Importance Splitting for Statistical Model Checking Properties

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Probabilistic model checking

Quantify temporal logical properties of stochastic systems

- *Numerical* model checking
  - precise
  - exhaustive exploration of state space
  - limited model size
- *Statistical* model checking (SMC)
  - statistical model of *executions*
  - results within confidence bounds
  - trades off tractability with precision

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Motivation

Objective :

- Standard Statistical technique for SMC: Monte Carlo.
- Rare events may cause serious problems and are difficult to simulate.
- Given a stochastic system, design a procedure for estimating a rare property in a reasonable time with SMC.
- Properties specified with time bounded temporal logic:

$$\phi = \alpha \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg \phi \mid \boldsymbol{X}\phi \mid \boldsymbol{F}^{t}\phi \mid \boldsymbol{G}^{t}\phi \mid \phi \boldsymbol{U}^{t} \phi$$

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Monte Carlo model checking

Goal: Given a Markovian system and a property φ, compute the probability γ that a path ω satisfies φ ($\gamma = P[\omega \models \varphi]$).

The behavior of the system with respect to the property can be modeled by a Bernoulli random variable Z.

property indicator function $z \in \{0,1\}$

probability measure function

$$\gamma \stackrel{\text{def}}{=} \mathrm{E}_f[Z] = \int_{\Omega} z(\omega) f(\omega) d\omega$$

$$\tilde{\gamma} = \frac{1}{N} \sum_{i=1}^{N} z(\omega_i)$$

sample traces generated under *f*

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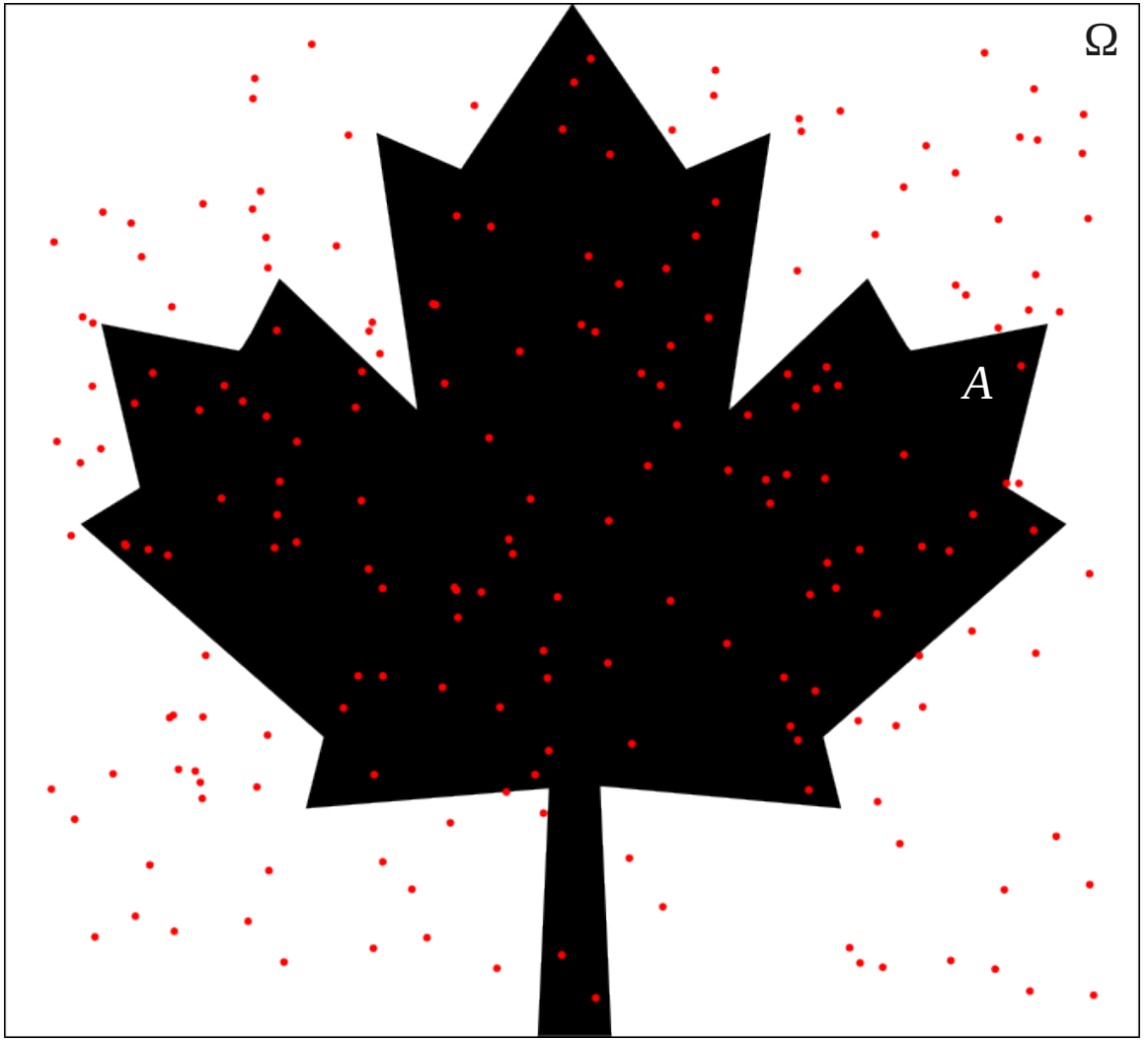
Monte Carlo estimation



$$A = \{\omega \in \Omega : z(\omega) = 1\}$$

$$\tilde{y} = \frac{1}{N} \sum_{i=1}^{N} z(\omega_i)$$

Absolute error = half the size of the confidence interval

$$\mathrm{AE} \propto \frac{\sqrt{\gamma(1-\gamma)}}{\sqrt{N}}$$

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Problems of rare events

- Occur with small probability (e.g. $< 10^{-6}$)
  - appear rarely in stochastic simulations
  - need very large number of trials to see single example
  - without seeing, cannot quantify how low the probability
- The absolute error is not useful $(\gamma \pm \epsilon)$ not useful if $\epsilon \gg \gamma$
  - Bounds (e.g. Chernoff) not useful when γ small
  - Unbounded *relative* error:

$$\mathrm{RE} = \frac{\sqrt{Var(z)}}{E(z)} = \frac{\sqrt{\gamma - \gamma^2}}{\gamma} \approx_{\gamma \to 0} \frac{1}{\sqrt{\gamma}}$$

