Low-Rank Dynamic Mode Decomposition: Optimal Solution in Polynomial Time

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Intractability

- **High-dimensional systems**

$$\begin{cases} x_t = f_t(x_{t-1}), \\ x_1 = \theta, \end{cases}$$

  with
  - state $x_t \in \mathbb{R}^n$, for $t = 1, \cdots, T$,
  - $f_t : \mathbb{R}^n \to \mathbb{R}^n$,
  - parameter $\theta \in \Theta \subseteq \mathbb{R}^n$.
- computing $\{x_t\}_t$ given $\theta \in \Theta$ leads to heavy computation.

Reduced Models

- **Low-rank linear model**

$$\begin{cases} \tilde{x}_t = A_k \tilde{x}_{t-1} \\ \tilde{x}_1 = \theta \end{cases}, \quad A_k \in \mathbb{R}^{n\times n}, \quad \mathrm{rank}(A_k) \leq k$$

- **Complexity decrease**
  - $\mathcal{O}(Tn^2)$: brute-force
  - $\mathcal{O}(Tk^2 + kn)$: if $A_k = RSL^\intercal$ such that

$$\tilde{x}_t = R(S)^{t-1} L^\intercal \theta,$$

    with $R, L \in \mathbb{C}^{n\times k}$ and $S \in \mathbb{C}^{k\times k}$ with $k \leq n$ determined *off-line*
  - $\mathcal{O}(kn)$: if S is a diagonal matrix

A_k: Data-driven Problem

- N **snapshots** with initial conditions $\{\theta_i\}_{i=1}^N$ in the set Θ.
- A_k **targets**

$$A_k^\star \in \underset{A:\mathrm{rank}(A)\leq k}{arg\ min} \sum_{t=2,i=1}^{T,N} \|x_t(\theta_i) - A x_{t-1}(\theta_i)\|_2^2,$$

given

- **Remark:** simple least square problem if $k \geq N(T-1)$.

Eigen Decomposition of $A_k^\star$

- **Low-rank dynamic mode decomposition (DMD)**

  Assume $A_k^\star$ diagonalisable, with $r = \mathrm{rank}(A_k^\star) \leq k$, then

$$\tilde{x}_t(\theta) = \underbrace{R}_{\text{matrix of modes}} \times \underbrace{(S)^{t-1} L^\intercal \theta}_{\text{coefficients}}$$

  where

  - columns of L and R in $\mathbb{C}^{n\times r}$ are left and right eigen-vectors of $A_k^\star$
  - $\mathrm{diag}(S) \in \mathbb{C}^r$ are the first r eigen-values.

Problematic

i) **Optimal solution** $A_k^\star$

- Non-convex problem where state-of-the-art polynomial solvers are all sub-optimal:
  - truncated approximation *[Tu14]*,
  - sub-space approximation *[Jovanic12][Chen12]*,
  - regularised approximation *[Li17]*,
  - convex relaxation *[Mishra13]*,
  - singular value projection *[Jain10]*.

Really out of reach in polynomial time?

ii) **Low-rank DMD**

- State-of-the-art DMD: Complexity to compute L and S is linear but cubic for R *[Tu14][William15]*.

Low-rank DMD within a linear complexity ?

First Contribution:
$A_k^\star$ in Polynomial Time

Preliminary Notations

- **Problem rewriting**

$$A_k^\star \in \underset{A:\mathrm{rank}(A)\leq k}{\textit{arg min}} \|\mathbf{Y} - A\mathbf{X}\|_F^2,$$

$x_{t-1}(\theta_i)$'s are the columns of $\mathbf{X} \in \mathbb{R}^{n\times m}$, with $m = N(T-1)$,
$x_t(\theta_i)$'s are the columns of $\mathbf{Y} \in \mathbb{R}^{n\times m}$.

- Columns of matrix $\mathcal{P} \in \mathbb{R}^{n\times k}$ are the k-first left singular vectors of

$$\mathbf{Z} = \mathbf{Y}\mathbb{P}_{\mathbf{X}^\intercal} \in \mathbb{R}^{n\times m}.$$

Solution of the Non-Convex Problem

Theorem

Closed-form optimal solution

$$A_k^\star = \mathcal{P}\mathcal{P}^\intercal \mathbf{Y}\mathbf{X}^\dagger.$$

Optimal approximation error

$$\|\mathbf{Y} - A_k^\star \mathbf{X}\|_F^2 = \|\mathbf{Y}(I_m - \mathbb{P}_{\mathbf{X}^\intercal})\|_F^2 + \sum_{i=k+1}^{m} \sigma_{\mathbf{Z},i}^2.$$

