

An axiomatization of the Token Game based on Petri Algebras

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Abstract. The firing rule of Petri nets relies on a residuation operation for the commutative monoid of non negative integers. We identify a class of residuated commutative monoids, called Petri algebras, for which one can mimic the token game of Petri nets to define the behaviour of generalized Petri nets whose flow relations and place contents are valued in such algebraic structures. The sum and its associated residuation capture respectively how resources within places are produced and consumed through the firing of a transition. We show that Petri algebras coincide with the positive cones of lattice-ordered commutative groups and constitute the subvariety of the (duals of) residuated lattices generated by the commutative monoid of non negative integers. We however exhibit a Petri algebra whose corresponding class of nets is strictly more expressive than the class of Petri nets. More precisely, we introduce a class of nets, termed lexicographic Petri nets, that are associated with the positive cones of the lexicographic powers of the additive group of real numbers. This class of nets is universal in the sense that any net associated with some Petri algebra can be simulated by a lexicographic Petri net. All the classical decidable properties of Petri nets however (termination, covering, boundedness, structural boundedness, accessibility, deadlock, liveness ...) are undecidable on the class of lexicographic Petri nets. Finally we turn our attention to bounded nets associated with Petri algebras and show that their dynamics can be reformulated in term of MV-algebras.

1 Introduction

The Petri net model is a graphical and mathematical modeling tool that, since its introduction in the early sixties, have come to play a pre-eminent role in the formal study of concurrent discrete-event dynamic systems. Petri nets are natural extensions of automata in which states are distributed (they can be represented as vectors of "local states") and for which the occurrence of an event relies on local conditions. More precisely a Petri net $(P, T, Pre, Post)$ consists of a finite set P of places, a finite set T of transitions (disjoint from P), and flow

relations $Pre, Post : P \times T \rightarrow \mathbb{N}$. Places can contain some tokens representing the resources available in this place for the current configuration. Such a (global) configuration of a Petri net is given as a vector $M : P \rightarrow \mathbb{N}$, called marking, indicating the number of tokens available in each place. Tokens are consumed and produced by the firing of transitions according to the so-called token game:

$$M [t] M' \Leftrightarrow \forall p \in P \quad M(p) \supseteq Pre(p, t) \\ \text{and } M'(p) = (M(p) - Pre(p, t)) + Post(p, t)$$

The token game of Petri nets says that in order for a transition t to fire in marking M it should be the case that each place contains enough resources as it is expressed by the condition $M(p) \supseteq Pre(p, t)$. Then the firing of transition t proceeds in two stages: a consumption of resources ($Pre(p, t)$ tokens are removed from place p) followed by a production of resources ($Post(p, t)$ tokens are added to place p). The notation $M [t] M'$ expresses the fact that transition t is allowed to fire in marking M and that firing t in marking M produces the new marking M' . Numerous techniques, supported and automated by software tools, can be used to verify that some required properties are met for systems specified using Petri nets. For instance the following problems have an effective solution on the class of Petri nets:

Reachability Is a given marking reachable (from a fixed initial marking) through a sequence of firable transitions?

Coverability Is a given marking M covered by some reachable marking M' ? (i.e. $\forall p \in P \quad M(p) \leq M'(p)$)

Place-boundedness Does there exist an upper bound on the number of tokens that a given place can hold in reachable markings?

Boundedness Are all places bounded on the set of reachable markings? (i.e. is the Petri net equivalent to a finite state machine?)

Deadlock Does there exist a reachable marking in which no transition can fire?

Liveness A transition t is live if for every reachable marking M there exists at least one marking, reachable from M , in which t can fire.

By combining these basic properties we can express complex static and dynamic requirements for the system. The ability, through the token game rule, to graphically visualise the behaviour of a Petri net helps the practitioner to uncover shortcomings in the specification or errors in the model. The Petri net technique thus promotes an iterative modelling process in which the model can incrementally be constructed by alternatively modify and re-analyse it.

Numerous extensions of this basic model of Petri nets have been introduced over the years. Some of them are high level nets that allows for more compact representations but does not increase the expressive power of Petri nets: these high level nets can be unfolded into equivalent, even though in general much larger, Petri nets. Some extensions however change more dramatically the semantics of the original model. For instance timing constraints may be added, as in timed Petri nets or stochastic Petri nets for the purpose of enabling performance analysis. With continuous Petri nets the discrete state transition rule

is replaced by a notion of trajectory using a continuum of intermediate states. In Fuzzy Petri nets one has a possibilistic measure of the firing of a transition in the given marking thus enabling to deal with uncertainty. Our purpose in this paper is to put forward an axiomatization of the token game of Petri nets. More precisely we identify a class of commutative residuated monoids, called Petri algebras, for which one can mimic the token game of Petri nets to define the behaviour of generalized Petri nets whose flow relations and place contents are valued in such algebraic structures. The sum and its associated residuation capture respectively how resources within places are produced and consumed through the firing of a transition. The class of usual Petri nets is associated with the commutative monoid of non negative integers. We show that Petri algebras coincide with the positive cones of lattice-ordered commutative groups and constitute the subvariety of the (duals of) residuated lattices generated by the commutative monoid of non negative integers. The basic Petri net model is thus associated with the generator of the variety of Petri algebras which shows that these extended nets share all algebraic properties of Petri nets, in particular they have the same equational and inequational theory. We however exhibit a Petri algebra whose corresponding class of nets is strictly more expressive than the class of Petri nets, i.e. their class of marking graphs is strictly larger. More precisely, we introduce a class of nets, termed lexicographic Petri nets, that are associated with the positive cones of the lexicographic powers of the additive group of real numbers. This class of nets is proved to be universal in the sense that any net associated with some Petri algebra can be simulated by a lexicographic Petri net. All the classical decidable properties of Petri nets however (termination, covering, boundedness, structural boundedness, accessibility, deadlock, liveness ...) are proved to be undecidable on the class of lexicographic Petri nets. This negative result put emphasize on the fact that algebraic considerations alone are not sufficient to capture the intrinsic nature of Petri nets. This might give us an understanding on the fact that all past attempts to find a strict extension of Petri nets with the same decidable properties did fail. Finally we turn our attention to bounded nets associated with Petri algebras and show that their dynamics can be reformulated in term of MV-algebras.

2 An axiomatization of the token game

2.1 Playing the token game

In order to obtain an axiomatization of the token game of Petri nets we represent the marking of a net as a map $M : P \rightarrow \prod_{p \in P} A_p$ that associates with each place $p \in P$ the local value of the current configuration $M(p) \in A_p$ in this place. Content of places are resources that are consumed and produced according to the token game. Thus we assume that each place $p \in P$ is associated with a commutative divisibility monoid $A_p = (A_p, \oplus, 0)$, i.e. a monoid such that

$$\text{the relation } a \sqsupseteq b \Leftrightarrow \exists c \cdot a = b \oplus c \text{ is an order relation} \quad (1)$$

The constant 0 represents the absence of resource and the binary operator \oplus the accumulation of resources in places. Immediate consequences of condition (1) are the following:

$$\begin{aligned} a \oplus b \sqsupseteq a, b \\ 0 \sqsubseteq a \\ a \oplus b = 0 \Rightarrow a = b = 0 \end{aligned}$$

Moreover we need to have a residuation operation \ominus such that $a \ominus b$ represents the residual resource obtained by subtracting b to a when $b \sqsubseteq a$. Thus the following should hold true:

$$b \sqsubseteq a \Rightarrow a = (a \ominus b) \oplus b \quad (2)$$

Usual Petri nets corresponds to the situation where, for every place p , $A_p = (\mathbb{N}, +, 0)$ is the commutative monoid of non negative integers with the truncated difference $n \ominus m = \max(0; n - m)$ as residuation. This operation is characterized by the universal property that for every non negative integer n, m and p

$$n + m \sqsupseteq p \Leftrightarrow n \sqsupseteq p \ominus m$$

Up to the reversal of the order relation, it is a *commutative residuated monoid* i.e. a commutative monoid $(A, \oplus, 0)$ with an order relation \leq and a residuation operation \ominus which is a right adjoint to the addition, in the sense that

$$a \oplus b \leq c \Leftrightarrow a \leq c \ominus b \quad (3)$$

It follows immediately from this definition that a commutative monoid is residuated if and only if its addition is order preserving in each argument and the inequation $a \oplus b \leq c$ has a largest solution for a (namely $c \ominus b$). In particular the residual is uniquely determined by the addition and the order relation. When the monoid is a divisibility monoid the order relation itself is defined in terms of the addition and thus the whole structure is characterized by its monoid reduct. The following result is known in the context of residuated lattices (i.e. if we further assume that each pair of elements admits a least upper bound and a greatest lower bound) but the proof given below, shows that it is also valid even if the monoid is not assumed to be lattice-ordered.

Proposition 1. *If $(A, \oplus, 0, \sqsubseteq)$ is a co-residuated commutative monoid, i.e. a monoid equipped with a residuation operation \ominus such that*

$$a \oplus b \sqsupseteq c \Leftrightarrow a \sqsupseteq c \ominus b \quad (4)$$

where 0 is the least element, then the following three conditions are equivalent

- (i) *It is a divisibility monoid: $a \sqsupseteq b \Leftrightarrow \exists c \cdot a = b \oplus c$*
- (ii) *It is an upper semi-lattice with: $a \sqcup b = (a \ominus b) \oplus b$*
- (iii) *$b \sqsubseteq a \Rightarrow a = (a \ominus b) \oplus b$*

Lemma 1. *Let $(A, \oplus, 0, \sqsubseteq)$ be a commutative ordered monoid where the neutral element is also the least element for the order relation, and let \ominus be some binary operation on A ; then the universal property*

$$a \oplus b \sqsupseteq c \Leftrightarrow a \sqsupseteq c \ominus b$$

is equivalent to the conjunction of the following properties:

- (i) $(a \oplus b) \ominus b \sqsubseteq a$
- (ii) $(a \ominus b) \oplus b \sqsupseteq a, b$
- (iii) $a \ominus (a \ominus b) \sqsubseteq a, b$
- (iv) *monotonies:* $a \sqsubseteq a' \wedge b \sqsubseteq b' \Rightarrow a \oplus b \sqsubseteq a' \oplus b' \wedge a \ominus b' \sqsubseteq a' \ominus b$

Proof. Let us assume that \ominus is left adjoint to the addition, i.e. the universal property (4) holds. Then from $a \oplus b \sqsupseteq a \oplus b$ it follows that $(a \oplus b) \ominus b \sqsubseteq a$ and from $a \ominus b \sqsupseteq a \ominus b$ it follows that $(a \ominus b) \oplus b \sqsupseteq a$. Combining the commutativity of addition with the adjunction it follows that $a \sqsupseteq c \ominus b \Leftrightarrow b \sqsupseteq c \ominus a$ and then in particular $b \sqsupseteq a \ominus (a \ominus b)$. Since the neutral element is also the least element for the order relation we deduce $a \ominus a = 0$ (because $a \oplus 0 \sqsupseteq a \Rightarrow 0 \sqsupseteq a \ominus a$) and then $a \oplus b \sqsupseteq b$ follows from $a \sqsupseteq b \ominus b = 0$ and $a \oplus b \sqsupseteq a$ by commutativity. From $a \ominus b \sqsupseteq 0 = b \ominus b$ it follows that $(a \ominus b) \oplus b \sqsupseteq b$ by adjunction. Similarly $a \ominus (a \ominus b) \sqsubseteq a$ follows from $a \ominus b \sqsupseteq 0 = a \ominus a$ by the equivalence $a \sqsupseteq c \ominus b \Leftrightarrow b \sqsupseteq c \ominus a$. Thus conditions (i) up to (iii) are established. Suppose $a \sqsubseteq a'$ then from $(a \oplus b) \ominus b \sqsubseteq a \sqsubseteq a'$ it follows by adjunction that $a \oplus b \sqsubseteq a' \oplus b$ and then $b \sqsubseteq b' \Rightarrow a \oplus b \sqsubseteq a \oplus b'$ by commutativity of the sum. By monotony of the sum one can now deduce $a \sqsubseteq a' \sqsubseteq (a' \ominus b) \oplus b \sqsubseteq (a' \ominus b) \oplus b'$ and then $a \ominus b' \sqsubseteq a' \ominus b$ when $a \sqsubseteq a' \wedge b \sqsubseteq b'$. For the converse direction, let us now assume that the conditions (i) up to (iv) are satisfied. Then $a \oplus b \sqsupseteq c$ entails that $a \sqsupseteq (a \oplus b) \ominus b \sqsupseteq c \ominus b$ and in the converse direction $a \sqsupseteq c \ominus b$ entails $a \oplus b \sqsupseteq (c \ominus b) \oplus b \sqsupseteq c$.