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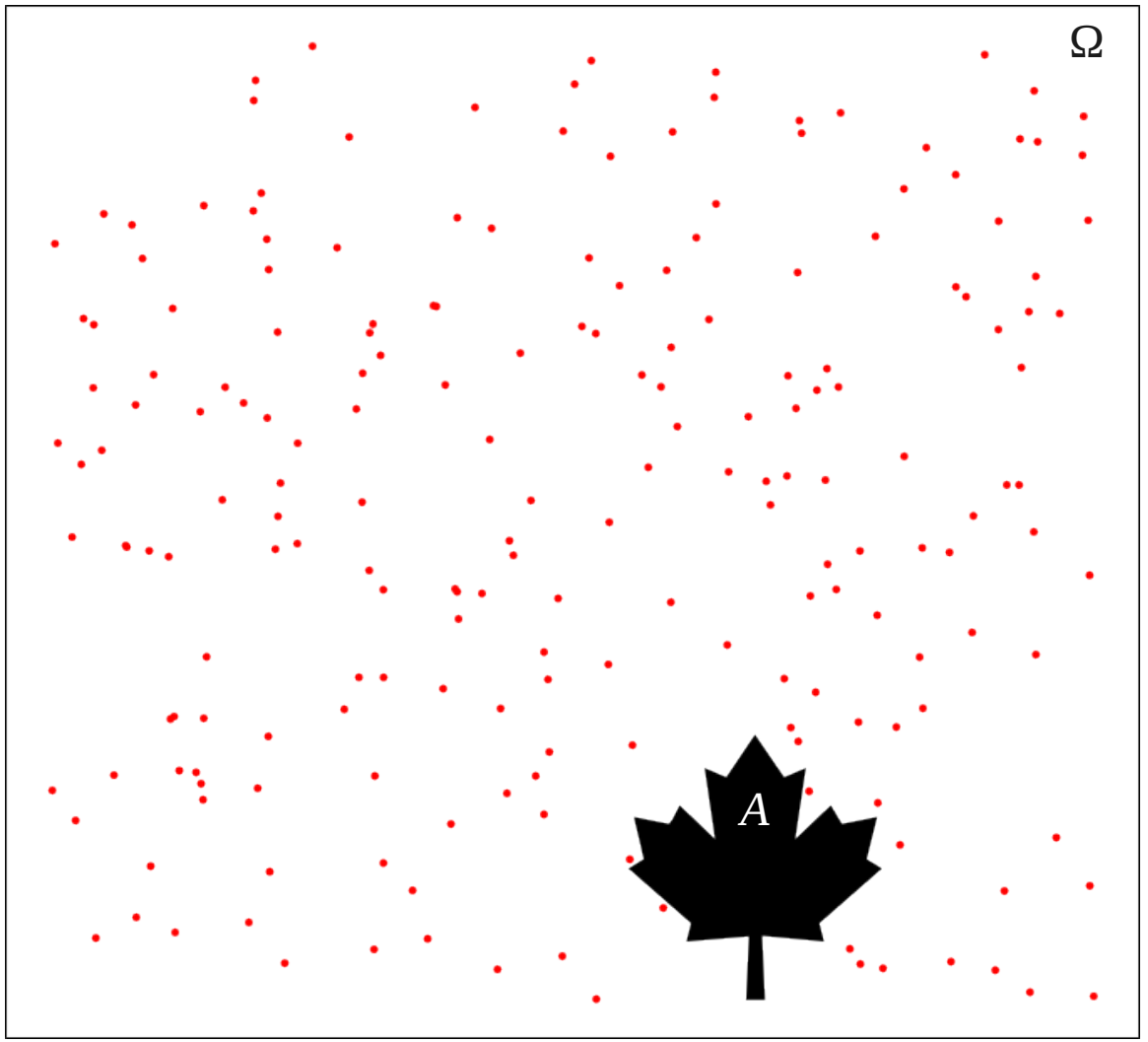
High variance



$$\mathrm{RE} \propto_{\gamma \to 0} \frac{1}{\sqrt{N\,\gamma}}$$

N very large to bound RE
with Monte Carlo simulation

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Importance sampling

Monte Carlo

$$\gamma = \int_{\Omega} z(\omega)\, f(\omega)\, d\omega$$

$$\tilde{\gamma}_{MC} = \frac{1}{N} \sum_{i=1}^{N} z(\omega_i)$$

traces generated under f

$$\gamma = \int_{\Omega} z(\omega) \frac{f(\omega)}{f'(\omega)} f'(\omega)\, d\omega$$

importance sampling distribution

likelihood ratio

$$\tilde{\gamma}_{IS} = \frac{1}{N} \sum_{i=1}^{N} z(\omega_i) \frac{f(\omega_i)}{f'(\omega_i)}$$

traces generated under f'

