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Abstract: The firing rule of Petri nets relies on a residuation operation for the commutative monoid of natural number. We identify a class of residuated commutative monoids, called Petri algebras, for which one can mimic the token game of Petri nets to define the behaviour of generalized Petri net whose flow relation and place contents are valued in such algebraic structures. The sum and its associated residuation capture respectively how resources within places are produced and consumed through the firing of a transition. We show that Petri algebras coincide with the positive cones of lattice-ordered commutative groups and constitute the subvariety of the (duals of) residuated lattices generated by the commutative monoid of natural number. We however exhibit a Petri algebra whose corresponding class of nets is strictly more expressive than the class of Petri nets. More precisely, we introduce a class of nets, termed lexicographic Petri nets, that are associated with the positive cones of the lexicographic powers of the additive group of real numbers. This class of nets is universal in the sense that any net associated with some Petri algebras can be simulated by a lexicographic Petri net. All the classical decidable properties of Petri nets however (termination, covering, boundedness, structural boundedness, accessibility, deadlock, liveness ...) are undecidable on the class of lexicographic Petri nets. Finally we turn our attention to bounded nets associated with Petri algebras and show that their dynamic can be reformulated in term of MV-algebras.

Key-words: Petri nets, token game, axiomatization

(Résumé : tsvp)

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Les algèbres de Petri

Résumé : La règle de franchissement d'un réseau de Petri repose sur l'opération de résiduation du monoïde des entiers naturels. On identifie une classe de monoïdes commutatifs résidués, appelés algèbres de Petri, permettant de simuler la règle de franchissement des réseaux de Petri afin de définir le comportement de réseaux étendus pour lesquels les contenus des places et les poids des relations de flots prennent leurs valeurs dans une telle algèbre. La somme et l'opération de résidu associée capturent respectivement la production et la consommation de ressources dans les places lors du franchissement des transitions. On montre que les algèbres de Petri coïncident avec les cônes positifs des groupes commutatifs ordonnés en treillis et qu'ils constituent la sous-variété de la variété des (duaux) des treillis résidués engendrée par le monoïde des entiers naturels. On peut néanmoins exhiber une algèbre de Petri dont la classe de réseaux associée est strictement plus expressive que la classe des réseaux de Petri. Plus précisémment, on introduit une classe de réseaux, appelés réseaux de Petri lexicographiques, associée aux cônes positifs de puissances lexicographiques du groupe additif des nombres réels. Cette classe de réseaux est universelle en ce sens que que tout réseau associé à une quelconque algèbre de Petri peut être simulé par un réseau de Petri lexicographique. Par ailleurs toutes les propriétés décidables des réseaux de Petri (terminaison, couverture, caractère borné, accessibilité, blocage, vivacité ...) sont indécidables sur la classe des réseaux lexicographiques. Finalement on s'intéresse au cas particulier des réseaux bornés et on montre que leur dynamique peut être reformulée en terme de MV-algèbres.

Mots-clé : Réseaux de Petri, règle de franchissement, axiomatisation

1. Introduction

The Petri net model is a graphical and mathematical modeling tool that, since its introduction in the early sixties, have come to play a pre-eminent role in the formal study of concurrent discrete-event dynamic systems. Petri nets are a natural extension of automata in which states are distributed (they can be represented as vectors of "local states") and for which the occurrence of an event relies on local conditions. More precisely a Petri net $(P, T, Pre, Post)$ consists of a finite set P of places, a finite set T of transitions (disjoint from P), and flow relations $Pre, Post : P \times T \to \mathbb{N}$. Places can contain some tokens representing the ressources available in this place for the current configuration. Such a (global) configuration of a Petri net is given as a vector $M : P \to \mathbb{N}$, called marking, indicating the number of tokens available in each place. Tokens are consumed and produced by the firing of transitions according to the so-called token game :

$$M\,[t\rangle\,M' \;\Leftrightarrow\; \forall p \in P \qquad M(p) \sqsupseteq Pre(p,t) \quad \text{and} \quad M'(p) = (M(p) - Pre(p,t)) + Post(p,t)$$

The token game of Petri net says that in order for a transition t to fire in marking M it should be the case that each place contains enough resources as it is expressed by the condition $M(p) \sqsupseteq Pre(p,t)$. Then the firing of transition t proceeds in two stages : a consumption of resources ($Pre(p,t)$ tokens are removed from place p) followed by a production of resources ($Post(p,t)$ tokens are added to place p). The notation $M\,[t\rangle\,M'$ expresses the fact that transition t is allowed to fire in marking M and that firing t in marking M produces the new marking M'. Numerous techniques, supported and automated by software tools, can be used to verify that some required properties are met for systems specified using Petri nets. For instance the following problems have an effective solution on the class of Petri nets :

- **Reachability:** Is a given marking reachable (from a fixed initial marking) through a sequence of firable transitions ?
- **Coverability:** Is a given marking M covered by some reachable marking M' ? (i.e. $\forall p \in P \quad M(p) \leq M'(p)$)
- **Place-boundedness:** Does there exists an upper bound on the number of tokens that a given place can hold in reachable markings ?
- **Boundedness:** Are all places bounded on the set of reachable markings ? (i.e. is the Petri net equivalent to a finite state machine ?)
- **Deadlock:** Does there exist a reachable marking in which no transition can fire ?
- **Liveness:** A transition t is live if for every reachable marking M there exists at least one marking, reachable from M, in which t can fire.

By combining these basic properties we can express complex static and dynamic requirements for the system. The ability, through the token game rule, to graphically visualise the behaviour of a Petri net helps the practitioner to uncover shortcomings in the specification or errors in the model. The Petri net technique thus promotes an iterative modelling process in which the model can incrementally be constructed by alternatively modify and re-analyse it.

