has to be evaluated for many different values $\mu \in \mathcal{P}$, the computational cost may become prohibitive. To circumvent this issue, model reduction intends to simplify the resolution of (1) by constraining u to belong to some low-dimensional subspace S. The choice of S should be made so that all the elements of the solution manifold $\mathcal{M} = \{u(\mu) \in \mathcal{H} : \mu \in \mathcal{P}\}$ are well-approximated by some element in S. Many techniques have been proposed in the literature to identify such subspaces: reduced basis [3], POD [1], etc. However, all these methods presuppose the knowledge of the solution manifold $\mathcal{M}$ (or the parameter set $\mathcal{P}$) although the latter may not always be available. On the other hand, since the advent of numerical acquisition, it has become quite common to have (incomplete) measurements of the elements of $\mathcal{M}$ at our disposal. In this work, we thus address the following question: can we build a good approximation subspace for $\mathcal{M}$ by exploiting these measurements?

More formally, the setup considered in our work is as follows. We assume that: *i)* for each $u \in \mathcal{M}$, we collect a set of observations $\{\langle w_i, u \rangle\}_{i=1}^m$, where $\{w_i\}_{i=1}^m$ is an orthogonal basis of some subspace W; *ii)* we have a "rough" prior knowledge of $\mathcal{M}$, that is we are given some $\hat{\Sigma}$ such that $\mathcal{M} \subseteq \hat{\Sigma}$. We assume that $\hat{\Sigma} = \{u : \mathrm{dist}(u, V) \leq \epsilon\}$ for some $\epsilon \geq 0$ and n-dimensional subspace V. We address the two following questions: *i)* how can we combine the observations $\{\langle w_i, u \rangle\}_{i=1}^m$ and the prior $\hat{\Sigma}$ to derive a "good" approximation subspace for $\mathcal{M}$? *ii)* can we derive guarantees on the quality of this approximation subspace?

The first question has a simple (theoretical) answer. Letting $\Sigma_{\mathrm{post}} \triangleq \cup_{u \in \mathcal{M}} (\hat{\Sigma} \cap H_u)$, with $H_u = \{u' : \langle w_i, u' \rangle = \langle w_i, u \rangle \text{ for } i = 1, \ldots, m\}$, it can be seen that $S^\star = \arg\min_{S:\dim(S)=j} \max_{u \in \Sigma_{\mathrm{post}}} \mathrm{dist}(u, S)$ is the best j-dimensional approximation subspace for $\mathcal{M}$ from a "worst-case" perspective. As for the second question, we provide an upper bound on the approximation quality achieved by $S^\star$ by elaborating on the recent results by Binev *et al.* [2]. More specifically, assuming that $\mathcal{M} \subseteq \{u : \mathrm{dist}(u, U) \leq \epsilon'\}$ where $0 \leq \epsilon' \leq \epsilon$ and $U \subseteq V$ is a k-dimensional subspace, we show that

$$\max_{u \in \mathcal{M}} \mathrm{dist}(u, S^\star) \leq C_j \, \epsilon', \tag{2}$$

for some constant $C_j \geq 1$. The value of C_j is related to the singular values of the projection operator from V to W. In particular, when V and W obey some simple non-degeneracy conditions, we have $C_j < \infty$ for $j \geq k$. This shows that, under a proper choice of V and W, one can essentially achieve the same reduction performance (up to some constant factor) as in the fully informed setup.

References

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