We see that under the assumptions of this lemma, each pair a, b of elements has an upper bound $(a \ominus b) \oplus b$ and a lower bound $a \ominus (a \ominus b)$. Proposition (1) states that the divisibility of the monoid is equivalent of the fact that the former is the least upper bound $a \sqcup b$. We will see below that the cancellability of the monoid is equivalent to the fact that the inequality in equation (i) in the above lemma is an equality and it entails that $a \ominus (a \ominus b)$ is the greatest lower bound $a \sqcap b$.

Proof (of Proposition 1). The implications $(ii) \Rightarrow (iii) \Rightarrow (i)$ are immediate. It remains to prove that $(i) \Rightarrow (ii)$. Let us therefore assume that $(A, \oplus, 0)$ is a commutative co-residuated divisibility monoid. The assumption of Lemma (1) are met and we can therefore deduce that $(a \ominus b) \oplus b \sqsupseteq a, b$. Suppose that $c \sqsupseteq a, b$ is some upper bound of a and b . By divisibility there exist some x and y such that $c = a \oplus x = b \oplus y$ and by adjunction it follows that $y \sqsupseteq (a \oplus x) \ominus b$ and thus $c = b \oplus y \sqsupseteq b \oplus ((a \oplus x) \ominus b) \sqsupseteq b \oplus (a \ominus b)$ (by monotony of \oplus and of \ominus w.r.t. its first argument and using the identity $a \oplus x \sqsupseteq a$).

In view of the above discussion we are let to set the following

Definition 1. A Petri pre-structure is a commutative co-residuated divisibility monoid i.e. a commutative monoid equipped with a residuation operation $(M, \oplus, 0, \ominus)$ satisfying the conditions (1) and (4).

The firing of a transition proceeds in two stages: a consumption of resources in the input places followed by a production of resources in the output places. More precisely, the transition relation $M [t] M'$ stating that transition t can fire in marking M and leads, when it is fired, to the new marking M' is given by:

$$M [t] M' \Leftrightarrow \forall p \in P \quad M(p) \supseteq \text{Pre}(p, t) \wedge M'(p) = (M(p) \ominus \text{Pre}(p, t)) \oplus \text{Post}(p, t)$$

A net is called *homogeneous* if all the algebras A_p are identical. We will stick to homogeneous nets until Section 3 where it will be noticed that the "multi-sorted" case add in fact no extra generality. By the way we also restrict our attention in this paper to commutative algebras. With non commutative monoids it would be possible [3] for example to take fifo nets [27] into account.

2.2 Associativity of the firing rule

For any non empty sequence of transitions $u = a_0 \dots a_{n-1} \in T^+$ we let $M [u] M'$ state the existence of markings $M = M_0, M_1, \dots, M_n = M'$ such that $M_i [a_i] M_{i+1}$ for every $0 \leq i < n$. Moreover we set $M [\varepsilon] M$ where $\varepsilon \in T^*$ is the empty sequence and M an arbitrary marking. We use $M [u]$ (respectively $[u] M'$) as a shorthand for $\exists M' \quad M [u] M'$ (resp. $\exists M \quad M [u] M'$). If $a, b \in T$ are transitions in a (n usual) Petri net we have the following equivalences

$$\begin{aligned} M [ab] &\Leftrightarrow M \supseteq \text{Pre}(a) \text{ and } (M - \text{Pre}(a)) + \text{Post}(a) \supseteq \text{Pre}(b) \\ &\Leftrightarrow M \supseteq \max(\text{Pre}(a); \text{Pre}(a) + (\text{Pre}(b) - \text{Post}(a))) \\ &\Leftrightarrow M \supseteq \text{Pre}(a) + \max(0; \text{Pre}(b) - \text{Post}(a)) \\ &\Leftrightarrow M \supseteq \text{Pre}(a) + (\text{Pre}(b) \ominus \text{Post}(a)) \end{aligned}$$

This suggest to let for any sequences $u, v \in T^*$

$$\text{Pre}(uv) = \text{Pre}(u) \oplus (\text{Pre}(v) \ominus \text{Post}(u))$$

and symmetrically

$$\text{Post}(uv) = (\text{Post}(u) \ominus \text{Pre}(v)) \oplus \text{Post}(v)$$

For these definitions to make sense however, it remains to show that they do not depend upon the specific chosen decomposition $w = uv$; otherwise stated, the product defined on $A \times A$ by

$$(x, y) \otimes (x', y') = (x \oplus (x' \ominus y), (y \ominus x') \oplus y')$$

should be associative.

Theorem 1. *For any Petri pre-structure, the following conditions are equivalent:*

- (i) *Operation \otimes is associative,*
- (ii) *the identity $(b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b))$ holds,*
- (iii) *the monoid is cancellable: $a \oplus b = a \oplus c \Rightarrow b = c$, and*
- (iv) *the identity $(a \oplus b) \ominus b = a$ holds.*

The remaining part of this section is devoted to the proof of this result. As already noticed the divisibility of the monoid (i.e. $a \sqsubseteq b \Leftrightarrow \exists c \cdot b = a \oplus c$) entails the following identities:

$$\begin{aligned} (\text{div1}) & : a \oplus b \sqsupseteq a \\ (\text{div2}) & : 0 \sqsubseteq a \\ (\text{div3}) & : a \oplus b = 0 \Rightarrow a = b = 0 \end{aligned}$$

the second of which states that the neutral element for \oplus is the least element. This property in conjunction with condition (4) has the following interesting consequences as it can be readily verified:

$$\begin{aligned} (\text{int1}) & : a \sqsubseteq b \Leftrightarrow a \ominus b = 0 \\ (\text{int2}) & : a \ominus 0 = a \\ (\text{int3}) & : 0 \ominus a = 0 \\ (\text{int4}) & : a \ominus a = 0 \\ (\text{int5}) & : a \ominus b \sqsubseteq a \end{aligned}$$

The equivalence between the cancellability of the monoid and the identity

$$(a \oplus b) \ominus b = a \tag{5}$$

is well-known for residuated lattices and it holds also trivially in our case: on one side $((a \oplus b) \ominus b) \oplus b = (a \oplus b) \sqcup b = a \oplus b$ shows that the identity $(a \oplus b) \ominus b = a$ follows from cancellability, conversely if this identity holds then $a \oplus b = a \oplus c \Rightarrow b = (a \oplus b) \ominus a = (a \oplus c) \ominus a = c$. The following proposition establishes the equivalence between the first two statements in Theorem (1).

Proposition 2. *In a Petri pre-structure associativity of \otimes is equivalent to the identity*

$$(b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b)) \tag{8}$$

Proof. For the one hand we compute $(x, y) \otimes [(x', y') \otimes (x'', y'')] =$

$$\begin{aligned} & = (x, y) \otimes [x' \oplus (x'' \ominus y'), (y' \ominus x'') \oplus y''] = \\ & = (x \oplus [(x' \oplus (x'' \ominus y')) \ominus y], [y \ominus (x' \oplus (x'' \ominus y'))] \oplus (y' \ominus x'') \oplus y'') \end{aligned}$$

and, on the other hand $[(x, y) \otimes (x', y')] \otimes (x'', y'') =$

$$\begin{aligned} & = [x \oplus (x' \ominus y), (y \ominus x') \oplus y'] \otimes (x'', y'') = \\ & = (x \oplus (x' \ominus y) \oplus [x'' \ominus ((y \ominus x') \oplus y')], [((y \ominus x') \oplus y') \ominus x''] \oplus y'') \end{aligned}$$

Associativity of \otimes is equivalent to the conjunction of both following identities

$$\begin{aligned} x \oplus [(x' \oplus (x'' \ominus y')) \ominus y] &= x \oplus (x' \ominus y) \oplus [x'' \ominus ((y \ominus x') \oplus y')] \\ [((y \ominus x') \oplus y') \ominus x''] \oplus y'' &= [y \ominus (x' \oplus (x'' \ominus y'))] \oplus (y' \ominus x'') \oplus y'' \end{aligned}$$

which are in fact equivalent to each other by commutativity of \oplus , and they can be both rewritten as:

$$x \oplus [(y \oplus (z \ominus t)) \ominus u] = x \oplus (y \ominus u) \oplus [z \ominus ((u \ominus y) \oplus t)] \quad (6)$$

By letting $x = t = 0$, $u = a$, $y = b$ and $z = c$ in the preceding identity it comes that

$$(b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b)) \quad (8)$$

Conversely, the latter suffices to establish identity (6) since:

$$\begin{aligned} x \oplus [(y \oplus (z \ominus t)) \ominus u] &= x \oplus (y \ominus u) \oplus [(z \ominus t) \ominus (u \ominus y)] \quad \text{By (8)} \\ &= x \oplus (y \ominus u) \oplus [z \ominus (t \oplus (u \ominus y))] \end{aligned}$$

The last equality comes from the identity $(a \ominus b) \ominus c = a \ominus (b \oplus c)$ which follows from the following equivalences

$$(a \ominus b) \ominus c \sqsubseteq d \Leftrightarrow a \ominus b \sqsubseteq c \oplus d \Leftrightarrow a \sqsubseteq b \oplus c \oplus d \Leftrightarrow a \ominus (b \oplus c) \sqsubseteq d$$

Proposition 3. *A Petri pre-structure satisfying the identity $(b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b))$ has a cancellable monoid reduct $(a \oplus b = a \oplus c \Rightarrow b = c)$ and it satisfies the following equivalence*

$$a \supseteq b \oplus c \Leftrightarrow a \supseteq b \text{ and } a \ominus b \supseteq c \quad (7)$$

Proof. By (int1) $a \supseteq b \oplus c$ is equivalent to $(b \oplus c) \ominus a = 0$, that is to say, by (8), to $(b \ominus a) \oplus (c \ominus (a \ominus b)) = 0$. By (div3) this latter condition is equivalent to the conjunction of $b \ominus a = 0$ with $c \ominus (a \ominus b) = 0$, and by (int1), to the conjunction of $a \supseteq b$ with $a \ominus b \supseteq c$. Now the conjunction of (7) with (4) allows us to conclude that

$$a \supseteq b \Rightarrow [a = b \oplus c \Leftrightarrow c = a \ominus b]$$

which establish the cancellability of the monoid: Let us assume $a \oplus b = a \oplus c$, let d be this value, then $d \supseteq a$ and $b = d \ominus a = c$.