Numerous extensions of this basic model of Petri nets have been introduced over the years. Some of them are high level nets that allows for more compact representations but does not increase the expressive power of Petri nets: these high level nets can be unfolded into equivalent, even though in general much larger, Petri nets. Some extensions however change more dramatically the semantics of the original model. For instance timing constraints may be added, as in timed Petri nets or stochastic Petri nets for the purpose of enabling performance analysis. With continuous Petri net the discrete state transition rule is replaced by a notion of trajectory using a continuum of intermediate states. In Fuzzy Petri nets one has a possibilistic measure of the firing of a transition in the given marking thus enabling to deal with incertainty. Our purpose in this paper is to put forward an axiomatisation of the token game of Petri nets. More precisely we identify a class of commutative residuated monoids, called Petri algebras, for which one can mimic the token game of Petri nets to define the behaviour of generalized Petri net whose flow relation and place contents are valued in such algebraic structures. The sum and its associated residuation capture respectively how resources within places are produced and consumed through the firing of a transition. The class of usual Petri nets is associated with the commutative monoid of natural numbers. We show that Petri algebras coincide with the positive cones of lattice-ordered commutative groups and constitute the subvariety of the (duals of) residuated lattices generated by the commutative monoid of natural number. The basic Petri net model is thus associated with the generator of the variety of Petri algebras which shows that these extended nets share all algebraic properties of Petri nets, in particular they have the same equational and inequational theory. We however exhibit a Petri algebra whose corresponding class of nets is strictly more expressive than the class of Petri nets, i.e. their class of marking graphs is strictly larger. More precisely, we introduce a class of nets, termed lexicographic Petri nets, that are associated with the positive cones of the lexicographic powers of the additive group of real numbers. This class of nets is proved to be universal in the sense that any net associated with some Petri algebra can be simulated by a lexicographic Petri net. All the classical decidable properties of Petri nets however (termination, covering, boundedness, structural boundedness, accessibility, deadlock, liveness ...) are proved to be undecidable on the class of lexicographic Petri nets. This negative result put emphasize on the fact that algebraic considerations alone are not sufficient to capture the intrinsic nature of Petri nets. This might give us an understanding on the fact that all past attempts to find a strict extension of Petri nets with the same decidable properties did fail. Finally we turn our attention to bounded nets associated with commutative Petri algebra and show that their dynamic can be reformulated in term of MV-algebras.

2. An axiomatisation of the token game

2.1. **Playing the token game.** In order to obtain an axiomatisation of the token game of Petri net we represent the marking of a net as a map $M : P \to \bigsqcup_{p\in P} A_p$ that associates with each place $p \in P$ the local value of the current configuration $M(p) \in A_p$ in this place. Content of places are resources that are consumed and produced according to the token game. Thus we assume that each place $p \in P$ is associated with a commutative divisibility monoid $A_p = (A_p, \oplus, 0)$, i.e. a monoid such that

(2.1) the relation $a \sqsupseteq b \;\Leftrightarrow\; \exists c \cdot a = b \oplus c$ is an order relation

The constant 0 represents the absence of resource and the binary operator $\oplus$ the accumulation of resources in places. Immediate consequences of condition (2.1) are the following:

$$\begin{array}{l} a \oplus b \sqsupseteq a, b \\ 0 \sqsubseteq a \\ a \oplus b = 0 \;\Rightarrow\; a = b = 0 \end{array}$$

Moreover we need to have a residuation operation $\ominus$ such that $a \ominus b$ represents the residual resource obtained by substracting b to a when $b \sqsubseteq a$. Thus the following should hold true:

$$b \sqsubseteq a \;\Rightarrow\; a = (a \ominus b) \oplus b \qquad (2.2)$$

Usual Petri nets corresponds to the situation where, for every place p, $A_p = (\mathbb{N}, +, 0)$ is the commutative monoid of whole numbers with the truncated difference $n \ominus m = \max(0; n - m)$ as residuation. This operation is characterized by the universal property that for every whole numbers n, m and p

$$n + m \sqsupseteq p \;\Leftrightarrow\; n \sqsupseteq p \ominus m$$

Up to the reversal of the order relation, it is a *commutative residuated monoid i.e.* a commutative monoid $(A, \oplus, 0)$ with an order relation $\leq$ and a residuation operation $\ominus$ which is a right adjoint to the addition, in the sense that

$$a \oplus b \leq c \;\Leftrightarrow\; a \leq c \ominus b \qquad (2.3)$$

It follows immediately from this definition that a commutative monoid is residuated if and only if its addition is order preserving in each argument and the inequation $a \oplus b \leq c$ has a largest solution for a (namely $c \ominus b$). In particular the residual is uniquely determined by the addition and the order relation. When the monoid is a divisibility monoid the order relation itself is defined in terms of the addition and thus the whole structure is caracterized by its monoid reduct. The following result is known in the context of residuated lattices (i.e. if we further assume that each pair of elements admit a least upper bound and a greatest lower bound) but the proof given below, shows that it is also valid even if the monoid is not assumed to be lattice-ordered.

Proposition 2.1. *If* $(A, \oplus, 0, \sqsubseteq)$ *is a commutative co-residuated monoid, i.e. a monoid equipped with a residuation operation* $\ominus$ *such that*

$$a \oplus b \sqsupseteq c \;\Leftrightarrow\; a \sqsupseteq c \ominus b \qquad (2.4)$$

then the following conditions are equivalent

(i) It is a divisibility monoid: $a \sqsupseteq b \;\Leftrightarrow\; \exists c \cdot a = b \oplus c$

(ii) It is an upper semi-lattice with: $a \sqcup b = (a \ominus b) \oplus b$

(iii) $b \sqsubseteq a \;\Rightarrow\; a = (a \ominus b) \oplus b$

Lemma 2.2. *Let $(A, \oplus, 0, \sqsubseteq)$ be a commutative ordered monoid where the neutral element is also the least element for the order relation, and let $\ominus$ be some binary operation on A ; then the universal property*

$$a \oplus b \sqsupseteq c \quad \Leftrightarrow \quad a \sqsupseteq c \ominus b$$

is equivalent to the conjunction of the following properties :

(i) $(a \oplus b) \ominus b \sqsubseteq a$

(ii) $(a \ominus b) \oplus b \sqsupseteq a,\, b$

(iii) $a \ominus (a \ominus b) \sqsubseteq a,\, b$

(iv) monotonies : $a \sqsubseteq a' \wedge b \sqsubseteq b' \;\Rightarrow\; a \oplus b \sqsubseteq a' \oplus b' \;\wedge\; a \ominus b' \sqsubseteq a' \ominus b$