Polynomial-Time Algorithm

Algorithm

inputs: $(\mathbf{X}, \mathbf{Y})$

1) *Compute the SVD of* $\mathbf{X} = U_{\mathbf{X}} \Sigma_{\mathbf{X}} V_{\mathbf{X}}^{\intercal}$
2) *Compute* $\mathbb{P}_{\mathbf{X}^{\intercal}} = V_{\mathbf{X}} \Sigma_{\mathbf{X}} \Sigma_{\mathbf{X}}^{\dagger} V_{\mathbf{X}}^{\intercal}$.
3) *Compute* $\mathbf{Z} = \mathbf{Y} \mathbb{P}_{\mathbf{X}^{\intercal}}$.
4) *Compute* $V_{\mathbf{Z}}$ *and* $\Sigma_{\mathbf{Z}}^2$ *(by eigen-decomposition of* $\mathbf{Z}^{\intercal}\mathbf{Z}$*)*
5) *Compute the columns of* $\mathcal{P}$ *(k-first columns of* $\mathbf{Z} V_{\mathbf{Z}} \Sigma_{\mathbf{Z}}^{\dagger}$*)*

output: $A_k^{\star} = \mathcal{P}\mathcal{P}^{\intercal} \mathbf{Y} V_{\mathbf{X}} \Sigma_{\mathbf{X}}^{\dagger} U_{\mathbf{X}}^{\intercal}$

Second Contribution:
Low-Rank DMD in $\mathcal{O}(n)$

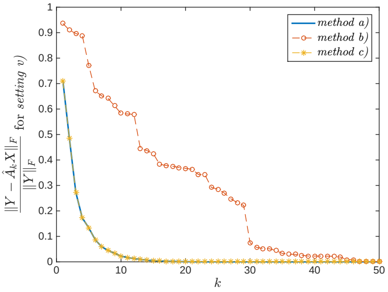
Low-rank DMD in $\mathcal{O}(n)$

- **Idea:**
  - $A_k^\star$ is the product of matrices in $\mathbb{R}^{n\times k}$ and in $\mathbb{R}^{n\times m}$.
  - We expect eigen-vectors of $A_k^\star$ to belong to a k-dimensional subspace.
- **Low-rank DMD (L, R and S).**
  - we show that matrices deduced in $\mathcal{O}(n)$ from eigen-decomposition of $\mathcal{P}^\intercal \mathcal{Q} \in \mathbb{R}^{k\times k}$ with $\mathcal{Q} = (\mathcal{P}^\intercal \mathbf{Y}\mathbf{X}^\dagger)^\intercal \in \mathbb{R}^{n\times k}$.

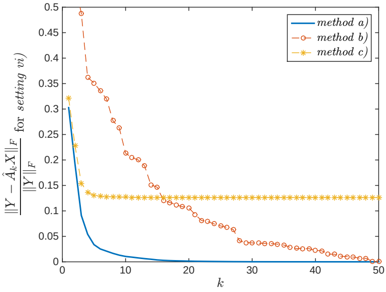
Numerical Evaluation

Physical Models

- Setup:
  - $n = 2^{10}$, $m = 50$, initial conditions living in a 10-dimensional subspace.
  - Algorithms: low-rank DMD (method a), truncated DMD *[Tu14]* (method b), low-rank projected DMD *[Jovanic12]* (method c).



advection-diffusion (linear)



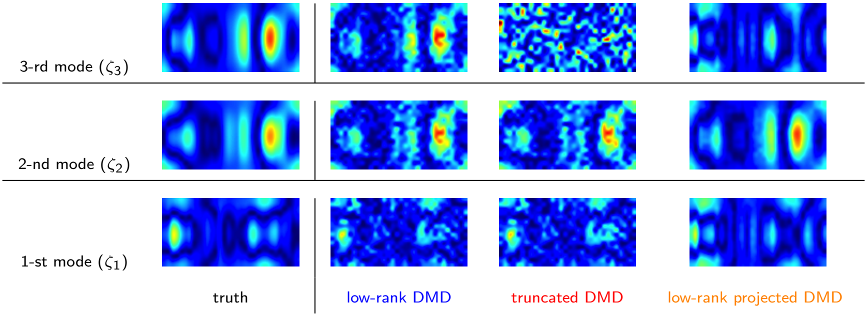
Rayleigh-Bénard convection

Robustness to Noise

- $n = 2^{10}$, $m = 50$, $x_t = Gx_{t-1}$ where

$$G = \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 \end{pmatrix} \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix}^\intercal .$$

- snapshots: additive noise with a PSNR of 20dB.



Conclusions

- Low-rank DMD: closed-form solution and approximation error
- Solver: linear complexity algorithm
- Numerical evaluation: importance in non-linear and noisy settings

Any questions ?

- Details on arXiv[1].
- Related poster[2] on upgrading kernel-based DMD.

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[1] *Patrick Héas, Cédric Herzet, "Low-Rank Dynamic Mode Decomposition: Optimal Solution in Polynomial-Time", arXiv:1610.02962, 2017*

[2] *Patrick Héas, Cédric Herzet, "Optimal Kernel-Based Dynamic Mode Decomposition", arXiv:1610.02962, 2017*