Cancellability of the monoid is then a necessary condition if we want to ensure associativity of the firing rule. Thus we let

Definition 2. *A Petri algebra is a Petri pre-structure with a cancellable monoid reduct; i.e. it is a co-residuated cancellable commutative monoid.*

Proposition 4. *A Petri algebra is a lower semi-lattice with $a \sqcap b = a \ominus (a \ominus b)$*

Proof. By Lemma (1) we have that $a \ominus (a \ominus b) \sqsubseteq a, b$. Let $c \sqsubseteq a, b$ be some lower bound of a and b . By monotony of residuation one has $a \ominus b \sqsubseteq a \ominus c$ from which it follows that $(a \ominus b) \oplus c \sqsubseteq (a \ominus c) \oplus c = a \sqcup c = a$ (since $c \sqsubseteq a$). Then by cancellability $c = (c \oplus (a \ominus b)) \ominus (a \ominus b) \sqsubseteq a \ominus (a \ominus b)$.

We recall that a commutative residuated lattice (see [5, 24]) consists of a set A equipped with a structure of commutative monoid $(A, \oplus, 0)$, of a lattice structure (A, \vee, \wedge) and of a residuation operation \ominus adjoint to the addition in the sense that

$$a \oplus b \leq c \Leftrightarrow a \leq c \ominus b$$

where \leq is the lattice order (i.e. $a \leq b \Leftrightarrow b = a \vee b$). By Propositions (1) and (4) the dual of a Petri algebra is a commutative residuated lattice. Thus the following identities, which are obtained by dualizing¹ known identities of commutative residuated lattices, hold (and moreover can be straightforwardly checked):

$$\begin{aligned} (RL1) \quad & a \oplus (b \sqcap c) = (a \oplus b) \sqcap (a \oplus c) \\ (RL2) \quad & (a \sqcup b) \ominus c = (a \ominus c) \sqcup (b \ominus c) \\ (RL3) \quad & a \ominus (b \sqcap c) = (a \ominus b) \sqcup (a \ominus c) \\ (RL4) \quad & (a \ominus b) \oplus b \sqsupseteq a \\ (RL5) \quad & a \ominus (a \ominus b) \sqsubseteq b \\ (RL6) \quad & a \oplus (b \ominus c) \sqsupseteq (a \oplus b) \ominus c \\ (RL7) \quad & (c \ominus b) \oplus (b \ominus a) \sqsupseteq c \ominus a \\ (RL8) \quad & c \ominus b \sqsupseteq (c \ominus a) \ominus (b \ominus a) \\ (RL9) \quad & b \ominus a \sqsupseteq (c \ominus a) \ominus (c \ominus b) \\ (RL10) \quad & c \ominus b \sqsupseteq (c \oplus a) \ominus (b \oplus a) \\ (RL11) \quad & (c \ominus a) \ominus b = c \ominus (a \oplus b) = (c \ominus b) \ominus a \end{aligned}$$

as well as the monotony property asserting that $x \oplus y$ is increasing w.r.t. both arguments and $x \ominus y$ is increasing w.r.t. x and decreasing w.r.t. y .

The last stage toward the proof of Theorem (1) is given by the following proposition

Proposition 5. *Petri algebras satisfy the following identity*

$$(b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b)) \quad (8)$$

the proof of which follows from a sequence of intermediate results starting from the following property:

Lemma 2. *In any Petri algebra*

1. If $a \sqsupseteq b$ then (i) $a = b \oplus c \Leftrightarrow a \ominus b = c$ and (ii) $a \sqsupseteq b \oplus c \Leftrightarrow a \ominus b \sqsupseteq c$
2. If $a \sqsupseteq c$ then (i) $a \ominus b = c \Leftrightarrow a \ominus c = a \sqcap b$ and (ii) $a \ominus b \sqsupseteq c \Leftrightarrow a \ominus c \sqsupseteq a \sqcap b$
3. If $a \sqsupseteq b$ and $a \sqsupseteq c$ then (i) $a = b \oplus c \Leftrightarrow a \ominus b = c \Leftrightarrow a \ominus c = b$ and (ii) $a \sqsupseteq b \oplus c \Leftrightarrow a \ominus b \sqsupseteq c \Leftrightarrow a \ominus c \sqsupseteq b$
4. $a \ominus b = (a \sqcup b) \ominus b$ and $b = (a \sqcup b) \ominus (a \ominus b)$
5. $b = (a \sqcap b) \oplus (b \ominus a)$ and $b \ominus a = b \ominus (a \sqcap b)$

Proof. 1.(ii): If $a \sqsupseteq b \oplus c$ then $a \ominus b \sqsupseteq (c \oplus b) \ominus b = c$ by cancellability. Conversely, let us assume that $a \ominus b \sqsupseteq c$, from $a \sqsupseteq b$ it follows from Prop.1 that $a = (a \ominus b) \oplus b$

¹ We inverse the order relation, which means that we replace \leq, \vee, \wedge respectively by $\sqsupseteq, \sqcap, \sqcup$ in every identity valid for the class of commutative residuated lattices.

and thus $a \sqsupseteq c \oplus b$. 1.(i) follows from 1.(ii) together with the adjunction :
 $a \sqsubseteq b \oplus c \Leftrightarrow a \oplus b \sqsubseteq c$

2.(i): If $a \oplus b = c$ then $a \oplus c = a \oplus (a \oplus b) = a \sqcap b$. Conversely, let us assume that $a \oplus c = a \sqcap b$, from $a \sqsupseteq c$ it follows that $c = a \sqcap c = a \oplus (a \oplus c) = a \oplus (a \sqcap b) = a \oplus b$. Actually $a \oplus (a \sqcap b) =$ [by RL3] $(a \oplus a) \sqcup (a \oplus b) =$ [by int4] $0 \sqcup (a \oplus b) =$ [by div2] $a \oplus b$

2.(ii): (Similar to 2.(i)) If $a \oplus b \sqsupseteq c$ then $a \oplus c \sqsupseteq a \oplus (a \oplus b) = a \sqcap b$. Conversely, let us assume that $a \oplus c \sqsupseteq a \sqcap b$, from $a \sqsupseteq c$ it follows that $c = a \sqcap c = a \oplus (a \oplus c) \sqsubseteq a \oplus (a \sqcap b) = a \oplus b$.

3: Follows from 1.

4: Follows from 3.(i). since $a \sqcup b = (a \oplus b) \oplus b$, $a \sqcup b \sqsupseteq a \sqsupseteq a \oplus b$ and $a \sqcup b \sqsupseteq b$.

5: Follows from 3.(i). since $a \sqcap b = b \oplus (b \oplus a)$, $b \sqsupseteq a \sqcap b$ and $b \sqsupseteq b \oplus a$.

Lemma 3. *The following conditions are equivalent in any Petri algebra*

1. $(b \oplus c) \oplus a = (b \oplus a) \oplus (c \oplus (a \oplus b))$
2. $c \oplus (a \oplus b) = ((b \oplus c) \oplus a) \oplus (b \oplus a)$
3. $[(b \oplus c) \oplus a] \oplus [c \oplus (a \oplus b)] = b \oplus a$

Proof. Let $\alpha = (b \oplus c) \oplus a$, $\beta = b \oplus a$ and $\gamma = c \oplus (a \oplus b)$. One has $\alpha \sqsupseteq \beta$ by monotony and (div1), and $\alpha \sqsupseteq \gamma$ follows from (RL8) and identity (5):

$$(b \oplus c) \oplus a \sqsupseteq ((b \oplus c) \oplus b) \oplus (a \oplus b) = c \oplus (a \oplus b)$$

Thus by Lemma 2.(3.i) it follows that $\alpha = \beta \oplus \gamma \Leftrightarrow \alpha \oplus \beta = \gamma \Leftrightarrow \alpha \oplus \gamma = \beta$.

Lemma 4. *In any Petri algebra*

1. $b \sqsupseteq c \Rightarrow a \oplus (b \oplus c) = (a \oplus b) \oplus c$
2. $c \sqsupseteq b \Rightarrow (a \oplus b) \oplus c = a \oplus (c \oplus b)$

Proof. If $b \sqsupseteq c$ then $(b \oplus c) \oplus c = b$ by Prop.1 and thus $a \oplus (b \oplus c) =$ [By 5] $(a \oplus (b \oplus c) \oplus c) \oplus c = (a \oplus b) \oplus c$. If $c \sqsupseteq b$ then $c \oplus (c \oplus b) = c \sqcap b = b$ and thus $a \oplus b = a \oplus (c \oplus (c \oplus b)) = (a \oplus c) \oplus (c \oplus b)$ using the preceding point with $c \sqsupseteq c \oplus b$. It follows $(a \oplus b) \oplus c = ((a \oplus c) \oplus (c \oplus b)) \oplus c =$ [By RL11] $((a \oplus c) \oplus c) \oplus (c \oplus b) =$ [By 5] $a \oplus (c \oplus b)$.

Lemma 5. *The following identity holds in any Petri algebra*

$$c \oplus (a \oplus b) = ((b \oplus c) \oplus a) \oplus (b \oplus a)$$

Proof. $((b \oplus c) \oplus a) \oplus (b \oplus a) =$

$$\begin{aligned} &= [(b \oplus c) \oplus ((a \sqcap b) \oplus (a \oplus b))] \oplus (b \oplus a) && \text{Lemma 2.(5)} \\ &= [((b \oplus c) \oplus (a \sqcap b)) \oplus (a \oplus b)] \oplus (b \oplus a) && \text{RL11} \\ &= [(c \oplus (b \oplus (a \sqcap b))) \oplus (a \oplus b)] \oplus (b \oplus a) && \text{Lemma 4.(1) since } b \sqsupseteq a \sqcap b \\ &= ((c \oplus (b \oplus a)) \oplus (a \oplus b)) \oplus (b \oplus a) && \text{Lemma 2.(5)} \\ &= ((c \oplus (b \oplus a)) \oplus (b \oplus a)) \oplus (a \oplus b) && \text{RL11} \\ &= c \oplus (a \oplus b) && (5) \end{aligned}$$

The proof of Proposition (5) and therefrom of Theorem (1) follows from lemma (3) and lemma (5).

Corollary 1. *Petri algebras satisfy the following equivalence*

$$a \sqsupseteq b \oplus c \Leftrightarrow a \sqsupseteq b \text{ and } a \ominus b \sqsupseteq c \quad (9)$$

Proof. Follows from Proposition (3) and Theorem (1)

Identity (8) is an internalization of (9) using the axiomatization of the order relation: $a \sqsubseteq b \Leftrightarrow a \ominus b = 0$.