Proof. Let us assume that $\ominus$is a left adjoint to the addition, i.e. the universal property (2.4) holds. Then from $a \oplus b \sqsupseteq a \oplus b$ it follows that $(a \oplus b) \ominus b \sqsubseteq a$ and from $a \ominus b \sqsupseteq a \ominus b$ if follows that $(a \ominus b) \oplus b \sqsupseteq a$. Combining the commutativity of addition with the adjunction if follows that $a \sqsupseteq c \ominus b \;\Leftrightarrow\; b \sqsupseteq c \ominus a$ and then in particular $b \sqsupseteq a \ominus (a \ominus b)$. Since the neutral element is also the least element for the order relation we deduce $a \ominus a = 0$ (because $a \oplus 0 \sqsupseteq a \Rightarrow 0 \sqsupseteq a \ominus a$) and then $a \oplus b \sqsupseteq a, b$. From $a \ominus b \sqsupseteq b \ominus b = 0$ it follows that $(a \ominus b) \oplus b \sqsupseteq b$ by adjunction and $a \ominus (a \ominus b) \sqsubseteq a$ using the equivalence $a \sqsupseteq c \ominus b \;\Leftrightarrow\; b \sqsupseteq c \ominus a$. Thus conditions (i) up to (iii) are established. Suppose $a \sqsubseteq a'$ then from $(a \oplus b) \ominus b \sqsubseteq a \sqsubseteq a'$ it follows by adjunction that $a \oplus b \sqsubseteq a' \oplus b$ and from $a \sqsubseteq a' \sqsubseteq (a' \ominus b) \oplus b$ we deduce $a \ominus b' \sqsubseteq a' \ominus b$. By monotony of the sum one can now deduce that $b \sqsubseteq (b \ominus a) \oplus a \sqsubseteq (b \ominus a) \oplus a'$ from which it follows that $b \ominus a' \sqsubseteq b \ominus a$. For the converse direction, let us now assume that the conditions (i) up to (iv) are satisfied. Then $a \oplus b \sqsupseteq c$ entails that $a \sqsupseteq (a \oplus b) \ominus b \sqsupseteq b \oplus c$ and in the converse direction $a \sqsupseteq c \ominus b$ entails $a \oplus b \sqsupseteq (c \ominus b) \oplus b \sqsupseteq c$. □

We see that under the assumptions of this lemma, each pair a, b of elements has an upper bound $(a \ominus b) \oplus b$ and a lower bound $a \ominus (a \ominus b)$. Proposition (2.1) states that the divisibility of the monoid is equivalent of the fact that the former is the least upper bound $a \sqcup b$. We will see below that the cancellability of the monoid is equivalent to the fact that the inequality in equation (i) in the above lemma is an equality and it entails that $a \ominus (a \ominus b)$ is the greatest lower bound $a \sqcap b$.

Proof. (of Proposition 2.1) The implications $(ii) \Rightarrow (iii) \Rightarrow (i)$ are immediate. It remains to prove that $(i) \Rightarrow (ii)$. Let us therefore assume that $(A, \oplus, 0)$ is a commutative co-residuated divisibility monoid. The assumption of Lemma (2.2) are met and we can therefore deduce that $(a \ominus b) \oplus b \sqsupseteq a,\, b$. Suppose that $c \sqsupseteq a,\, b$ is some upperbound of a and b. By divisiblity there exist some x and y such that $c = a \oplus x = b \oplus y$ and by adjunction it follows that $y \sqsupseteq (a \oplus x) \ominus b$ and thus $c = b \oplus y \sqsupseteq b \oplus ((a \oplus x) \ominus b) \sqsupseteq b \oplus (a \ominus b)$ (by monotony of $\oplus$and using the identity $a \oplus x \sqsupseteq a$). □

In view of the above discussion we are let to set the following

Definition 2.3. A *Petri pre-structure* a commutative monoid equipped with a residuation operation $(M, \oplus, 0, \ominus)$ satisfying the conditions (2.1) and (2.4).

The firing of a transition proceeds in two stages: a consumption of ressources in the input places followed by a production of ressources in the output places. More precisely, the transition relation $M\,[t\rangle\, M'$ stating that transition t can fire in marking M and leads, when it is fired, to the new marking M' is given by:

$$M\,[t\rangle\, M' \;\Leftrightarrow\; \forall p \in P \;\; M(p) \sqsupseteq Pre(p,t) \;\wedge\; M'(p) = (M(p) \ominus Pre(p,t)) \oplus Post(p,t)$$

A net is called ***homogeneous*** if all the algebras A_p are identical. We will stick to homogeneous nets until Section 3 where it will be noticed that the "multi-sorted" case add in fact no extra generality. By the way we also restrict our attention in this paper to commutative algebras. With non commutative monoids it would be possible [2] for example to take fifo nets [11] into account.

2.2. **Associativity of the firing rule.** For any non empty sequence of transitions $u = a_0 \ldots a_{n-1} \in T^+$we let $M\,[u\rangle\, M'$ state the existence of markings $M = M_0, M_1, \ldots, M_n = M'$ such that $M_i\,[a_i\rangle\, M_{i+1}$ for every $0 \leq i < n$. Moreover we set $M\,[\varepsilon\rangle\, M$ where $\varepsilon \in E^*$ is the empty sequence and M an arbitrary marking. We use $M\,[u\rangle$ (respectively $[u\rangle\, M'$) as a shorthand for $\exists M' \;\; M\,[u\rangle\, M'$ (resp. $\exists M \;\; M\,[u\rangle\, M'$). If $a, b \in T$ are transitions in a (n usual) Petri net we have the following equivalences

$$\begin{aligned} M\,[ab\rangle \quad &\Leftrightarrow \quad M \sqsupseteq Pre(a) \text{ and } (M - Pre(a)) + Post(a) \sqsupseteq Pre(b) \\ &\Leftrightarrow \quad M \sqsupseteq \max(Pre(a); Pre(a) + (Pre(b) - Post(a))) \\ &\Leftrightarrow \quad M \sqsupseteq Pre(a) + \max(0; Pre(b) - Post(a)) \\ &\Leftrightarrow \quad M \sqsupseteq Pre(a) + (Pre(b) \ominus Post(a)) \end{aligned}$$

This suggest to let for any sequences $u, v \in T^*$

$$Pre(uv) = Pre(u) \oplus (Pre(v) \ominus Post(u))$$

and symmetrically

$$Post(uv) = (Post(u) \ominus Pre(v)) \oplus Post(v)$$

For these definitions to make sense however, it remains to show that they do not depend upon the specific chosen decomposition $w = uv$; otherwise stated, the product defined on $A \times A$ by

$$(x, y) \otimes (x', y') = (x \oplus (x' \ominus y), (y \ominus x') \oplus y')$$

should be associative.