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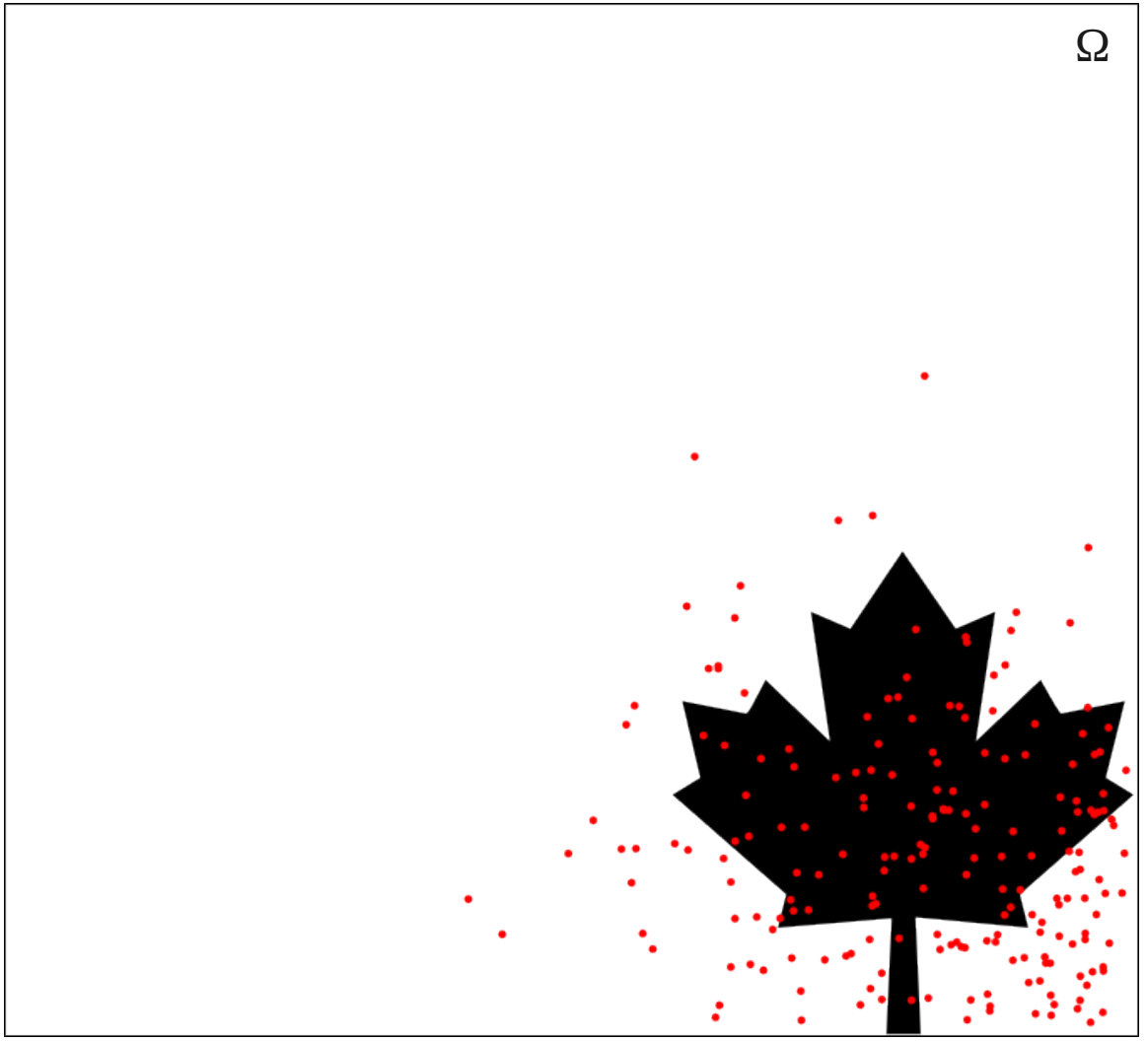
‘Tilted’ simulation



$$\tilde{y} = \frac{1}{N} \sum_{i=1}^{N} z(\omega'_i) \frac{f(\omega')}{f'(\omega'_i)}$$

traces generated under f'
(importance sampling dist.)

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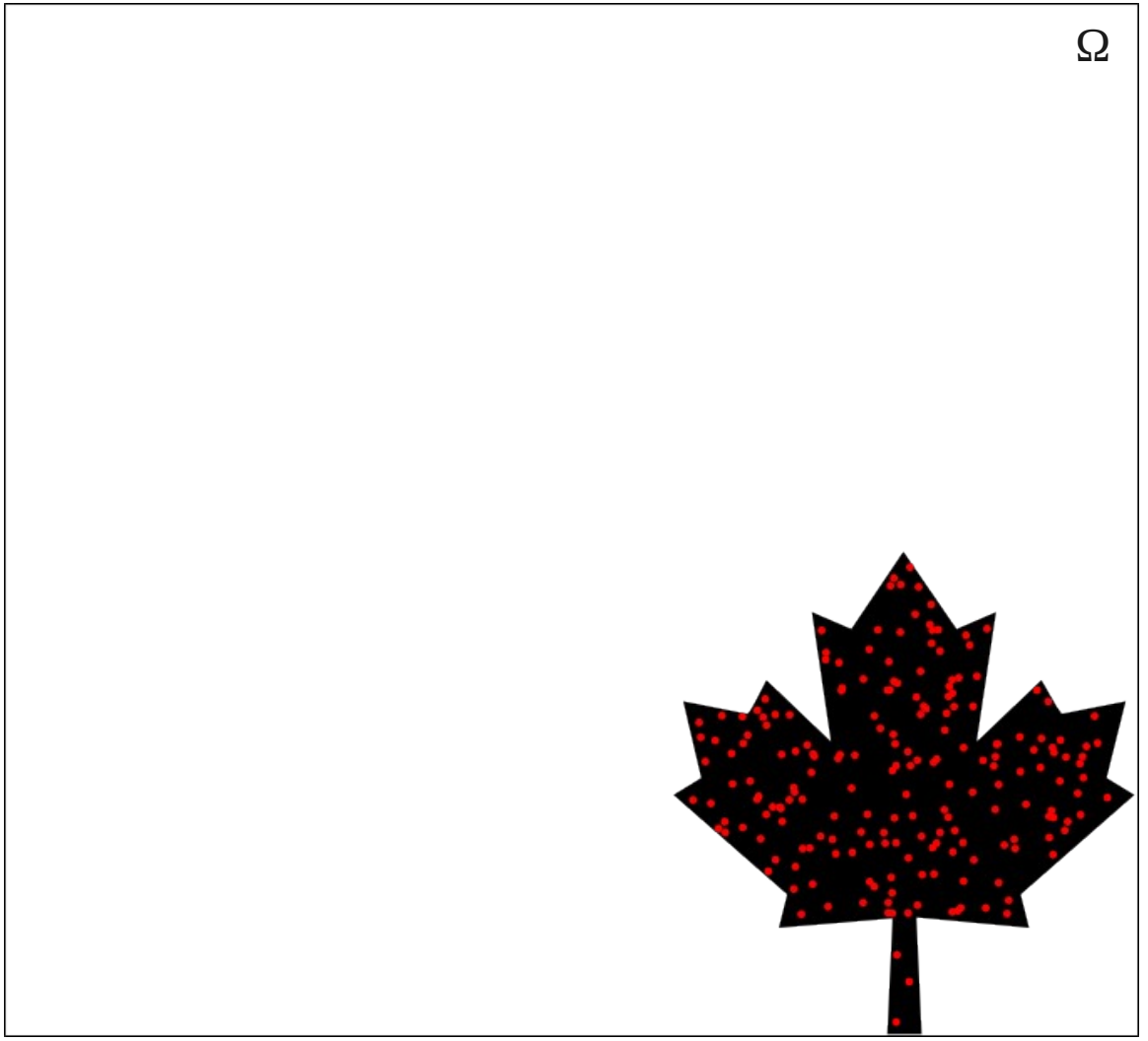
Optimal importance sampling



$$\tilde{y} = \frac{1}{N} \sum_{i=1}^{N} z(\omega'_i) \frac{f(\omega')}{f'(\omega'_i)}$$

$$f^{opt} = \frac{z f}{y}$$

f conditioned on the rare event

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Limitations of Importance Sampling

- Quantifying the performance of apparently “good” distributions is an open problem.
- Problem of accuracy with long simulations: likelihood ratio vanishes and variance of the estimators increases.