2.3 Additional properties of the firing rule

Let us consider a net over a Petri algebra A , then we can inductively define the applications $Pre, Post : P \times T^* \rightarrow A$ by letting $\varphi(p, u) = (Pre(p, u), Post(p, u))$ where $\varphi(p, -) : T^* \rightarrow A \times A$ is the unique monoid morphism such that the images $\varphi(p, t) = (Pre(p, t), Post(p, t))$ of the generators $t \in T$ be given by the flow relations of the net. Then

$$\begin{aligned} Pre(p, \varepsilon) &= Post(p, \varepsilon) = 0 \\ Pre(p, uv) &= Pre(p, u) \oplus (Pre(p, v) \ominus Post(p, u)) \\ Post(p, uv) &= (Post(p, u) \ominus Pre(p, v)) \oplus Post(p, v) \end{aligned}$$

Theorem 2. *The generalized transition relation $M [u] M'$ stating the existence of a sequence u of transitions leading from M to M' is given by any of the three following equivalent conditions*

1. $\forall p \in P \quad M(p) \sqsupseteq Pre(p, u) \text{ and } M'(p) = (M(p) \ominus Pre(p, u)) \oplus Post(p, u)$
2. $\forall p \in P \quad M'(p) \sqsupseteq Post(p, u) \text{ and } M(p) = (M'(p) \ominus Post(p, u)) \oplus Pre(p, u)$
3. $\forall p \in P \quad M(p) \sqsupseteq Pre(p, u) ; M'(p) \sqsupseteq Post(p, u) \text{ and } M(p) \ominus Pre(p, u) = M'(p) \ominus Post(p, u)$

Proof. On the one hand we have to show that the three above conditions are equivalent, and on the other hand that they do correspond to the generalized transition relation $M [u] M'$. To simplify the presentation of the proof, we adopt vectorial notations using the product algebra A^P which satisfies the same properties that A for the corresponding pointwise defined operations. It amounts to consider a net with only one place whose algebra is the product of the algebras of the original net.

Concerning the former point we thus have to establish the equivalence between the following assertions.

1. $M \sqsupseteq Pre(u) \text{ and } M' = (M \ominus Pre(u)) \oplus Post(u)$
2. $M' \sqsupseteq Post(u) \text{ and } M = (M' \ominus Post(u)) \oplus Pre(u)$
3. $M \sqsupseteq Pre(u) ; M' \sqsupseteq Post(u) \text{ and } M \ominus Pre(u) = M' \ominus Post(u)$

For that purpose we recall the equivalence (9)

$$a \sqsupseteq b \oplus c \Leftrightarrow a \sqsupseteq b \text{ and } a \ominus b \sqsupseteq c$$

which allows us to conclude that

$$[M' \sqsupseteq (M \ominus \text{Pre}(u)) \oplus \text{Post}(u)] \Leftrightarrow [M' \sqsupseteq \text{Post}(u) \text{ and } M' \ominus \text{Post}(u) \sqsupseteq M \ominus \text{Pre}(u)]$$

which, in turn, in conjunction with

$$[(M \ominus \text{Pre}(u)) \oplus \text{Post}(u) \sqsupseteq M'] \Leftrightarrow [M' \ominus \text{Post}(u) \sqsubseteq M \ominus \text{Pre}(u)]$$

that itself is an instance of condition (4) proves the equivalence between assertions (1) and (3) above. The equivalence between assertions (2) and (3) are proven similarly.

As far as the second point is concerned let us temporarily denote $M[u]M'$ the relation given by any of the three assertions that we have proved to be equivalent. We recall the identities

$$\begin{aligned} \text{Pre}(\varepsilon) &= \text{Post}(\varepsilon) = 0 \\ \text{Pre}(uv) &= \text{Pre}(u) \oplus (\text{Pre}(v) \ominus \text{Post}(u)) \\ \text{Post}(uv) &= (\text{Post}(u) \ominus \text{Pre}(v)) \oplus \text{Post}(v) \end{aligned}$$

We deduce that $M[\varepsilon]M' \Leftrightarrow M = M' \Leftrightarrow M[\varepsilon]M'$ where $\varepsilon \in T^*$ is the empty word, and by definition $M[t]M' \Leftrightarrow M[t]M'$ for $t \in T$ any transition. It remains to prove that $M[uv]M' \Leftrightarrow \exists M'' \cdot M[u]M''$ and $M''[v]M'$ in order to be able to deduce that $\{M[u]M' \mid u \in T^*\}$ is the transitive closure of the elementary transition relation $\{M[t]M' \mid t \in T\}$ and that it consequently coincides with $\{M[u]M' \mid u \in T^*\}$. For that purpose, let us observe that

$$(0, M) \otimes (\text{Pre}(u), \text{Post}(u)) = (\text{Pre}(u) \ominus M, (M \ominus \text{Pre}(u)) \oplus \text{Post}(u))$$

and thus

$$M[u]M' \Leftrightarrow (0, M') = (0, M) \otimes (\text{Pre}(u), \text{Post}(u))$$

(we recall that $a \sqsupseteq b \Leftrightarrow b \ominus a = 0$). By definition of the maps Pre and Post one has

$$(\text{Pre}(uv), \text{Post}(uv)) = (\text{Pre}(u), \text{Post}(u)) \otimes (\text{Pre}(v), \text{Post}(v))$$

which allows us to conclude, on the basis of the associativity of \otimes , that $(\exists M'' \cdot M[u]M''$ and $M''[v]M') \Rightarrow M[uv]M'$. In order to establish the converse direction it remains to verify that $M[uv] \Leftrightarrow (M[u]$ and $M * u[v])$ where $M * u$ stands for $(M \ominus \text{Pre}(u)) \oplus \text{Post}(u)$ and where $M[u]$ stands for $\exists M' \cdot M[u]M'$ that is to say $M \sqsupseteq \text{Pre}(u)$. By definition of \otimes one has $M[uv] \Leftrightarrow M \sqsupseteq \text{Pre}(uv) = \text{Pre}(u) \oplus (\text{Pre}(v) \ominus \text{Post}(u))$, which by (7) is equivalent to $[M \sqsupseteq \text{Pre}(u)$ and $M \ominus \text{Pre}(u) \sqsupseteq \text{Pre}(v) \ominus \text{Post}(u)]$ which, in turn by adjunction (equation 4) is equivalent to $[M \sqsupseteq \text{Pre}(u)$ and $(M \ominus \text{Pre}(u)) \oplus \text{Post}(u) \sqsupseteq \text{Pre}(v)]$, i.e. $M[u]M'$ and $M * u[v]$.

2.4 The variety of Petri algebras

We have so far identified the set of conditions that should be fulfilled by Petri algebras so that we can play the token game and the resulting firing rule is associative. To sum up, such a structure is a dual of a commutative residuated lattice whose joins and meets are given by the formulas $a \sqcup b = a \ominus (a \ominus b)$ (Prop. 1) and $a \sqcap b = b \oplus (a \ominus b)$ (Prop. 4). Moreover since the a Petri algebra is a divisibility monoid, this lattice is integral in the sense that the neutral element for the sum is also the least element of the lattice. Finally the underlying monoid is cancellable and this condition is equivalent to the identity (5).

Proposition 6. *A Petri algebra consists of a set A equipped with a structure of a commutative monoid $(A, \oplus, 0)$ and a structure of a lattice (A, \sqcup, \sqcap) such that:*

1. *the neutral element is the least element of the lattice: $0 \sqsubseteq a$.*
2. *Sum admits an adjoint \ominus characterized by the universal property: $a \oplus b \sqsupseteq c \Leftrightarrow a \sqsupseteq c \ominus b$*
3. *$(a \oplus b) \ominus b = a$.*
4. *$a \sqcap b = a \ominus (a \ominus b)$.*
5. *$a \sqcup b = b \oplus (a \ominus b)$.*

Proof. It remains to prove that a structure satisfying the conditions stated in the proposition is a Petri algebra. It is a co-residuated commutative monoid (Condition 2). Integrality (Condition 1) together with Condition (5) implies by Prop. (1) that it is a divisibility monoid which is a cancellable monoid due to Condition (3).

Recall that a commutative GMV-algebra ([4, 21]) is a commutative residuated lattice such that $a \vee b = a \ominus ((a \ominus b) \wedge 0)$; it also satisfies $a \wedge b = ((a \ominus b) \wedge 0) \oplus b$. A GMV-algebra is said to be *integral* when the neutral element is the greatest element of the lattice (hence the least element for the inverse order relation \sqsupseteq). Thus we conclude that Petri algebras coincide with the (duals of) *integral, cancellative and commutative GMV-algebras*. These algebras form a subvariety of the variety of residuated lattices and the following result is a direct consequence of [24, Theorem 5.6 and corollaries].

Theorem 3. *Petri algebras coincide with the positive cones of lattice-ordered abelian groups. Moreover lattice-ordered abelian groups constitute the subvariety of lattice-ordered groups generated by the group \mathbb{Z} of integers, and their positive cones (i.e. Petri algebras) is the subvariety of residuated lattices generated by \mathbb{N} .*

To give a better understanding of this result we recall that if $(G, +, 0)$ is an abelian group equipped with an order relation compatible with its addition (i.e. addition is increasing w.r.t. both of its arguments) then the positive cone $P = \{x \in G \mid x \geq 0\}$ is a submonoid of G which is *zerosumfree* ($x + y = 0 \Rightarrow x = y = 0$). Conversely any zerosumfree submonoid P of G induces a compatible order relation on G whose corresponding positive cone is P : $x \sqsubseteq y \Leftrightarrow \exists z \in P \cdot y = x + z$. Of course a submonoid of a group is necessarily cancellable. If moreover G is a lattice-ordered group, meaning that each pair $\{a; b\}$ admits a

join $a \sqcup b$ and a meet $a \sqcap b$ for the corresponding order relation, then the positive cone is residuated with $a \ominus b = (a - b) \sqcup 0$ (where $a - b = a + (-b)$) and is also lattice-ordered. For instance the monoid $(\mathbb{N}, +, 0)$ is the positive cone of the group $(\mathbb{Z}, +, 0)$ for the usual order relation on integers; this is a total order and thus in particular a lattice and therefore the preceding considerations apply. We know that through linear algebra techniques this group of integers plays a central role in the study of the structural properties of Petri nets ([34]: the fundamental equation, linear invariants, siphons, traps ...) and as well in the algorithmic implementation of Petri nets synthesis based on the theory of regions [2].

3 Lexicographic Petri nets

3.1 Subdirectly irreducible Petri algebras

We define a (generalized) Petri net as a structure $\mathcal{N} = (P, T, Pre, Post, M_0)$ where P is a finite set of places with a Petri algebra A_p associated with each place $p \in P$, T is a finite set of transitions disjoint from P and $Pre, Post : P \times T \rightarrow \bigsqcup_{p \in P} A_p$, the flow relations, are such that $\forall p \in P \forall t \in T \quad Pre(p, t), Post(p, t) \in A_p$. A marking is a map $M : P \rightarrow \bigsqcup_{p \in P} A_p$ that associates with each place $p \in P$ the local value of the current configuration $M(p) \in A_p$ in this place. M_0 is some fixed marking, called the *initial marking*. The transition relation $M [t] M'$ stating that transition t can fire in marking M and leads, when it is fired, to the new marking M' is given by:

$$M [t] M' \Leftrightarrow \forall p \in P \quad M(p) \supseteq Pre(p, t) \wedge M'(p) = (M(p) \ominus Pre(p, t)) \oplus Post(p, t)$$

This relation can be extended inductively to sequences $u \in T^*$ of transitions by letting $M [\varepsilon] M$ for every marking M and $M [t \cdot u] M'$ if and only if there exists some marking M'' such that $M [t] M''$ and $M'' [u] M'$ for every $t \in T$ and $u \in T^*$. We let $Reach(\mathcal{N}) = \{M \mid \exists u \in T^* \quad M_0 [u] M\}$ stand for the set of *reachable markings*, and the *marking graph* of a generalized net $\mathcal{N} = (P, T, Pre, Post, M_0)$ is then the labelled graph $\Gamma_{\mathcal{N}} = (V, A, v_0)$ whose set of vertices is given by the set $V = Reach(\mathcal{N})$ of reachable markings with $v_0 = M_0$ and whose set of arcs $A \subseteq V \times T \times V$ is the restriction of the transition relation to the set of reachable markings: $A = \{(M, t, M') \mid M, M' \in V \wedge M [t] M'\}$.