Theorem 2.4. *For any Petri pre-structure, the following conditions are equivalent:*
(i) Operation $\otimes$ *is associative,*
(ii) the identity $(b \oplus c) \ominus a = (b \ominus (a \ominus c)) \oplus (c \ominus a)$ *holds,*
(iii) the monoid is cancellable: $a \oplus b = a \oplus c \;\Rightarrow\; b = c$*, and*
(iv) the identity $(a \oplus b) \ominus b = a$ *holds.*

The remaining part of this section is devoted to the proof of this result. As already noticed the divibility of the monoid (i.e. $a \sqsubseteq b \;\Leftrightarrow\; \exists c \cdot b = a \oplus c$) entails the following identities

$$\begin{array}{lll} (div1) & : & a \oplus b \sqsupseteq a \\ (div2) & : & 0 \sqsubseteq a \\ (div3) & : & a \oplus b = 0 \ \Rightarrow \ a = b = 0 \end{array}$$

the second of which states that the neutral element for $\oplus$ is the least element. This property in conjunction with condition (2.4) has the following interesting consequences as it can be readily verified:

$$\begin{array}{lll} (int1) & : & a \sqsubseteq b \ \Leftrightarrow \ a \ominus b = 0 \\ (int2) & : & a \ominus 0 = a \\ (int3) & : & 0 \ominus a = 0 \\ (int4) & : & a \ominus a = 0 \\ (int5) & : & a \ominus b \sqsubseteq a \end{array}$$

The equivalence between the cancellability of the monoid and the identity

$$(a \oplus b) \ominus b \ = \ a \qquad (2.5)$$

is well-known for residuated lattices and it holds also trivially in our case: on one side $((a \oplus b) \ominus b) \oplus b = (a \oplus b) \sqcup b = a \oplus b$ shows that the identity $(a \oplus b) \ominus b = a$ follows from cancellability, conversely if this identity holds then $a \oplus b = a \oplus c \ \Rightarrow \ b = (a \oplus b) \ominus a = (a \oplus c) \ominus a = a$. The following proposition establishes the equivalence between the first two statements in Theorem (2.4).

Proposition 2.5. *In a Petri pre-structure associativity of* $\otimes$ *is equivalent to the identity*

$$(b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b)) \qquad \textit{(2.8)}$$

Proof. For the one hand we compute $(x, y) \otimes [(x', y') \otimes (x", y")] =$

$$\begin{array}{l} = (x, y) \otimes [x' \oplus (x" \ominus y'), (y' \ominus x") \oplus y"] = \\ = (x \oplus [(x' \oplus (x" \ominus y')) \ominus y], [y \ominus (x' \oplus (x" \ominus y'))] \oplus (y' \ominus x") \oplus y") \end{array}$$

and, on the other hand $[(x, y) \otimes (x', y')] \otimes (x", y") =$

$$\begin{array}{l} = [x \oplus (x' \ominus y), (y \ominus x') \oplus y'] \otimes (x", y") = \\ = (x \oplus (x' \ominus y) \oplus [x" \ominus ((y \ominus x') \oplus y')], [((y \ominus x') \oplus y') \ominus x"] \oplus y") \end{array}$$

Associativity of $\otimes$ is equivalent to the conjunction of both following identities

$$\begin{array}{lll} x \oplus [(x' \oplus (x" \ominus y')) \ominus y] & = & x \oplus (x' \ominus y) \oplus [x" \ominus ((y \ominus x') \oplus y')] \\ [((y \ominus x') \oplus y') \ominus x"] \oplus y" & = & y \ominus (x' \oplus (x" \ominus y'))] \oplus (y' \ominus x") \oplus y" \end{array}$$

which are in fact equivalent to each other by commutativity of $\oplus$, and they can be both rewritten as:

$$x \oplus [(y \oplus (z \ominus t)) \ominus u] = x \oplus (y \ominus u) \oplus [z \ominus ((u \ominus y) \oplus t)] \qquad (2.6)$$

By letting $x = t = 0$, $u = a$, $y = b$ et $z = c$ *in the preceding identity it comes that*

$$(b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b)) \qquad \textit{(2.8)}$$

Conversely, the latter suffices to establish identity (2.6) since:

$$x \oplus [(y \oplus (z \ominus t)) \ominus u] =$$

$$\begin{aligned} &= x \oplus (y \ominus u) \oplus [(z \ominus t) \ominus (u \ominus y)] \quad \text{By (2.8)} \\ &= x \oplus (y \ominus u) \oplus [z \ominus (t \oplus (u \ominus y))] \end{aligned}$$

The last equality comes from the identity $(a \ominus b) \ominus c = a \ominus (b \oplus c)$ which follows from the following equivalences

$$(a \ominus b) \ominus c \sqsubseteq d \Leftrightarrow a \ominus b \sqsubseteq c \oplus d \Leftrightarrow a \sqsubseteq b \oplus c \oplus d \Leftrightarrow a \ominus (b \oplus c) \sqsubseteq d$$

□

Proposition 2.6. *A Petri pre-structure satisfying the identity* $(b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b))$ *has a cancellable monoid reduct* ($a \oplus b = a \oplus c \Rightarrow b = c$).

Proof. We first observe that

(2.7) $$a \sqsupseteq b \oplus c \quad \Leftrightarrow \quad a \sqsupseteq b \text{ and } a \ominus b \sqsupseteq c$$

Actually by (int1) $a \sqsupseteq b \oplus c$ is equivalent to $(b \oplus c) \ominus a = 0$, that is to say, by (2.8), to $(b \ominus a) \oplus (c \ominus (a \ominus b)) = 0$. By (div3) this latter condition is equivalent to the conjunction of $b \ominus a = 0$ with $c \ominus (a \ominus b) = 0$, or else, by (int1), to the conjunction of $a \sqsupseteq b$ with $a \ominus b \sqsupseteq c$. Now the conjunction of (2.7) together with (2.4) allows us to conclude that

$$a \sqsupseteq b \quad \Rightarrow \quad [a = b \oplus c \Leftrightarrow c = a \ominus b]$$

which establish the cancellability of the monoid: Let us assume $a \oplus b = a \oplus c$, let d be this value, then $d \sqsupseteq b, c$ and $b = d \ominus a = c$. □

Cancellability of the monoid is then a necessary condition if we are to ensure associativity of the firing rule. Thus we let

Definition 2.7. A Petri algebra is a Petri pre-structure with a cancellable monoid reduct.