=> need of an alternative technique: Importance Splitting.

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Basics of Importance Splitting

Let A a rare event and $(A_k)_{0 \le k \le n}$ a sequence of nested events:

$$A_0 \supset A_1 \supset \ldots \supset A_n = A$$

$$\gamma \stackrel{\text{def}}{=} P(A) = P(A_0) P(A_1 | A_0) P(A_2 | A_1) \ldots P(A_n | A_{n-1})$$

Bayes formula

$$\forall k \quad P(A_k \mid A_{k-1}) = \gamma_k \ge \gamma$$

Less rare

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Generation of traces in Importance Splitting

- Assuming a set of increasing *levels k*, generate traces starting from a distribution of the initial states.
- Simulations are stopped as soon as they reach the next level k+1.
- The final states become the empirical distribution of initial states for the next level (level k+1).
- Failed traces discarded. Successful traces continue from where they stopped.
- Avoid a reduction of simulations by resampling the discarded traces from empirical distribution of level k+1.

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Illustration 1

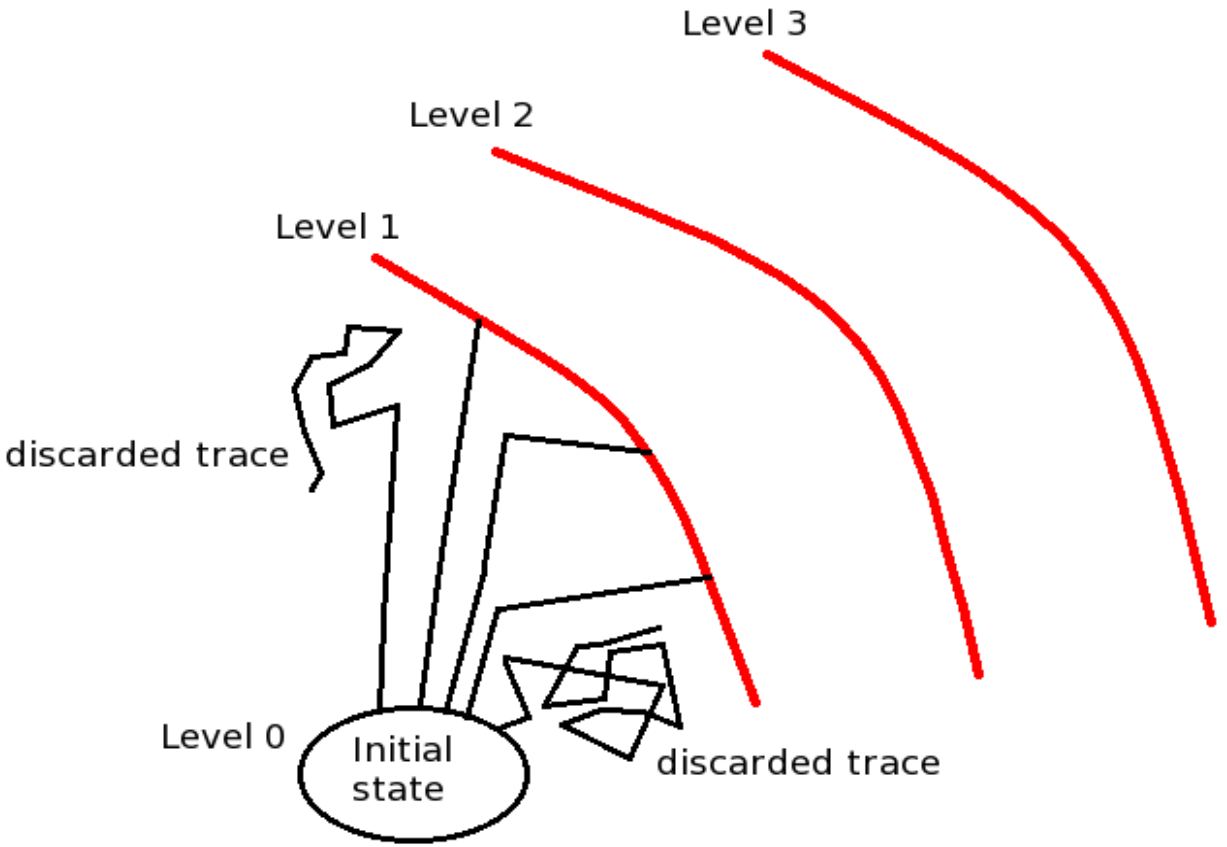

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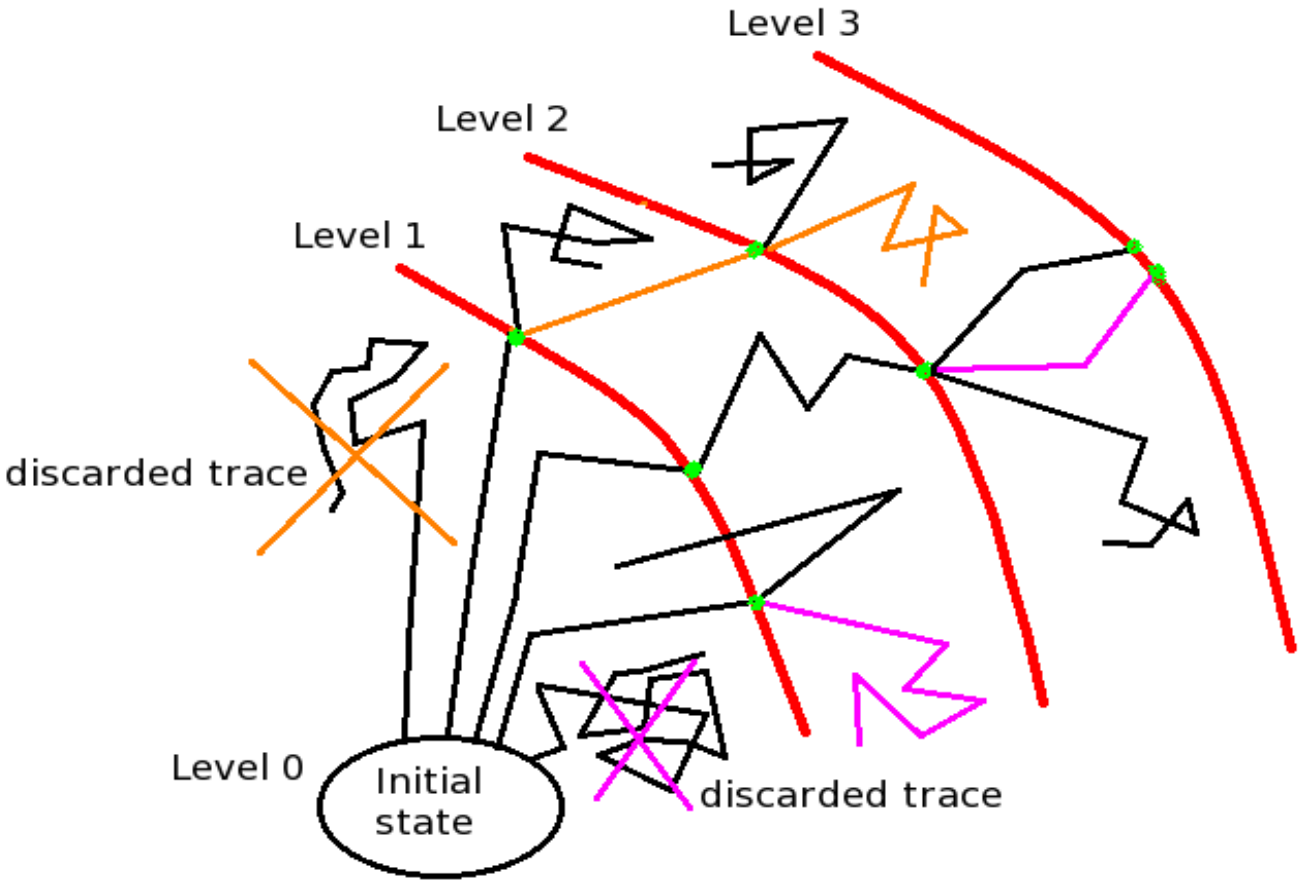
Illustration 2



P(reaching Level 3)=3/5*2/5*2/5

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Importance Splitting for (Temporal) Logic

Idea: given a rare property ϕ, define a set of levels based on a sequence of temporal properties such that:

$$(\phi_k)_{0 \le k \le n} : \ \phi_0 \Leftarrow \phi_1 \Leftarrow \ldots \Leftarrow \phi_n = \phi$$

Thus,

$$\gamma = P(\omega \vDash \phi_0) \prod_{k=1}^{n} P(\omega \vDash \phi_k \mid \omega \vDash \phi_{k-1})$$

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Level-based Score functions

- Goal: Generalise the concept of levels.
- Definition 1:

  Let $J_0 \supset J_1 \supset ... \supset J_n$ be a set of nested intervals of $\mathbb{R}$.

  Let $\phi_0 \supset \phi_1 \supset ... \supset \phi_n$ be a set of nested properties.

  $S : \Omega \to \mathbb{R}$ is a level-based score function of property ϕ iff $\forall k$:

$$\omega \models \phi_k \Leftrightarrow S(\omega) \in J_k \text{ and } \forall i, j \in \{0, ..., |\omega|\}: i < j \Rightarrow S(\omega_{\leq i}) \leq S(\omega_{\leq j})$$