Definition 3. *Two generalized Petri nets are termed equivalent when they have isomorphic marking graphs.*

We immediately see that a place p whose type A_p is a subalgebra of a product of Petri algebras ($A_p \subseteq A_1 \times \dots \times A_n$) can be replaced by n places p_1, \dots, p_n with respective types A_1, \dots, A_n without changing the marking graph (at least up to isomorphism). A classical result of universal algebra due to Birkhoff says that any algebra of a variety is a subdirect product of subdirectly irreducible algebras ². Thus we can assume without loss of generality that all algebras

² An algebra A is a subdirect product of a family of similar algebras $\{A_i \mid i \in I\}$ if A is a subalgebra of the product algebra $\prod_{i \in I} A_i$ such that for every index $i \in I$ the

A_p are subdirectly irreducible algebras in the variety of Petri algebras. Now any $M(p)$ belongs to the subalgebra of A_p generated by the set $\{M_0(p)\} \cup \bigcup_{t \in T} \{Pre(p, t), Post(p, t)\}$. Thus:

Theorem 4. *Every generalized Petri net is equivalent to a generalized Petri net all of whose types are finitely generated subalgebras of subdirectly irreducible Petri algebras.*

Let $Irr(V)$ denote the set of subdirectly irreducible algebras of a variety V , then if V is a subvariety of W one has $Irr(W) \cap V = Irr(V)$; using the fact that the subdirectly irreducible commutative GMV-algebras are chains (simply ordered sets) we deduce that

Proposition 7. *subdirectly irreducible Petri algebras are chains.*

An algebra is subdirectly irreducible if and only if it admits a least non trivial congruence [5]. Now we know [6, 7] that the congruences of Petri algebras are in bijective correspondance with their convex submonoids. On the one direction we can associate each congruence θ of a Petri algebra A with the class of the neutral element which is a convex submonoid $M_\theta = [0]_\theta$ of A . Conversely we associate each such monoid M to the congruence $\theta_M = \{(a, b) \in A^2 \mid b \oplus a, a \oplus b \in M\}$. The correspondances $\theta \mapsto M_\theta$ et $M \mapsto \theta_M$ are inverses to each other and they establish an isomorphism between the lattice of congruences of A and the lattice of the convex submonoids of A . Moreover for every $a \in A$, the principal congruence generated by the equation $a = 0$ corresponds to the convex submonoid generated by a . A Petri algebra is then subdirectly irreducible if and only if it admits a least non trivial convex submonoid. Let us assume that A is a totally ordered Petri algebra. Let

$$M(x) = \{y \in A \mid \exists k \in \mathbb{N} \cdot y \sqsubseteq k \cdot x = \underbrace{x \oplus \cdots \oplus x}_{k \text{ times}}\}$$

denote the principal convex submonoid generated by $x \in A$. $M(x)$ is non-trivial if and only if $x \neq 0$. Now if x is some element of a convex submonoid M of A , one necessarily have $M(x) \subseteq M$; thus a minimal convex submonoid is principal and is generated by any of its non null elements. Since A is totally ordered and $x \leq y \Rightarrow M(x) \subseteq M(y)$ we deduce that A admits at most one minimal non-trivial submonoid. $M(x)$ is minimal if and only if $y \ll x \Rightarrow y = 0$ where relation \ll is given by

$$y \ll x \Leftrightarrow \forall k \in \mathbb{N} \cdot k \cdot y \sqsubset x$$

morphism $p_i : A \rightarrow A_i$, obtained by composition of the embedding of A into $\prod_{i \in I} A_i$ with the canonical projection $\prod_{i \in I} A_i \rightarrow A_i$, is surjective. The set $\{(A_i, p_i) \mid i \in I\}$ is a representation of A as subdirect product of the algebras A_i . Symetrically such a representation can be given as a family of surjective morphisms $p_i : A \rightarrow A_i$ jointly injective, that is to say such that the product $\langle p_i \rangle : A \rightarrow \prod_{i \in I} A_i$ is injective. An algebra A is say to be subdirectly irreducible if for every representation $\{(A_i, p_i) \mid i \in I\}$ of A as a subdirect product of algebras A_i there exists at least one index i for which the corresponding morphism p_i is an isomorphism.

Otherwise stated $y \ll x$ if and only if $y \sqsubset x$ and $M(y)$ is strictly included in $M(x)$. Therefore A has no non trivial minimal submonoid if and only if for every $x \in A \setminus \{0\}$ one can find some $y \in A \setminus \{0\}$ such that $y \ll x$. Under that condition one can form an infinite strictly decreasing chain thus proving that the order relation \ll is not well-founded³. Conversely if this order is well-founded then any non-empty subset of A , and thus in particular $A \setminus \{0\}$ if A is not trivial, admits a least element for this order which shows the existence of a minimal non-trivial submonoid. We thus have established the following:

Theorem 5. *A Petri algebra is subdirectly irreducible if and only if it is a chain and the order relation $y \ll x \Leftrightarrow \forall k \in \mathbb{N} \cdot k \cdot y \sqsubset x$ is well-founded.*

3.2 Lexicographic Petri nets

The lexicographic product $\mathbb{G} \circ \mathbb{H}$ of two ordered groups \mathbb{G} and \mathbb{H} is the product group $\mathbb{G} \times \mathbb{H}$ equipped with the lexicographic order relation:

$$(x, y) \leq_{\mathbb{G} \circ \mathbb{H}} (x', y') \Leftrightarrow x <_{\mathbb{G}} x' \text{ or } (x = x' \text{ and } y \leq_{\mathbb{H}} y')$$

If \mathbb{G} and \mathbb{H} are totally ordered abelian groups then the same holds for their lexicographic product. This product is associative and we can define inductively $L_n(\mathbb{G}) = (\mathbb{G}^n)^+$ for every simply ordered abelian group \mathbb{G} and integer $n \in \mathbb{N}$ by letting $\mathbb{G}^0 = \{0\}$ be the trivial group and $\mathbb{G}^{n+1} = \mathbb{G}^n \circ \mathbb{G}$ and where \mathbb{G}^+ denote the positive cone of group \mathbb{G} . The group \mathbb{G}^n naturally embeds into \mathbb{G}^m when $n \leq m$; the projective limit of this sequence of embeddings is the group \mathbb{G}^ω whose elements are the infinite sequences of elements in \mathbb{G} , with componentwise composition and the lexicographic order relation defined as follows: $u \leq_{lex} v \Leftrightarrow u <_{lex} v$ or $u = v$ where

$$u <_{lex} v \Leftrightarrow \exists n \in \mathbb{N} \forall m \leq n \ u_m = v_m \text{ and } u_n <_{\mathbb{G}} v_n$$

The inductive limit, or "union" $\bigcup_{n < \omega} \mathbb{G}^n$, is the subgroup of \mathbb{G}^ω consisting of the sequences u of finite support ($supp(u) = \sup \{k \in \mathbb{N} \mid u_k \neq 0\} < \omega$) with \mathbb{G}^n identified with the subgroup of $u \in \mathbb{G}^\omega$ such that $supp(u) \leq n$.

Definition 4. *The set $Lex(\mathbb{G})$ of lexicographic Petri nets based on a totally ordered abelian group \mathbb{G} is the set of (homogeneous) generalized Petri nets of type $(\mathbb{G}^\omega)^+$. $Lex(\mathbb{G}, n) \subseteq Lex(\mathbb{G})$ is the set of n -dimensional lexicographic Petri nets with type $L_n(\mathbb{G}) = (\mathbb{G}^n)^+ \subseteq (\mathbb{G}^\omega)^+$, i.e. all flow arc inscriptions and initial place contents, and hence all place contents in every accessible marking, are element in $(\mathbb{G}^n)^+$.*

³ We recall that an order relation is well-founded if it satisfies any of the following equivalent conditions:

1. every decreasing infinite sequence is stationary,
2. every strictly decreasing sequence is finite,
3. every non-empty subset admits a least element.

An ordinal, or well-ordered set, is a set with a total and well-founded order.

If \mathcal{K} and \mathcal{L} are subclasses of generalized Petri nets we let $\mathcal{K} \lesssim \mathcal{L}$ when every net in \mathcal{K} is equivalent to some net in \mathcal{L} . This is a pre-order relation, we let \approx denote its associated equivalence relation and \lesssim the corresponding strict relation: $\mathcal{K} \lesssim \mathcal{L}$ when every net in \mathcal{K} is equivalent to some net in \mathcal{L} but there exists some net in \mathcal{L} not equivalent to any net in \mathcal{K} . Notice that $Lex(\mathbb{G}, n) \lesssim Lex(\mathbb{H}, m)$ when $\mathbb{G} \subseteq \mathbb{H}$ and $n \leq m$; and that $Lex(\mathbb{Z}, 1)$ is the class of Petri nets.

Lemma 6. *Any finitely generated subalgebra of a subdirectly irreducible Petri algebra A is isomorphic to a subalgebra of the positive cone of some finite power of the additive group of real numbers: $A \subseteq (\mathbb{R}^n)^+$.*

Proof. A is a chain and we can partition its set of generators into $G = G_1 \cup \dots \cup G_n$ such that (i) $\forall i < j \leq n \ a_i \in G_i \wedge a_j \in G_j \Rightarrow a_i \ll a_j$ and (ii) all element in G_i are incomparable with respect to \ll : $a_i, a'_i \in G_i \Rightarrow \exists k \in \mathbb{N} \ a_i \sqsubseteq k \cdot a'_i$. Let A_i be the subalgebra of A generated by G_i . A_i is a simply ordered archimedean ($a \ll b \Rightarrow a = 0$) divisibility monoid (Archimedean systems of Magnitudes) and thus is isomorphic to an ordered submonoid of the additive monoid of positive real numbers ([5, 19]), hence $A_i \subseteq \mathbb{R}^+$. Now we check that A is a subalgebra of the lexicographic product of the A_i 's: $A \subseteq A_n \circ \dots \circ A_1$. For each decomposition $a = \sum_{g \in G} \lambda_g \cdot g$ of an element $a \in A$ in terms of the generators ($\lambda_g \in \mathbb{N}$) we obtain a related decomposition $a = a_n \oplus \dots \oplus a_1$ with $a_i \in A_i$ by letting $a_i = \sum_{g \in G_i} \lambda_g \cdot g$. Let us verify that such a decomposition is unique. We shall assume $n = 2$, the proof extends by induction for larger n . Thus we assume $a_1 \oplus a_2 = a'_1 \oplus a'_2$ with $a_i, a'_i \in A_i$. Since the order relation is total and the addition is monotonic we may assume that $a_1 \sqsubseteq a'_1$ and $a'_2 \sqsubseteq a_2$. Then

$$\begin{aligned} a_1 \oplus a_2 = a'_1 \oplus a'_2 &\Leftrightarrow a'_1 = (a_1 \oplus a_2) \ominus a'_2 \text{ By Lemma 2} \\ &\Leftrightarrow a'_1 = a_1 \oplus (a_2 \ominus a'_2) \text{ By Lemma 4} \\ &\Leftrightarrow a'_1 \ominus a_1 = a_2 \ominus a'_2 \text{ By Lemma 2} \end{aligned}$$

Now $a'_1 \ominus a_1 \in A_1$ and $a_2 \ominus a'_2 \in A_2$ and the only way that these elements can be comparable is when they are null, i.e. $a_1 = a'_1$ and $a_2 = a'_2$ as required. We let $\varphi_i : A \rightarrow A_i$ be the map such that $\varphi_i(a) = a_i$ where $a = a_n \oplus \dots \oplus a_1$ with $a_i \in A_i$. These maps are monoid morphisms, we let $\varphi : A \rightarrow A_n \circ \dots \circ A_1$ denote the monoid morphism obtained by collecting them: $\varphi(a) = (\varphi_n(a), \dots, \varphi_1(a))$. By construction φ is an injection with left-inverse $\psi : A_n \circ \dots \circ A_1 \rightarrow A$ given by $\psi(x_n, \dots, x_1) = x_n \oplus \dots \oplus x_1$. Thus φ induces an isomorphism between the monoid reduct of A and a submonoid of the monoid reduct of $A_n \circ \dots \circ A_1$. Now by definition of the lexicographic order we have that this injection φ preserves and reflects the respective order relations. Since residuation in a Petri algebra is characterized by the monoid structure and the order relation, we deduce that the image of A by φ is a subalgebra of $A_n \circ \dots \circ A_1$.