Proposition 2.8. *A Petri algebra is a lower semi-lattice with* $a \sqcap b = a \ominus (a \ominus b)$

Proof. By Lemma (2.2) we have that $a \ominus (a \ominus b) \sqsubseteq a, b$. Let $c \sqsubseteq a, b$ be some lower bound of a and b. By monotony of residuation one has $a \ominus b \sqsubseteq a \ominus c$ from which it follows that $(a \ominus b) \oplus c \sqsubseteq (a \ominus c) \oplus c = a \sqcup c = a$ (since $c \sqsubseteq a$). Then by cancellability $c = (c \oplus (a \ominus b)) \ominus (a \ominus b) \sqsubseteq a \ominus (a \ominus b)$. □

We recall that a commutative residuated lattice (see [4, 10]) consists of a set A equipped with a structure of commutative monoid $(A, \oplus, 0)$, of a lattice structure $(A, \vee, \wedge)$ and of a residuation operation $\ominus$ adjoint to the addition in the sense that

$$a \oplus b \leq c \quad \Leftrightarrow \quad a \leq c \ominus a$$

where $\leq$ is the lattice order (i.e. $a \leq b \Leftrightarrow b = a \vee b$). By Propositions (2.1) and (2.8) the dual of a Petri algebra is a commutative residuated lattice. Thus the following identities, which are just obtained by dualizing[1] known identities of commutative residuated lattices, holds (and moreover can be straightforwardly checked):

$$
\begin{array}{ll}
(RL1) & a \oplus (b \sqcap c) = (a \oplus b) \sqcap (a \oplus c) \\
(RL2) & (a \sqcup b) \ominus c = (a \ominus c) \sqcup (b \ominus c) \\
(RL3) & a \ominus (b \sqcap c) = (a \ominus b) \sqcup (a \ominus c) \\
(RL4) & (a \ominus b) \oplus b \sqsupseteq a \\
(RL5) & a \ominus (a \ominus b) \sqsubseteq b \\
(RL6) & a \oplus (b \ominus c) \sqsupseteq (a \oplus b) \ominus c \\
(RL7) & (c \ominus b) \oplus (b \ominus a) \sqsupseteq c \ominus a \\
(RL8) & c \ominus b \sqsupseteq (c \ominus a) \ominus (b \ominus a) \\
(RL9) & b \ominus a \sqsupseteq (c \ominus a) \ominus (c \ominus b) \\
(RL10) & c \ominus b \sqsupseteq (c \oplus a) \ominus (b \oplus a) \\
(RL11) & (c \ominus a) \ominus b = c \ominus (a \oplus b) = (c \ominus b) \ominus a
\end{array}
$$

as well as the monotony property asserting that $x \oplus y$ is increasing w.r.t. both arguments and $x \ominus y$ is increasing w.r.t. x and decreasing w.r.t. y .

The last stage toward the proof of Theorem (2.4) is given by the following proposition

Proposition 2.9. *Petri algebras satisfy the following identity*

$$(b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b)) \tag{2.8}$$

the proof of which follows from a sequence of intermediate results starting from the following property:

Lemma 2.10. *In any commutative Petri algebra*

1. *If $a \sqsupseteq b$ then (i) $a = b \oplus c \Leftrightarrow a \ominus b = c$ and (ii) $a \sqsupseteq b \oplus c \Leftrightarrow a \ominus b \sqsupseteq c$*
2. *If $a \sqsupseteq c$ then (i) $a \ominus b = c \Leftrightarrow a \ominus c = a \sqcap b$ and (ii) $a \ominus b \sqsupseteq c \Leftrightarrow a \ominus c \sqsupseteq a \sqcap b$*
3. *If $a \sqsupseteq b$ and $a \sqsupseteq c$ then (i) $a = b \oplus c \Leftrightarrow a \ominus b = c \Leftrightarrow a \ominus c = b$ and (ii) $a \sqsupseteq b \oplus c \Leftrightarrow a \ominus b \sqsupseteq c \Leftrightarrow a \ominus c \sqsupseteq b$*
4. *$a \ominus b = (a \sqcup b) \ominus b$ and $b = (a \sqcup b) \ominus (a \ominus b)$*
5. *$b = (a \sqcap b) \oplus (b \ominus a)$ and $b \ominus a = b \ominus (a \sqcap b)$*

Proof. 1.(ii). : If $a \sqsupseteq b \oplus c$ then $a \ominus b \sqsupseteq (c \oplus b) \ominus b = c$. Conversely, let us assume that $a \ominus b \sqsupseteq c$, from $a \sqsupseteq b$ it follows that $a = (a \ominus b) \oplus b$ and thus $a \sqsupseteq c \oplus b$. 1.(i) follows from 1.(ii) together with the adjunction : $a \sqsubseteq b \oplus c \Leftrightarrow a \ominus b \sqsubseteq c$

2.(i). : If $a \ominus b = c$ then $a \ominus c = a \ominus (a \ominus b) = a \sqcap b$. Conversely, let us assume that $a \ominus c = a \sqcap b$, from $a \sqsupseteq c$ it follows that $c = a \ominus (a \ominus c) = a \ominus (a \sqcap b) = a \ominus b$. Actually $a \ominus (a \sqcap b) =$ [by RL3] $(a \ominus a) \sqcup (a \ominus b) =$ [by int4] $0 \sqcup (a \ominus b) =$ [by div2] $a \ominus b$

2.(ii). : If $a \ominus b \sqsupseteq c$ then $a \ominus c \sqsupseteq a \ominus (a \ominus b) = a \sqcap b$. Conversely, let us assume that $a \ominus c \sqsupseteq a \sqcap b$, from $a \sqsupseteq c$ it follows that $c = a \ominus (a \ominus c) \sqsubseteq a \ominus (a \sqcap b) = a \ominus b$.

3. : Follows from 1.

4. : Follows from 3.(i). since $a \sqcup b = (a \ominus b) \oplus b$, $a \sqcup b \sqsupseteq a \sqsupseteq a \ominus b$ and $a \sqcup b \sqsupseteq b$.

5. : Follows from 3.(i). since $a \sqcap b = b \ominus (b \ominus a)$, $b \sqsupseteq a \sqcap b$ and $b \sqsupseteq b \ominus a$. □

[1] We inverse the order relation, which means that we replace $\leq, \vee, \wedge$ respectively by $\sqsupseteq$, $\sqcap$, $\sqcup$ in every identity valid for the class of commutative residuated lattice.