- Example: given a set of nested properties, a simple score function may be defined as follows:

$$S(\omega) = \sum_{k=1}^{n} 1(\omega \models \phi_k)$$

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General score functions

- Definition 2:

  Let $J_0 \supset J_1 \supset \ldots \supset J_n$ be a set of nested intervals of $\mathbb{R}$.

  Let $\Omega = \Omega_0 \supset \Omega_1 \supset \ldots \supset \Omega_n$ be a set of nested subsets of Ω.

  $S : \Omega \to \mathbb{R}$ is a general score function of property ϕ iff $\forall k$:

  (i) $\omega \in \Omega_k \Leftrightarrow S(\omega) \in J_k$

  (ii) $\omega \vDash \phi \Leftrightarrow \omega \in \Omega_n$

  (iii) $\forall i, j \in \{0, \ldots, |\omega|\}: i < j \Rightarrow S(\omega_{\leq i}) \leq S(\omega_{\leq j})$

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Score functions

- Goal: Generalise the concept of levels.
- Level-based score functions: Mapping from logical properties to the real numbers which give information on the number of satisfied sub-formulae.

$$\text{Example: } S(\omega) = \sum_{k=1}^{n} 1(\omega \models \phi_k)$$

- General score functions: Mapping from sets of paths to the real numbers s.t. higher scores assigned to paths that satisfy the overall property.

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Use of Heuristics

- Level-based score functions correlate logic to score.
- General score functions requires:
  - higher scores assigned to paths that satisfy the overall property.
  - Score of a path's prefix is non decreasing with increasing prefix length.
- In some case, the shortest paths satisfying a rare property are the most likely.

  => possibility to exploit the length of a path to improve a score function based on coarse logical levels.

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Simple decomposition

- When $\phi = \Lambda_{j=1}^{n} \psi_j$, a decomposition into nested properties is:

$$\phi_i = \Lambda_{j=1}^{i} \psi_j, \quad \forall i \in \{1, \ldots, n\} \text{ with } \phi_0 \equiv True$$

- Possibility to choose an arbitrary order of sub-formulae:

Example: Given $\phi = a \wedge b \wedge c$,

$$\phi_3 = a \wedge b \wedge c, \ \phi_2 = a \wedge b, \ \phi_1 = a$$

$$\phi_3 = a \wedge b \wedge c, \ \phi_2 = b \wedge c, \ \phi_1 = c$$

Both decompositions are valid.