By Theorem 4 we deduce the following result.

Theorem 6. *Every generalized Petri net is equivalent to some lexicographic Petri net, more precisely:*

$$GenPetri \approx Lex(\mathbb{R})$$

We have in fact proved a more precise result, namely that every generalized Petri net is equivalent to a finite dimensional lexicographic Petri net based on the additive group of real numbers, i.e. $GenPetri \approx \bigcup_{n < \omega} Lex(\mathbb{R}, n)$.

Proposition 8. $Petri = Lex(\mathbb{Z}, 1) \lesssim GenPetri$

Proof. We provide an example showing that $Lex(\mathbb{Z}, 1) \lesssim Lex(\mathbb{Z}, 2)$, i.e. that lexicographic Petri nets based on the group of integers of dimension 2 are already strictly more expressive than the class of Petri nets. Let us consider the net of

Fig. 1. A two dimensional lexicographic Petri net

type

$$L_2 = (\mathbb{Z} \circ \mathbb{Z})^+ = \{(n, m) \mid (n = 0 \text{ and } m \geq 0) \text{ or } (n > 0 \text{ and } m \in \mathbb{Z})\}$$

depicted in Fig. 1. This net has a unique place $P = \{p\}$ and two transitions $T = \{a; b\}$ with the following flow relations $Pre(p, a) = (1, 0)$, $Pre(p, b) = (0, 1)$ and $Post(p, a) = Post(p, b) = (0, 0)$. From the initial marking $(1, 0)$ transition a can fire once $(1, 0) [a] (0, 0)$ and transition b can fire an infinite number of time leading to the infinite firing sequence

$$(1, 0) [b] (1, -1) [b] (1, -2) \dots [b] (1, -n) \dots$$

and there are no other transitions in the marking graph of the net. Suppose there exists some Petri net with an isomorphic marking graph. Since transition b can fire an infinite number of time, and

$$M_0 [b^n] M_n \Rightarrow \forall p \in P M_n(p) = M_0(p) - n \times (Pre(p, b) - Post(p, b))$$

we deduce that for every place p it is the case that $Post(p, b) \supseteq Pre(p, b)$ and thus $M_n(p) \supseteq M_0(p)$. By monotony of the firing rule, any transition that can fire in the initial marking M_0 can also fire in any of the markings M_n obtained by firing b . Transition a is in contradiction with this property.

By generalizing on the above we could establish that

$$n < m < \omega \Rightarrow Lex(\mathbb{Z}, n) \lesssim Lex(\mathbb{Z}, m) \lesssim Lex(\mathbb{Z})$$

it can also be shown that

$$Lex(\mathbb{Z}, n) \approx Lex(\mathbb{Q}, n) \lesssim Lex(\mathbb{R}, n)$$

3.3 Lexicographic Petri nets and Petri nets with inhibitor arcs

It appears to be difficult to obtain strict extensions of the class of Petri nets that preserve all of its decidable properties. Many of these extensions, like the class of Petri nets with inhibitor arcs, are indeed Turing-powerful.

Definition 5. A Petri net with inhibitor arcs is a structure $\mathcal{N} = (P, T, Pre, Post, M_0)$ where $P, T, Post,$ and M_0 (initial marking) are defined as for the usual Petri nets. Pre is a map from $P \times T$ to $\mathbb{N} \cup \{\zeta\}$ where ζ is a special symbol. We say that we have an inhibitor arc from place p to transition t when $Pre(p, t) = \zeta$. Transition t is firable in marking $M \in \mathbb{N}^P$ when

$$\forall p \in P \quad [Pre(p, t) \in \mathbb{N} \wedge M(p) \geq Pre(p, t)] \vee [Pre(p, t) = \zeta \wedge M(p) = 0]$$

i.e. they are enough resources in the input places $\bullet t = \{p \in P \mid Pre(p, t) > 0\}$ of transition t and all of its inhibiting places $\circ t = \{p \in P \mid Pre(p, t) = \zeta\}$ are empty. Then the firing of transition t leads to a new marking M' defined as follows:

$$\forall p \in P \quad M'(p) = \begin{cases} M(p) - Pre(p, t) + Post(p, t) & \text{if } Pre(p, t) \in \mathbb{N} \\ M(p) + Post(p, t) & \text{if } Pre(p, t) = \zeta \end{cases}$$

We let $M[t]M'$ express the fact that transition t can fire in marking M and leads to marking M' when it fires. This relation can be extended inductively to sequences $u \in T^*$ of transitions by letting $M[\varepsilon]M$ for every marking M and $M[t \cdot u]M'$ if and only if there exists some marking M'' such that $M[t]M''$ and $M''[u]M'$ for every $t \in T$ and $u \in T^*$. The set of reachable markings is $Reach(\mathcal{N}) = \{M \mid \exists u \in T^* M_0[u]M\}$, and the marking graph of a Petri net with inhibitor arcs $\mathcal{N} = (P, T, Pre, Post, M_0)$ is the labelled graph $\Gamma_{\mathcal{N}} = (V, A, v_0)$ whose set of vertices is given by the set $V = Reach(\mathcal{N})$ of reachable markings with $v_0 = M_0$ and whose set of arcs $A \subseteq V \times T \times V$ is the restriction of the transition relation to the set $A = \{(M, t, M') \mid M, M' \in V \wedge M[t]M'\}$ of reachable markings.

Fig. 2. A Petri net with inhibitor arcs and its marking graph

Figure (2) displays a Petri net with inhibitor arcs and its marking graph. Inhibitor arcs restrict the firing of transitions but have no effect on the content

of places when transitions are allowed to occur. This means that the marking graph of a Petri net with inhibitor arcs \mathcal{N} is a subgraph of the marking graph of the induced Petri net \mathcal{N}° obtained by removing the inhibitor arcs.

Fig. 3. A translation from Petri nets with inhibitor arcs into lexicographic Petri nets

Theorem 7. *Lexicographic Petri nets are a strict extension of the class of Petri nets with inhibitor arcs.*

Proof. The translation of a Petri net with inhibitor arcs \mathcal{N} into an equivalent lexicographic Petri net $\overline{\mathcal{N}}$, illustrated in Fig. (3), consists in splitting every place p with initial marking $m \in \mathbb{N}$ of the original net into two places denoted p and p' with initial markings $(0, m) \in (\mathbb{Z} \circ \mathbb{Z})^+$ and $(1, -m) \in (\mathbb{Z} \circ \mathbb{Z})^+$ respectively. The restriction of this lexicographic Petri net to set of places p of the first family is isomorphic to \mathcal{N}° , i.e. the original net in which the inhibitor arcs have been removed. We thus have a graph morphism $\Gamma_{\overline{\mathcal{N}}} \rightarrow \Gamma_{\mathcal{N}^\circ}$ sending a marking of the lexicographic Petri net to its restriction to the set of places of the original net \mathcal{N} . Now the set of reachable markings of $\overline{\mathcal{N}}$ satisfies the following property:

$$\mathcal{K}(M) \equiv \forall p \in P \quad \exists m_p \in \mathbb{N} \quad M(p) = (0, m_p) \wedge M(p') = (1, -m_p)$$

This property holds for the initial marking by definition and is preserved by the firing of a transition due to the fact that

$$x - y = \max(0, x - y) - \max(0, y - x)$$

Therefore this graph morphism is injective and one has a pair of graphs monomorphisms:

$$\Gamma_{\mathcal{N}} \xhookrightarrow{\iota} \Gamma_{\mathcal{N}^\circ} \xhookrightarrow{\iota'} \Gamma_{\overline{\mathcal{N}}}$$

The inhibitor arcs in \mathcal{N} and the places of the second family in $\overline{\mathcal{N}}$ (i.e. places of the form p' for $p \in P$) serve the same purpose: cutting off a subset of transitions in $\Gamma_{\mathcal{N}^\circ}$. It just remains to show that they cut off exactly the same transitions. Place p' inhibits the firing of transition t in some marking M satisfying property \mathcal{K} if and only if $(1, -m_p) \sqsubseteq \text{Pre}(p', t)$; this is possible only if $\text{Pre}(p', t) = (1, 0)$ in $\overline{\mathcal{N}}$ i.e. $\text{Pre}(p, t) = \varsigma$ in \mathcal{N} (there is an inhibitor arc from p to t) and $m_p \neq 0$ and these conditions are also clearly sufficient conditions:

$$(1, -m_p) \sqsubseteq \text{Pre}(p', t) \quad \Leftrightarrow \quad \text{Pre}(p, t) = \varsigma \wedge m_p \neq 0$$

which means there exists for the associated marking in \mathcal{N} an inhibitor arc from a non-empty place to that transition. Figure (4) shows the lexicographic Petri net associated with the Petri net with inhibitor arcs of Fig. (2).

Fig. 4. A lexicographic Petri net equivalent to the Petri net with inhibitor arcs of Fig.2

In order to show that lexicographic Petri nets are strictly more expressive than Petri nets with inhibitor arcs, it suffices to observe that there exists no Petri net with inhibitor arcs whose language is the language of the lexicographic Petri net of Fig. (5) i.e. the prefix closure of $b^* + ab$. Suppose on the contrary

Fig. 5. A lexicographic Petri net with no equivalent Petri net with inhibitor arcs

that such a Petri net with inhibitor arcs exists. Since transition b can fire an arbitrary number of time in the initial marking it should be the case that for every place p either there exists an inhibitor arc from p to b and no output arc

from b to p or $Post(p, b) \sqsupseteq Pre(p, b)$. Thus it follows that b can fire an arbitrary number of times in every marking at which it can fire once; which contradicts the fact that after the firing of transition a transition b can fire only once.

Corollary 2. *Reachability, Coverability, Place-boundedness, Boundedness, Deadlock and Liveness are undecidable for the class of lexicographic Petri nets.*

4 Bounded nets

4.1 Nets associated with MV-algebras

A net is bounded if we can find an upper bound on the possible values of places in any accessible marking. Let us start our study on the algebraization of the dynamics of bounded nets by the following observation.

Proposition 9. *Any non trivial Petri algebra is an unbounded lattice.*

Proof. Let us consider a bounded and non trivial Petri algebra. Let $\infty \neq 0$ stand for its greatest element. One has

1. $\infty \ominus \infty = 0$ because more generally from $x \ominus x \sqsubseteq 0 \Leftrightarrow x \sqsubseteq 0 \oplus x = x$ it follows $x \ominus x = 0$.