Lemma 2.11. *The following conditions are equivalent in any Petri algebra*

$$\begin{array}{ll} 1. & (b \oplus c) \ominus a = (b \ominus a) \oplus (c \ominus (a \ominus b)) \\ 2. & c \ominus (a \ominus b) = ((b \oplus c) \ominus a) \ominus (b \ominus a) \\ 3. & [(b \oplus c) \ominus a] \ominus [c \ominus (a \ominus b)] = b \ominus a \end{array}$$

Proof. Let $\alpha = (b \oplus c) \ominus a$, $\beta = b \ominus a$ and $\gamma = c \ominus (a \ominus b)$. One has $\alpha \sqsupseteq \beta$ by monotony and (div1), and $\alpha \sqsupseteq \gamma$ follows from (RL8) and identity (2.5):

$$(b \oplus c) \ominus a \sqsupseteq ((b \oplus c) \ominus b) \ominus (a \ominus b) \sqsupseteq c \ominus (a \ominus b)$$

Thus by Lemma 2.10.(3.i) if follows that $\alpha = \beta \oplus \gamma \;\Leftrightarrow\; \alpha \ominus \beta = \gamma \;\Leftrightarrow\; \alpha \ominus \gamma = \beta$. □

Lemma 2.12. *In any Petri algebra*

$$\begin{array}{l} b \sqsupseteq c \;\Rightarrow\; a \oplus (b \ominus c) = (a \oplus b) \ominus c \\ c \sqsupseteq b \;\Rightarrow\; (a \oplus b) \ominus c = a \ominus (c \ominus b) \end{array}$$

Proof. If $b \sqsupseteq c$ then $(b \ominus c) \oplus c = b$ and thus $a \oplus (b \ominus c) = (a \oplus (b \ominus c) \oplus c) \ominus c = (a \oplus b) \ominus c$. If $c \sqsupseteq b$ then $c \ominus (c \ominus b) = b$ and thus $a \oplus b = a \oplus (c \ominus (c \ominus b)) = (a \oplus c) \ominus (c \ominus b)$ using the preceding point and because $c \sqsupseteq c \ominus b$. It follows $(a \oplus b) \ominus c = ((a \oplus c) \ominus (c \ominus b)) \ominus c = ((a \oplus c) \ominus c) \ominus (c \ominus b) = a \ominus (c \ominus b)$. □

Lemma 2.13. *The following identity holds in any Petri algebra*

$$c \ominus (a \ominus b) = ((b \oplus c) \ominus a) \ominus (b \ominus a)$$

Proof. $((b \oplus c) \ominus a) \ominus (b \ominus a) =$

$=$	$[(b \oplus c) \ominus ((a \sqcap b) \oplus (a \ominus b))] \ominus (b \ominus a)$	**Lemma 2.10.(5)**
$=$	$[((b \oplus c) \ominus (a \sqcap b)) \ominus (a \ominus b)] \ominus (b \ominus a)$	**RL11**
$=$	$[(c \oplus (b \ominus (a \sqcap b)) \ominus (a \ominus b)] \ominus (b \ominus a)$	**Lemma 2.12.(1) since** $b \sqsupseteq a \sqcap b$
$=$	$((c \oplus (b \ominus a)) \ominus (a \ominus b)) \ominus (b \ominus a)$	**Lemma 2.10.(5)**
$=$	$((c \oplus (b \ominus a)) \ominus (b \ominus a)) \ominus (a \ominus b)$	**RL11**
$=$	$c \ominus (a \ominus b)$	**(2.5)**

□

The proof of Proposition (2.9) and therefrom of Theorem (2.4) follows from lemma (2.11) and lemma (2.13).

Corollary 2.14. *Petri algebras satisfies the following equivalence*

$$a \sqsupseteq b \oplus c \quad \Leftrightarrow \quad a \sqsupseteq b \text{ and } a \ominus b \sqsupseteq c \tag{2.9}$$

Proof. $a \sqsupseteq b \oplus c \quad \Leftrightarrow$

$\Leftrightarrow$	$(b \oplus c) \ominus a = 0$	By int1
$\Leftrightarrow$	$(b \ominus a) \oplus (c \ominus (a \ominus b))$	By 2.8
$\Leftrightarrow$	$b \ominus a = 0$ and $c \ominus (a \ominus b) = 0$	By div3
$\Leftrightarrow$	$a \sqsupseteq b$ and $a \ominus b \sqsupseteq c$	By int1

□

Identity (2.8) is an internalization of (2.9) using the axiomatization of the order relation: $a \sqsubseteq b \Leftrightarrow a \ominus b = 0$.

2.3. **Additional properties of the firing rule.** Let us consider a net over a Petri algebra A, then we can inductively define the applications Pre, $Post : P \times T^* \rightarrow A$ by letting $\varphi(p,u) = (Pre(p,u), Post(p,u))$ where $\varphi(p,-) : T^* \rightarrow A \times A$ is the unique monoid morphism such that the images $\varphi(p,t) = (Pre(p,t), Post(p,t))$ of the generators $t \in T$ be given by the flow relations of the net. Then the following holds:

$$\begin{aligned} &Pre(p,\varepsilon) = Post(p,\varepsilon) = 0 \\ &Pre(p,uv) = Pre(p,u) \oplus (Pre(p,v) \ominus Post(u,p)) \\ &Post(p,uv) = (Post(p,u) \ominus Pre(p,v)) \oplus Post(p,v) \end{aligned}$$

Theorem 2.15. *The generalized transition relation $M\,[u\rangle\,M'$ stating the existence of a sequence u of transitions leading from M to M' is given by any of the three following equivalent conditions*

1. $\forall p \in P \;\; M(p) \sqsupseteq Pre(p,u)$ *and* $M'(p) = (M(p) \ominus Pre(p,u)) \oplus Post(p,u)$

2. $\forall p \in P \;\; M'(p) \sqsupseteq Post(p,u)$ *and* $M(p) = (M'(p) \ominus Post(p,u)) \oplus Pre(p,u)$

3. $\forall p \in P \;\; M(p) \sqsupseteq Pre(p,u)$; $M'(p) \sqsupseteq Post(p,u)$ *and* $M(p) \ominus Pre(p,u) = M'(p) \ominus Post(p,u)$

Proof. On the one hand we have to show that the three above conditions are equivalent, and on the other hand that they do correspond to the generalized transition relation $M\,[u\rangle\,M'$. To simplify we shall adopt here vectorial notations by using the product algebra A^P which satisfy the same properties than A for the corresponding pointwise defined operations. It amounts to consider a net with only one place whose algebra is the product of the algebras of the original net.

Concerning the former point we thus have to establish the equivalence between the following assertions.