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Natural decomposition

- Many rare events are defined with a natural notion of level, when some quantity of the system reaches a particular value.
- In Computational systems: might refer to a loop counter, a number of software objects, etc...
- In physical systems: might refer to a temperature, a distance, a number of molecules...
- Natural levels defined by nested atomic properties:

$$\phi_i = (l > l_i) \text{ with } l \text{ a state variable and } \omega \models \phi_n \Leftrightarrow l \geq l_n$$

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Decomposition of temporal operators

$(i)\ \ (\phi_n \Rightarrow \phi_{n-1}) \Rightarrow (\boldsymbol{S}\phi_n \Rightarrow \boldsymbol{S}\phi_{n-1})$ with $\boldsymbol{S} \in \{\boldsymbol{F}^{\leq t}, \boldsymbol{G}^{\leq t}, \boldsymbol{X}, \boldsymbol{F}^{\leq t}\boldsymbol{G}^{\leq s}\}$

$(ii)\ \ (\phi_n \Rightarrow \phi_{n-1} \wedge \psi_m \Rightarrow \psi_{m-1}) \Rightarrow (\phi_n \boldsymbol{U} \psi_m \Rightarrow \phi_{n-1} \boldsymbol{U} \psi_{n-1})$

$(iii)\ \ (\phi_n \Rightarrow \phi_{n-1}) \Rightarrow (\forall \omega \vDash \boldsymbol{G}^{\leq t}\phi_n : \exists t' \geq t \mid \omega \vDash \boldsymbol{G}^{\leq t'}\phi_{n-1})$

$(iv)\ \ (\phi_n \Rightarrow \phi_{n-1}) \Rightarrow (\forall \omega \vDash \boldsymbol{F}^{\leq t}\phi_n : \exists t' \leq t \mid \omega \vDash \boldsymbol{F}^{\leq t'}\phi_{n-1})$

$(v)\ \ (t' \geq t \wedge s' \leq s) \Rightarrow (\boldsymbol{F}^{\leq t}\boldsymbol{G}^{\leq s}\phi_n \Rightarrow \boldsymbol{F}^{\leq t'}\boldsymbol{G}^{\leq s'}\phi_n)$

$(vi)\ \ (\phi_n \Rightarrow \phi_{n-1}) \Rightarrow (\forall \omega \vDash \boldsymbol{F}^{\leq t}G^{\leq s}\phi_n : \exists t' \leq t \wedge s' \geq s \mid \omega \vDash \boldsymbol{F}^{\leq t'}\boldsymbol{G}^{\leq s'}\phi_{n-1})$

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Two algorithms

- Fixed level algorithm:
  - Exploits a score function based on “logical” levels

- Adaptive level algorithm:
  - Given a score function, finds itself the “best” levels
  - Requires a score function refined enough.

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Fixed level algorithm

Let $(\tau_k)_{1 \le k \le M}$ be the sequence of thresholds

Let **stop** be a termination condition

for $1 \le k \le M$ **do**

$\forall\, 1 \le j \le N$, using prefix $\tilde{\omega}_j^k$, generate path ω_j^k until $S(\omega_j^k) \ge \tau_k \vee$ **stop**

$I_k = \{\ j : S(\omega_j^k) \ge \tau_k\ \}$ and $\tilde{\gamma}_k = \dfrac{|I_k|}{N}$

$\forall\, j \in I_k,\ \tilde{\omega}_j^{k+1} = \omega_j^k$

$\forall\, j \notin I_k$, let $\tilde{\omega}_j^{k+1}$ be a copy of ω_i^k with $i \in I_k$ chosen randomly

$$\tilde{\gamma} = \prod_{k=1}^{M} \tilde{\gamma}_k$$

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Fluctuation analysis

- Estimator unbiased
- Confidence interval based on the relative variance:

$$\sqrt{N}\frac{\tilde{\gamma}-\gamma}{\gamma} \underset{n\to\infty}{\overset{D}{\to}} \boldsymbol{N}(0,\sigma^2) \text{ where } \boldsymbol{N} \text{ denotes a Gaussian distribution}$$

$$\text{with } \sigma^2 \geq \sum_{k=0}^{n-1} \frac{1-\gamma_k}{\gamma_k}$$

- Inequality arises because the independence of initial states diminishes with increasing levels.
- Several possibilities minimise this dependence effect.

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Idealized version

- relative variance of the estimator: $\sigma^2 = \sum_{k=0}^{n-1} \frac{1-\gamma_k}{\gamma_k}$

- For a fixed number of levels, this variance is minimal if all the conditional probabilities are equal:

$$\underset{p_0,\cdots p_{n-1}}{argmin} \sum_{k=0}^{n-1} \frac{1-\gamma_k}{\gamma_k} \quad \text{s.t.} \quad \prod_{k=0}^{n-1} \gamma_k = p$$

- It corresponds to the case where the levels are evenly spaced in terms of probability of success.
- Hence, the idea of an adaptive algorithm.

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Adaptive level algorithm

Let N_k be the predefined number of paths to keep per iteration

Let τ_ϕ be the minimum score of paths that satisfy ϕ

$k=1$. $\forall\, 1 \le j \le N$, generate path ω_j^k

repeat (until $\tau_k \ge \tau_\phi$)

Let $T = \{ S(\omega_j^k),\ \forall\, j \in \{1, \ldots, N\} \}$

Find minimum $\tilde{\tau}_k$ s.t. $|\{\tau \in T : \tau > \tilde{\tau}_k\}| \ge N_k$

$I_k = \{\ j : S(\omega_j^k) \ge \tau_k\ \}$ and $\tilde{\gamma}_k = \dfrac{|I_k|}{N}$