2. $\infty \ominus 0 = \infty$ because $\infty \ominus 0 \sqsubseteq x \Leftrightarrow \infty \sqsubseteq 0 \oplus x = x$.

3. $\infty \oplus x = \infty$ because $\infty \oplus x \sqsupseteq y \Leftrightarrow \infty \sqsupseteq y \ominus x$.

4. $x \ominus \infty = 0$ because $x \ominus \infty \sqsubseteq y \Leftrightarrow x \sqsubseteq y \oplus \infty = \infty$.

Then the identity $(a \oplus b) \ominus b = a$ is not satisfied for $a = b = \infty$ since the left-hand side evaluates to 0.

However we can enforce boundedness by modifying the rule of the token game. Let us consider first the case of the usual Petri nets: assume that each place $p \in P$ is associated with a capacity $k_p \in \mathbb{N}$ and that we want to ensure that the value of a place p of a Petri net is bounded from above by its capacity k_p . For that purpose we modify the firing rule as follows (where all computations are performed in \mathbb{Z})

$$M[t] M' \Leftrightarrow \forall p \in P \begin{cases} M(p) \sqsupseteq Pre(p, t) \wedge (M(p) - Pre(p, t)) + Post(p, t) \sqsubseteq k_p \\ M'(p) = (M(p) - Pre(p, t)) + Post(p, t) \end{cases}$$

this rule can be reformulated as:

$$M[t] M' \Leftrightarrow \forall p \in P \begin{cases} Pre(p, t) \sqsubseteq M(p) \sqsubseteq (k_p + Pre(p, t)) - Post(p, t) \\ M'(p) = (M(p) - Pre(p, t)) + Post(p, t) \end{cases}$$

Petri algebras are the positive cones G^+ of lattice-ordered abelian groups $G = (G, +, 0, \sqcup, \sqcap)$. It is an algebra with the following operations:

- **the sum:** restriction of the group operation $x \oplus y = x + y$
- **the truncated difference:** $x \ominus y = (x - y) \sqcup 0$

Let $k \sqsupseteq 0$ be some element of this positive cone; we suppose that it is a *strong unit* in the sense that $\forall g \in G \exists n \in \mathbb{N} \cdot n \cdot k \sqsupseteq g$. We can, by modifying the firing rule, ensure that the values of places stay within the interval $I = [0, k] = \{g \in G \mid 0 \sqsubseteq g \sqsubseteq k\}$. This interval with induced order is a bounded lattice that can be equipped with the following operations:

- **the truncated sum:** $x \boxplus y = (x + y) \sqcap k$
- **the truncated difference:** $x \ominus y = (x - y) \sqcup 0$
- **the product:** $x \bullet y = ((x + y) - k) \sqcup 0$
- **the implication:** $x \rightarrow y = ((y + k) - x) \sqcap k$
- **the negation:** $\neg x = x \rightarrow 0 = k \ominus x$

The following properties are met:

1. It is a bounded lattice: $0 \sqsubseteq x \sqsubseteq k$
2. $(I, \boxplus, 0)$ is an abelian monoid whose unit is an absorbing element for the product: $0 \bullet x = 0$.
3. (I, \bullet, k) is an abelian monoid whose unit k is an absorbing element for the sum: $k \boxplus x = k$.
4. Negation is an involution ($\neg\neg x = x$) such that $k = \neg 0$ (hence $0 = \neg k$) and $x \bullet y = \neg(\neg x \boxplus \neg y)$ (hence $x \boxplus y = \neg(\neg x \bullet \neg y)$)
5. $x \sqsubseteq y \Leftrightarrow (x \rightarrow y) = k \Leftrightarrow (x \ominus y) = 0$
6. \bullet is a residuated monoid: $x \bullet y \sqsubseteq z \Leftrightarrow x \sqsubseteq y \rightarrow z$
7. \boxplus is a co-residuated monoid: $x \boxplus y \sqsupseteq z \Leftrightarrow x \sqsupseteq z \ominus y$
8. $(y \rightarrow z) = z \boxplus \neg y$ and $z \ominus y = y \bullet \neg z$
9. $x \sqcup y = (y \rightarrow x) \rightarrow x = (x \rightarrow y) \rightarrow y = x \boxplus (y \ominus x) = y \boxplus (x \ominus y)$
10. $x \sqcap y = x \bullet (x \rightarrow y) = y \bullet (y \rightarrow x) = x \ominus (x \ominus y) = y \ominus (y \ominus x)$

Such a structure is called an MV-algebra. MV-algebras are generalizations of boolean algebras used in the algebraic analysis of Lukasiewicz infinite-valued propositional logic and this class of algebras admits several equivalent definitions [14, 31]. For instance an MV-algebra is a bounded commutative residuated lattice ($x \bullet y \sqsubseteq z \Leftrightarrow x \sqsubseteq y \rightarrow z$) such that $x \sqcup y = (y \rightarrow x) \rightarrow x$. An MV-algebra is also an abelian monoid $(A, \bullet, 1)$ together with an involution $\neg : A \rightarrow A$ such that if we let $x \boxplus y = \neg(\neg x \bullet \neg y)$, $0 = \neg 1$ and $(x \rightarrow y) = \neg x \boxplus y$ then $x \bullet 0 = 0$ and $(y \rightarrow x) \rightarrow x = (x \rightarrow y) \rightarrow y$.

Definition 6. An MV-Petri net is a structure $\mathcal{N} = (P, T, Pre, Post, M_0)$ where P is a finite set of places with an MV-algebra A_p associated with each place $p \in P$ (the unit k_p of A_p is called the capacity of place p), T is a finite set of transitions disjoint from P , and $Pre, Post : P \times T \rightarrow \bigsqcup_{p \in P} A_p$, the flow relations, are such that $\forall p \in P \forall t \in T \quad Pre(p, t), Post(p, t) \in A_p$. A marking is a map $M : P \rightarrow \bigsqcup_{p \in P} A_p$ that associates with each place $p \in P$ the local value of the current configuration $M(p) \in A_p$ in this place. M_0 is some fixed marking, called the initial marking. The transition relation $M[t]M'$ stating that transition t can fire in marking M and leads, when it is fired, to the new marking M' is given by:

$$M[t]M' \Leftrightarrow \forall p \in P \begin{cases} Pre(p, t) \sqsubseteq M(p) \sqsubseteq Post(p, t) \rightarrow Pre(p, t) \\ M'(p) = (M(p) \ominus Pre(p, t)) \boxplus Post(p, t) \end{cases} \quad (10)$$

This relation can be extended inductively to sequences $u \in T^*$ of transitions by letting $M [\varepsilon] M$ for every marking M and $M [t \cdot u] M'$ if and only if there exists some marking M'' such that $M [t] M''$ and $M'' [u] M'$ for every $t \in T$ and $u \in T^*$. We let $\text{Reach}(\mathcal{N}) = \{M \mid \exists u \in T^* M_0 [u] M\}$ stand for the set of reachable markings, and the marking graph of an MV-Petri net $\mathcal{N} = (P, T, \text{Pre}, \text{Post}, M_0)$ is then the labelled graph $\Gamma_{\mathcal{N}} = (V, \Lambda, v_0)$ whose set of vertices is given by the set $V = \text{Reach}(\mathcal{N})$ of reachable markings with $v_0 = M_0$ and whose set of arcs $\Lambda \subseteq V \times T \times V$ is the restriction of the transition relation to the set of reachable markings: $\Lambda = \{(M, t, M') \mid M, M' \in V \wedge M [t] M'\}$.

Bounded Petri nets A bounded Petri net \mathcal{N} where place $p \in P$ has capacity k_p is an MV-Petri net where the MV-algebra associated with place p is the $[0, k_p]$ interval of the group \mathbb{Z} of integers. Indeed under the assumptions that $\text{Pre}(p, t)$, $\text{Post}(p, t)$ and $M(p)$ are elements of the interval $[0, k_p]$ one has $M(p) \sqsubseteq (k_p + \text{Pre}(p, t) - \text{Post}(p, t))$ if and only if $M(p) \sqsubseteq \text{Post}(p, t) \rightarrow \text{Pre}(p, t) = [(k_p + \text{Pre}(p, t) - \text{Post}(p, t)) \sqcap k_p]$ and one has $M'(p) = (M(p) \ominus \text{Pre}(p, t)) \boxplus \text{Post}(p, t) = [((M(p) - \text{Pre}(p, t)) \sqcup 0) + \text{Post}(p, t)] \sqcap k_p = (M(p) - \text{Pre}(p, t)) + \text{Post}(p, t)$ when both $\text{Pre}(p, t) \sqsubseteq M(p)$ and $(M(p) - \text{Pre}(p, t)) + \text{Post}(p, t) \sqsubseteq k_p$ are satisfied (i.e. $\text{Pre}(p, t) \sqsubseteq M(p) \sqsubseteq (k_p + \text{Pre}(p, t) - \text{Post}(p, t))$).

One-safe nets The boolean algebra $2 = \{0, 1\}$ is an MV-algebra where $x \boxplus y = x \sqcup y$ and $x \bullet y = x \sqcap y$. Let P be the set of places of a net and $B = 2^P = \wp(P)$ the corresponding product structure, the firing rule of MV-Petri nets can be reformulated as:

$$M [a] M' \Leftrightarrow M \supseteq \text{Pre}(a) \wedge M \cap \text{Post}(a) \subseteq \text{Pre}(a) \wedge M' = (M \setminus \text{Pre}(a)) \cup \text{Post}(a)$$

which is the usual firing rule of one-safe nets.

One-safe Fuzzy nets If we replace the boolean algebra $2 = \{0, 1\}$ by the interval $[0, 1]$ of the additive group of real numbers, i.e. with $x \boxplus y = \min(1, x + y)$ and $x \bullet y = \max(0, x + y - 1)$ then we obtain the firing rule of some kind of one-safe "fuzzy" nets:

$$M [a] M' \Leftrightarrow \forall p \in P \begin{cases} \text{Pre}(a, p) \sqsubseteq M(p) \sqsubseteq \text{Pre}(a, p) + 1 - \text{Post}(a, p) \\ M'(p) = \min(1, M(p) + \text{Post}(a, p) - \text{Pre}(a, p)) \end{cases}$$

4.2 Complementary places

Mundici [31] proved that MV-algebras coincide with $[0, k]$ intervals of abelian lattice-ordered groups where k is a strong unit. More precisely if $G = (G, +, 0)$ is an abelian lattice-ordered group with strong unit k then $\Gamma(G, k) = (A, \boxplus, 0, \bullet, k, \neg)$ where $G = [0, k] = \{x \in G \mid 0 \sqsubseteq x \sqsubseteq k\}$, $x \boxplus y = (x + y) \sqcap k$, $x \bullet y = ((x + y) - k) \sqcup 0$, and $\neg x = k - x$ is an MV-algebra such that the restriction of the order relation of the group on the unit interval $[0, k]$ coincides with the order relation of the

MV-algebra: $x \sqsubseteq y \Leftrightarrow x \ominus y = 0$ (where $x \ominus y = x \bullet \neg y = (x - y) \sqcup 0$). Moreover Γ extends into an equivalence between the respective categories of abelian lattice-ordered groups with strong unit and MV-algebras, i.e. Γ induces a bijective correspondence between isomorphism classes of abelian lattice-ordered groups with strong unit and isomorphism classes of MV-algebras. Now we have seen that Petri algebras corresponds bijectively, up to isomorphism, to the positive cones of abelian lattice-ordered groups, we thus have a Petri algebra canonically associated with each MV-algebra. The following result shows that, by using complementary places, we can simulate an MV-Petri net by a generalized Petri net defined on the associated Petri algebra.