1. $M \sqsupseteq Pre(u)$ and $M' = (M \ominus Pre(u)) \oplus Post(u)$
2. $M' \sqsupseteq Post(u)$ and $M = (M' \ominus Post(u)) \oplus Pre(u)$
3. $M \sqsupseteq Pre(u)$; $M' \sqsupseteq Post(u)$ and $M \ominus Pre(u) = M' \ominus Post(u)$

For that purpose we recall the equivalence (2.9)

$$a \sqsupseteq b \oplus c \quad \Leftrightarrow \quad a \sqsupseteq b \text{ and } a \ominus b \sqsupseteq c$$

which allows us to conclude that

$$[M' \sqsupseteq (M \ominus Pre(u)) \oplus Post(u)] \;\Leftrightarrow\; [M' \sqsupseteq Post(u) \text{ and } M' \ominus Post(u) \sqsupseteq M \ominus Pre(u)]$$

which, in turn, in conjunction with

$$[(M \ominus Pre(u)) \oplus Post(u) \sqsupseteq M'] \quad \Leftrightarrow \quad [M' \ominus Post(u) \sqsubseteq M \ominus Pre(u)]$$

that itself is an instance of condition (2.3) proves the equivalence between assertions (1) and (3) above. The equivalence between assertions (2) and (3) are proven similarly.

As far as the second point is concerned let us temporarily denote $M[u]M'$ the relation given by any of the three assertions that we have proved to be equivalent. We recall the identities

$$\begin{aligned} &Pre(\varepsilon) = Post(\varepsilon) = 0 \\ &Pre(uv) = Pre(u) \oplus (Pre(v) \ominus Post(u)) \\ &Post(uv) = (Post(u) \ominus Pre(v)) \oplus Post(v) \end{aligned}$$

We deduce that $M[\varepsilon]M' \Leftrightarrow M = M' \Leftrightarrow M\,[\varepsilon\rangle\,M'$ where $\varepsilon \in T^*$ is the empty word, and by definition $M[t]M' \Leftrightarrow M\,[t\rangle\,M'$ for $t \in T$ any transition. It remains to prove that $M[uv]M' \Leftrightarrow \exists M" \cdot M[u]M"$ and $M"[v]M'$ in order to be able to deduce that $\{M[u]M' \mid u \in T^*\}$ is the transitive closure of the elementary transition relation $\{M\,[t\rangle\,M' \mid t \in T\}$ and that it consequently coincides with $\{M\,[u\rangle\,M' \mid u \in T^*\}$. For that purpose, let us observe that

$$(0, M) \otimes (Pre(u), Post(u)) = (Pre(u) \ominus M, (M \ominus Pre(u)) \oplus Post(u))$$

and thus

$$M[u]M' \Leftrightarrow (0, M') = (0, M) \otimes (Pre(u), Post(u))$$

(we recall that $a \sqsupseteq b \Leftrightarrow b \ominus a = 0$). By definition of the maps Pre and $Post$ one has

$$(Pre(uv), Post(uv)) = (Pre(u), Post(u)) \otimes (Pre(v), Post(v))$$

which allows us to conclude, on the basis of the associativity of $\otimes$, that $(\exists M" \cdot M[u]M"$ and $M"[v]M') \Rightarrow M[uv]M'$. In order to establish the converse direction it remains to verify that $M[uv] \Leftrightarrow M[u]$ and $M * u[v])$ where $M * u$ stands for $(M \ominus Pre(u)) \oplus Post(u)$ and where $M[u]$ stands for $\exists M' \cdot M[u]M'$ that is to say $M \sqsupseteq Pre(u)$. By definition of $\otimes$ one has $M[uv] \Leftrightarrow M \sqsupseteq Pre(uv) = Pre(u) \oplus (Pre(v) \ominus Post(u))$, which by (2.7) is equivalent to $[M \sqsupseteq Pre(u)$ **and** $M \ominus Pre(u) \sqsupseteq Pre(v) \ominus Post(u)]$ which, in turn by adjunction (equation 2.4) is equivalent to $[M \sqsupseteq Pre(u)$ **and** $(M \ominus Pre(u)) \oplus Post(u)) \sqsupseteq Pre(v)]$, i.e. $M[u]M'$ **and** $M * u[v]$. □

2.4. **Petri algebras.** We have so far identify the set of conditions that should be fulfilled by Petri algebras so that we can play the token game and the resulting firing rule is associative. To sum up, these structures are duals of commutative residuated lattices whose meet and joins are given by the formulas $a \sqcup b = a \ominus (a \ominus b)$ and $a \sqcap b = b \oplus (a \ominus b)$. Moreover this lattice is integral in the sense that the neutral element for the sum is also the least element of the lattice. Finally the underlying monoid is cancellable and this condition is equivalent to the identity (2.5).

Proposition 2.16. *A Petri algebra consists of a set A equipped both with a structure of a commutative monoid $(A, \oplus, 0)$ and a structure of a lattice $(A, \sqcup, \sqcap)$ such that:*

1. *the neutral element is the least element of the lattice:* $0 \sqsubseteq a$ *.*
2. *Sum admit an adjoint $\ominus$ characterized by the universal property:*

$$a \oplus b \sqsupseteq c \Leftrightarrow a \sqsupseteq c \ominus b$$

3. $(a \oplus b) \ominus b = a$.
4. $a \sqcap b = a \ominus (a \ominus b)$.
5. $a \sqcup b = b \oplus (a \ominus b)$.

Proof. It just remains to prove that a structure satisying the conditions stated in the proposition verify to the following properties.

1. $a \sqsupseteq b \;\Leftrightarrow\; \exists c \cdot a = b \oplus c$
2. $a \sqsupseteq b \;\Rightarrow\; a = b \oplus (a \ominus b)$.

The latter is an immediate consequence of $a \sqcup b = b \oplus (a \ominus b)$ and it entails the left to right implication of the former: $a \sqsupseteq b \;\Rightarrow\; \exists c \cdot a = b \oplus c$. The converse implication follows from the identity $a \oplus b \sqsupseteq a$ which in turn can be justified as follows. Identity $a \oplus 0 = a$ entails by adjunction that $a \ominus a \sqsubseteq 0$ (and thus $a \ominus a = 0$). Now the identity $a \ominus a \sqsubseteq b$ entails by adjunction the required property: $a \oplus b \sqsupseteq a$. □

Recall that a commutative GMV-algebra ([3, 9]) is a commutative residuated lattice such that $a \vee b = a \ominus ((a \ominus b) \wedge 0)$; it also satisfies $a \wedge b = ((a \ominus b) \wedge 0) \oplus b$. A GMV-algebra is said to be *integral* when the neutral element is the greteast element of the lattice (hence the least element for the inverse order relation $\sqsubseteq$). Thus we conclude that Petri algebras coincide with the (duals of) *integral, cancellative and commutative GMV-algebras.* These algebras form a sub-variety of the variety of residuated lattices and the following result is a direct consequence of [10, Theorem 5.6 and corollaries].