$\forall\, j \in I_k,\ \omega_j^{k+1} = \omega_j^k$

for $j \notin I_k$ **do**

Choose randomly $l \in I_k$

$\tilde{\omega}_j^{k+1} = max\{\omega \in pref(\omega_l^k) : S(\omega) < \tau_k\}$

generate path ω_j^{k+1} with prefix $\tilde{\omega}_j^{k+1}$

$M, k = k, k+1$

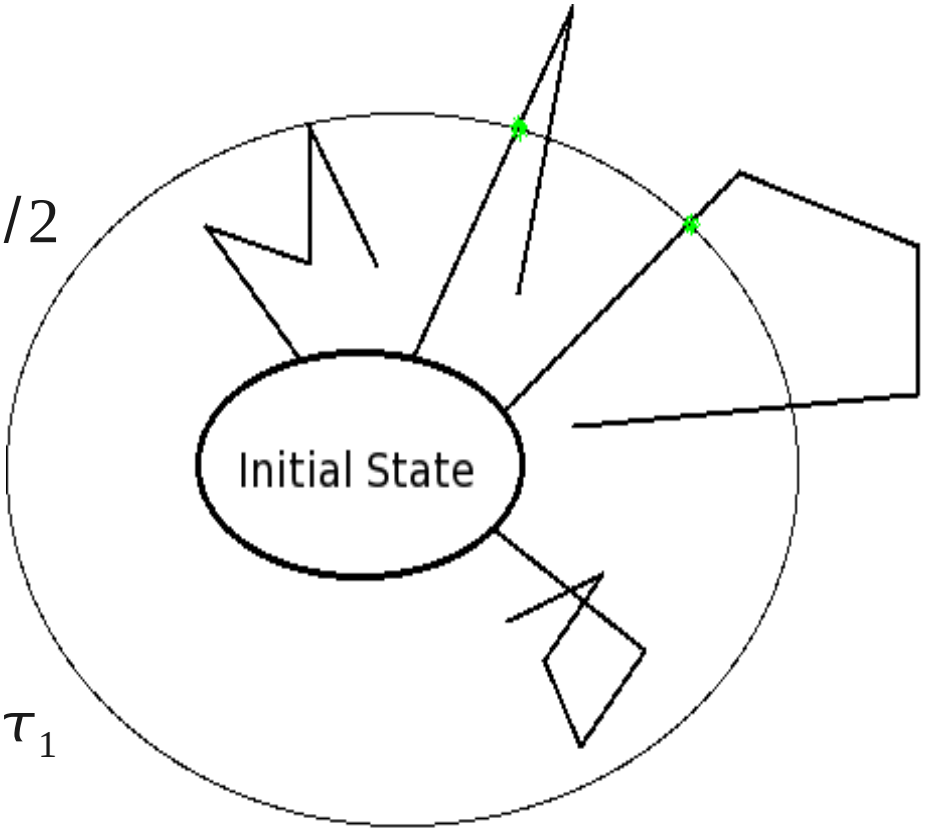
$$\tilde{\gamma} = \prod_{k=1}^{M} \tilde{\gamma}_k$$

$^{|\omega|}$

Adaptive algorithm illustration
1st step: trace generation

Predefine $\gamma_0 = 1/2$

Estimate τ_1 s.t. $P(Score(\omega) > \tau_1) = 1/2$



Ellipse corresponding to $Score(\omega) = \tau_1$

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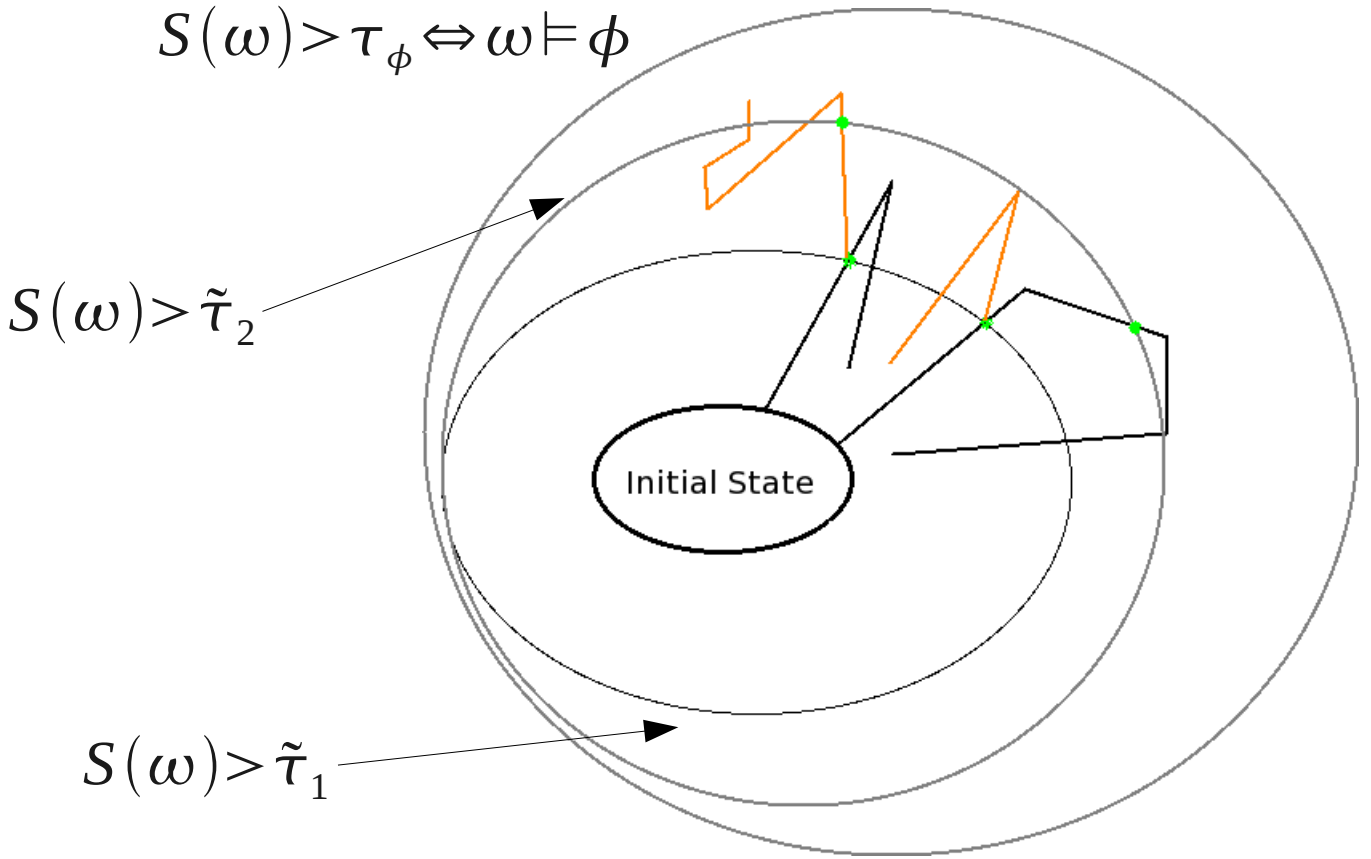
Adaptive algorithm illustration