Theorem 8. *Any MV-Petri net can be simulated by a generalized Petri net.*

Lemma 7. *For every elements a, b , and c in an MV-algebra one has*

$$b \sqsubseteq \neg a \sqsubseteq b \boxplus c \quad \Rightarrow \quad (a \boxplus b) \bullet c = (a \bullet (c \boxplus b)) \boxplus (b \bullet c) \quad (11)$$

Proof. Since every MV-algebra is a subdirect product of totally ordered MV-algebras we can assume without loss of generality that the given MV-algebra is totally ordered. Condition $b \sqsubseteq \neg a$ is equivalent to $a \bullet b = 0$ i.e. $a + b \sqsubseteq k$ in the underlying group. Condition $\neg a \sqsubseteq b \boxplus c$ is equivalent to $a \boxplus b \boxplus c = k$, i.e. $a + b + c \sqsupseteq k$ in the underlying group. On the one hand

$$(a \boxplus b) \bullet c = ((a + b) \sqcap k) \bullet c = (a + b) \bullet c = (a + b + c - k) \sqcup 0 = a + b + c - k$$

On the other hand

$$\begin{aligned} a \bullet (c \boxplus b) &= a \bullet [(c + b) \sqcap k] \\ &= (a + [(c + b) \sqcap k] - k) \sqcup 0 \\ &= \begin{cases} a & \text{if } c + b \sqsupseteq k \\ a + c + b - k & \text{if } c + b \sqsubseteq k \end{cases} \end{aligned}$$

and

$$b \bullet c = (b + c - k) \sqcup 0 = \begin{cases} b + c - k & \text{if } b + c \sqsupseteq k \\ 0 & \text{if } b + c \sqsubseteq k \end{cases}$$

and thus $(a \boxplus b) \bullet c = (a \bullet (c \boxplus b)) \boxplus (b \bullet c)$ as required.

The above property may be rephrased as:

$$b \sqsubseteq \neg a \sqsubseteq b \boxplus \neg c (= c \rightarrow b) \quad \Rightarrow \quad (a \boxplus b) \ominus c = (a \ominus (c \ominus b)) \boxplus (b \ominus c) \quad (12)$$

and thus appears as a weakening of equation (8) which ensures the associativity of the firing rule of generalized Petri nets.

Proof (of Theorem 8). The translation of a bounded net \mathcal{N} into an equivalent generalized Petri net $\overline{\mathcal{N}}$, illustrated in Fig. (6), consists in splitting every place p with initial marking $m \in A_p$ of the bounded net into two places denoted p and p' with initial markings $m \in (\mathbb{G}_p)^+$ and $k_p - m \in (\mathbb{G}_p)^+$ respectively where

Fig. 6. Translation from bounded nets to generalized Petri nets

(\mathbb{G}_p, k_p) is the lattice-ordered abelian group with strong unit associated with the MV-algebra $A_p \simeq \Gamma(\mathbb{G}_p, k_p)$. I.e. one has

$$\begin{aligned}\overline{Pre}(p, t) &= Pre(p, t) \\ \overline{Post}(p, t) &= Post(p, t) \\ \overline{Pre}(p', t) &= Post(p, t) \ominus Pre(p, t) \\ \overline{Post}(p', t) &= Pre(p, t) \ominus Post(p, t)\end{aligned}$$

First one has $x \sqsubseteq m \sqsubseteq y \rightarrow x = \neg y \boxplus x$ if and only if $m \sqsupseteq x$ and $\neg m \sqsupseteq y \bullet \neg x = y \ominus x$; hence transition t can fire in marking M in \mathcal{N} if and only if the homonymous transition t can fire in $\overline{\mathcal{N}}$ in the marking \overline{M} such that $\overline{M}(p) = M(p)$ and $\overline{M}(p') = \neg M(p) = k_p - M(p)$. Finally $M [t] M'$ in \mathcal{N} if and only if $\overline{M} [t] \overline{M}'$ in $\overline{\mathcal{N}}$ due to the fact that

$$x \sqsubseteq m \sqsubseteq y \rightarrow x \quad \Rightarrow \quad \neg[(m \ominus x) \boxplus y] = [\neg m \ominus (y \ominus x)] \boxplus (x \ominus y)$$

which follows from (12) since $\neg[(m \ominus x) \boxplus y] = (\neg m \boxplus x) \bullet \neg y = (\neg m \boxplus x) \ominus y$.

5 Conclusion

5.1 Contribution of the paper

In this paper we have put forward an axiomatization of the token game of Petri nets by identifying a class of commutative residuated monoids, called Petri algebras, for which one can generalize the rule of token game of Petri nets to define the behaviour of generalized Petri net whose flow relation and place contents are valued in such algebras. This rule is given by

$$M [t] M' \Leftrightarrow M \sqsupseteq Pre(t) \wedge M' = (M \ominus Pre(t)) \oplus Post(t)$$

where the sum and its associated residuation capture respectively how resources within places are produced and consumed through the firing of a transition. The above rule extends to sequences of transitions by letting

$$\begin{aligned}Pre(uv) &= Pre(u) \oplus (Pre(v) \ominus Post(u)) \\ Post(uv) &= (Post(u) \ominus Pre(v)) \oplus Post(v)\end{aligned}$$

subject to the condition that the associated binary operation

$$(x, y) \otimes (x', y') = (x \oplus (x' \ominus y), (y \ominus x') \oplus y')$$

be associative. We prove this latter condition to be equivalent to the cancellability of the sum which is also characterized by the identity:

$$(b \oplus c) \ominus a = (b \ominus (a \ominus c)) \oplus (c \ominus a)$$

The firing relation can then be expressed as

$$M [u] M' \Leftrightarrow (0, M') = (0, M) \otimes (Pre(u), Post(u))$$

from which the reversibility of the firing rule follows, namely the three following statements are proved equivalent:

1. $M \sqsupseteq Pre(u)$ and $M' = (M \ominus Pre(u)) \oplus Post(u)$
2. $M' \sqsupseteq Post(u)$ and $M = (M' \ominus Post(u)) \oplus Pre(u)$
3. $M \sqsupseteq Pre(u)$; $M' \sqsupseteq Post(u)$ and $M \ominus Pre(u) = M' \ominus Post(u)$

We have proved that Petri algebras coincide with the positive cones of lattice-ordered abelian groups. We know that lattice-ordered abelian groups constitute the subvariety of lattice-ordered groups generated by the group \mathbb{Z} of integers, and that their positive cones (i.e. Petri algebras) form the subvariety of residuated lattices generated by \mathbb{N} . We have introduced a class of nets, termed lexicographic Petri nets, that are associated with the positive cones of the lexicographic powers of the additive group of real numbers. This class of nets is proved to be universal in the sense that any generalized Petri net can be simulated by a lexicographic Petri net. All the classical decidable properties of Petri nets however (termination, covering, boundedness, structural boundedness, accessibility, deadlock, liveness ...) are proved to be undecidable on the class of lexicographic Petri nets, even if restricted to integer coefficients, by showing that this class is a strict extension of the class of Petri nets with inhibitor arcs.

Finally we have turned our attention to bounded nets associated with Petri algebras and have shown that their dynamics can be reformulated in term of MV-algebras and that any such MV-Petri net can be simulated by some generalized Petri net.

5.2 Comparaison with related works

This study is not the first attempt to generalize Petri nets using cones of groups [8, 25, 26], however the originality of the present work is to derive such structures from an axiomatization of the token game of Petri nets. Various axiomatizations of Petri nets token game have also been introduced using variants of linear logic. Linear logic, introduced by J.-Y. Girard [23], is generally described as a resource concious logic because resources are consumed or produced under rules of inference like tokens in Petri nets. In [9, 22, 29] the authors deal with a multiplicative fragment of linear logic. In [22, 29] each transition is considered as one axiom added to linear logic, when [9] translates a Petri net into a unique

linear logic formula. These works provide new tools for analysing Petri nets or working with them and concern accessibility as well as composition mechanisms and concurrency. In [28, 29] the correspondance between Petri nets and linear logic is given by an isomorphism between two categories in which places take their value in a free commutative monoid. In [30] one finds generalisations of the algebra of commutative monoids, they suggest also abelian groups but they consider this choice too restrictive. G. Winskel and U. Engberg [17, 18] use a semantic approach and consider Petri nets as models of intuitionistic linear logic. Their work is based on the fact that a Petri net induces a preorder on its set of markings. This preorder is compatible with the binary operation of the free monoid over the set of places of the net. Thus, this monoid is preordered and one can construct a quantale from any Petri net as the set of the downward closed subsets of this preorder. Quantales [33] are distributive lattices used in particular in the phase semantics [23] of linear logic. This allows G. Winskel and U. Engberg to give an interpretation of many connectives of linear intuitionistic logic. Another approach [32] using PCLL, the partially commutative linear logic, an intuitionistic logic which contains both commutative and non commutative connectives, permits to represent step transitions of Petri nets where a step is a serie-parallel pomset of events. The greatest part of these works consider a marking as a multiset of places or as an element of the free commutative monoid generated by the set of places, i.e. a map from P , the set of places, to \mathbb{N} . Henceforth they don't propose an algebraic generalisation of Petri nets in our sense, and the token game is exactly that of classical Petri nets.

5.3 Directions for further research

We have put the basis for a uniform presentation of various families of Petri nets by recasting them as particular instances of a generic class of Petri nets parametric in algebraic structures representing some concrete notion of resources. However the present approach, centered on the notion of resources, is probably too concrete to be of practical interest in many situations. For instance even though one can describe continuous Petri nets in this framework the obtained semantics is too much extensional. We see two directions that can be used in order to derive more abstract representations for the behaviour of these generic nets. First, one can abstract of the flow arc inscriptions. Such an inscription takes its value in some algebra of abstract properties. A precondition then appears as a guard stating that some property has to be satisfied by the resources contained in the corresponding place and a postcondition is interpreted as adding resources to enforce some property. Unfortunately, it seems that the inductive definitions of the flow relations *Pre* and *Post* strangely forces the algebra of flow arc inscriptions to be a Petri algebra. If one want to continue in this direction, one has to leave them and consequently lose the three characterizations of the generalized transition relation given in Theorem 2. Second, one can abstract on the firing relation itself by giving the measure in some adequate semiring of the "firability" of a transition in some marking.

A second line of research is to recast the theory of regions [1, 2] in the context of nets over Petri algebras. For that purpose the marking graphs of these generalized Petri nets should also be presented as transition relations enriched over such structures. It will probably be easier to solve that problem for MV-Petri nets; indeed an MV-algebra gives rise to two De Morgan dual semi-rings, one of which is used for the definition of the firing relation of the nets while their marking graphs can be expressed as transition systems with coefficients in the other semiring.

Finally, we have hinted at the fact that MV-Petri nets might be used for the description of uncertainty in dynamic systems. Several definitions of fuzzy Petri nets and of possibilistic Petri nets have been introduced already in the literature [12, 13, 11]. These definitions were generally put forward in an *ad hoc*, and sometimes informal, manner to meet with very specific applications and without paying the attention to the underlying algebraic structures. As a result it seems that even now there is no general agreement on which definition should be favored.

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