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Fluctuation analysis

- For simplicity, let us write: $\gamma = r\,\gamma_0^{\,M}$

  with γ_0 the predefined conditionnal probability,

  M the number of levels

  r the number of traces ω in the last iteration s.t.: $S(\omega) \geq \tau_\phi$

$$\sqrt{N}\,\frac{\tilde{\gamma}-\gamma}{\gamma} \underset{n\to\infty}{\overset{D}{\to}} \mathbf{N}(0,\sigma^2) \qquad \text{with } \sigma^2 \geq M\,\frac{1-\gamma_0}{\gamma_0} + \frac{1-r}{r}$$

$$E[\tilde{\gamma}] - \gamma \sim \frac{\gamma}{N}\,\frac{M(1-\gamma_0)}{\gamma_0}$$

- Positive Bias of order O(1/N) => Good news!

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Bias, variance and Confidence interval

- $E[\tilde{y}] = p(1+O(N^{-1}))$
- $Var(\tilde{y}) = \frac{p^2}{N}(M\frac{1-\gamma_0}{\gamma_0}+\frac{1-r}{r})+o(N^{-1})$
- => Hence, the variance is reduced if γ_0 is chosen large.
- Confidence interval of level (1-α) based on the relative variance:

$$[\ \tilde{y}(\frac{1}{1+z_\alpha\sigma N^{-1/2}}),\ \tilde{y}(\frac{1}{1-z_\alpha\sigma N^{-1/2}})$$

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Example: Dining philosophers

Model: 150 philosophs

Property of interest:

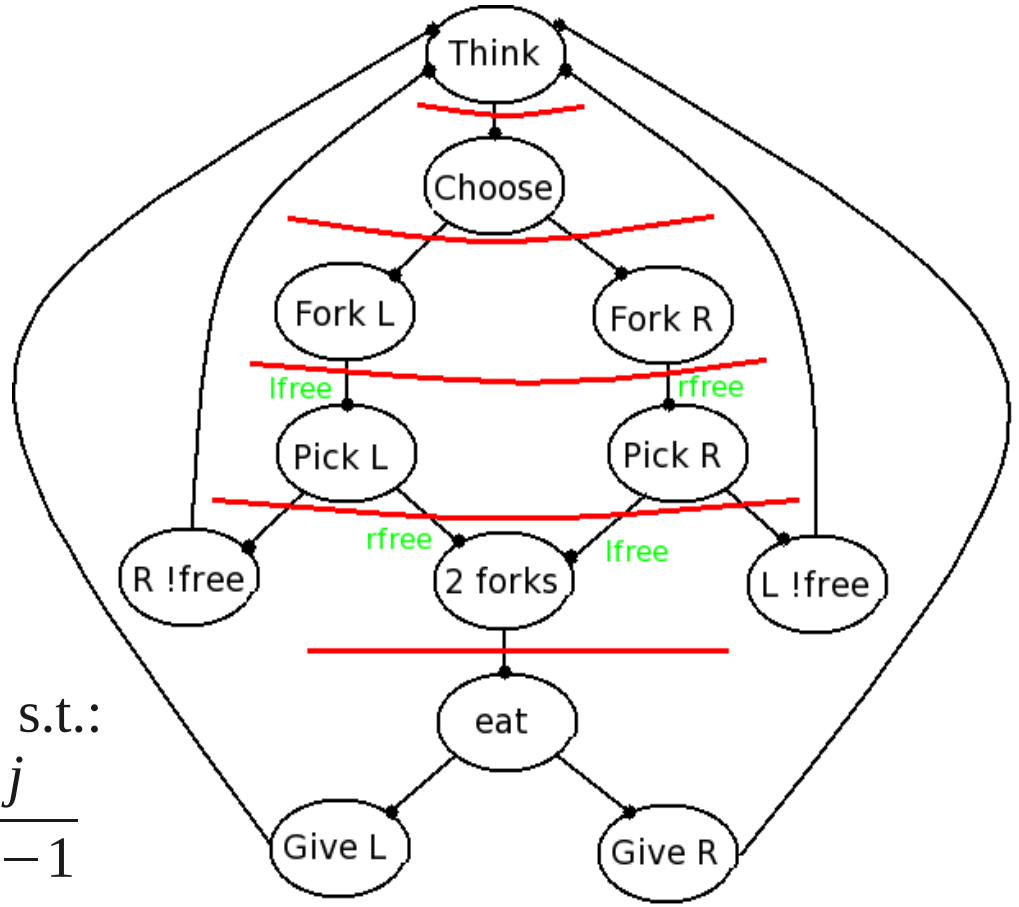
$$\boldsymbol{F}^{\leq K}\ \text{phil}_i\ \textit{eat}$$

$$S_1(\omega) = \sum_{k=1}^{n} 1\,(\omega \vDash \phi_k)$$

$$S_2(\omega) = \max_{1 \leq j \leq K} \Psi\,(\omega_{\leq j})$$

where j is prefix's length and Ψ s.t.:

$$\Psi\,(\omega_{\leq j}) = S_1(\omega_{\leq j}) - \frac{S_1(\omega_{\leq j}) - j}{S_1(\omega_{\leq j}) - K - 1}$$



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Experimental results

- Description of the model: more than 10^{96} states
- MC probability based on 10^{7} samples: $1.4 * 10^{-6}$
- With N=1000 samples, score function S_2 and $\gamma_0 \approx 1 - \frac{1}{N}$
  - Probability estimator: $1.581 * 10^{-6}$
  - Variance of the estimator: $0.119 * 10^{-6}$
  - Around 100 levels found adaptively.
- Roughly, the number of samples required for IS is between 1000 and 10000 times less important than with MC. => Gain of time

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Ongoing work

- Quantifying performance of importance splitting:
  - Define more complex score functions to improve efficiency
  - Real case studies (biology, robotics?)
- Continuing the development of PLASMA

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