Theorem 2.17. *Petri algebras coincide with the positive cones of lattice-ordered abelian groups. Moreover lattice-ordered abelian groups constitute the subvariety of lattice-ordered groups generated by the group $\mathbb{Z}$ of integer, and their positive cones (i.e. Petri algebras) is the subvariety of residuated lattices generated by $\mathbb{N}$.*

To give a better understanding of this result we recall that if $(G, +, 0)$ is an abelian group equipped with an order relation compatible with its addition (i.e. addition is increasing w.r.t. both of its arguments) then the positive cone $P = \{x \in G \mid x \geq 0\}$ is a sub-monoid of G which is *zerosumfree* ($x + y = 0 \;\Rightarrow\; x = y = 0$). Conversely any zerosumfree sub-monoid P of G induces a compatible order relation on G whose corresponding positive cone is P: $x \sqsubseteq y \;\Leftrightarrow\; \exists z \in P \cdot y = x + z$. Of course a sub-monoid of a group is necessarily cancellable. If moreover G is a lattice-ordered group, meaning that each pair $\{a; b\}$ admits a join $a \sqcup b$ and a meet $a \sqcap b$ for the corresponding order relation, then the positive cone is residuated with $a \ominus b = (a - b) \sqcup 0$ (where $a - b = a + (-b)$) and is also lattice-ordered. For instance the monoid $(\mathbb{N}, +, 0)$ is the positive cone of the group $(\mathbb{Z}, +, 0)$ for the usual order relation on integers; this is a total order and thus in particular a lattice and therefore the preceeding considerations apply. We know that through linear algebra techniques this group of integer plays a central role in the study of the structural properties of Petri nets ([13]: the fundamental equation, linear invariants, siphons, traps ...) and as well in the algorithmic implementation of Petri nets synthesis based on the theory of regions [1].

3. Lexicographic Petri nets

3.1. **Sub-irreducible Petri algebras.** We define a (generalized) Petri net as a structure $\mathcal{N} = (P, T, Pre, Post, M_0)$ where P is a finite set of places with a Petri algebra A_p associated with each place $p \in P$, T is a finite set of transitions disjoint from P and $Pre, Post : P \times T \to \bigsqcup_{p \in P} A_p$, the flow relations, are such that $\forall p \in P \; \forall t \in$

T $Pre(p,t)$, $Post(p,t) \in A_p$. A marking is a map $M : P \to \bigsqcup_{p\in P} A_p$ that associates with each place $p \in P$ the local value of the current configuration $M(p) \in A_p$ in this place. M_0 is some fixed marking, called the *initial marking*. The transition relation $M\,[t\rangle\,M'$ stating that transition t can fire in marking M and leads, when it is fired, to the new marking M' is given by:

$$M\,[t\rangle\,M' \;\Leftrightarrow\; \forall p \in P \;\; M(p) \sqsupseteq Pre(p,t) \;\wedge\; M'(p) = (M(p) \ominus Pre(p,t)) \oplus Post(p,t)$$

This relation can be extended inductively to sequences $u \in T^*$ of transitions by letting $M\,[\varepsilon\rangle\,M$ for every marking M and $M\,[t\cdot u\rangle\,M'$ if and only if there exists some marking M'' such that $M\,[t\rangle\,M''$ and $M''\,[u\rangle\,M'$ for every $t \in T$ and $u \in T^*$. The set of *reachable markings* is $Reach(\mathcal{N}) = \{M \mid \exists u \in T^*\; M_0\,[u\rangle\,M\}$, and the *marking graph* of a generalized net $\mathcal{N} = (P, T, Pre, Post, M_0)$ is the labelled graph $\Gamma_{\mathcal{N}} = (V, \Lambda, v_0)$ whose set of vertices is given by the set $V = Reach(\mathcal{N})$ of reachable markings with $v_0 = M_0$ and whose set of arcs $\Lambda \subseteq V \times T \times V$ is the restriction of the transition relation to the set of reachable markings: $\Lambda = \{(M,t,M') \mid M, M' \in V \;\wedge\; M\,[t\rangle\,M'\}$. Two generalized Petri nets are termed *equivalent* when they have isomorphic marking graphs.

We immediately see that a place p whose type A_p is a sub-algebra of a product of Petri algebras ($A_p \subseteq A_1 \times \cdots \times A_n$) can be replaced by n places $p_1, \ldots, p_n$ with respective types $A_1, \ldots, A_n$ without changing the marking graph (at least up to isomorphism). A classical result of universal algebra says that any algebra of a variety is a sub-direct product of subdirectly irreducible algebras [2]. Thus we can assume without loss of generality that all algebras A_p are subdirectly irreducible algebras in the variety of Petri algebras. Now any $M(p)$ belongs to the sub-algebra of A_p generated by the set $\{M_0(p)\} \cup \bigcup_{t\in T} \{Pre(p,t),\, Post(p,t)\}$. Thus:

Theorem 3.1. *Every generalized Petri net is equivalent to a generalized Petri net all of whose types are subdirectly irreducible and finitely generated Petri algebras.*

Let $Irr\,(V)$ denote the set of sub-direct irreducible algebras of a variety V , then if V is a subvariety of W one has $Irr(W) \cap V = Irr(V)$; using the fact that the subdirectly irreducible commutative GMV-algebras are chains (simply ordered sets) we deduce that

Proposition 3.2. *subdirectly irreducible Petri algebras are chains.*

An algebra is subdirectly irreducible if and only if it admits a least non trivial congruence [4]. Now we know [5, 6] that the congruences of Petri algebras are in bijective correspondance with their convex sub-monoids. On the one direction we can associate each congruence θ of a Petri algebra A with the class of the neutral element which is a convex sub-monoid

[2] An algebra A is a sub-direct product of a family of similar algebras $\{A_i \mid i \in I\}$ if A is a sub-algebra of the product algebra $\prod_{i\in I} A_i$ such that for every index $i \in I$ the morphism $p_i : A \to A_i$, obtained by composition of the embedding of A into $\prod_{i\in I} A_i$ with the canonical projection $\prod_{i\in I} A_i \to A_i$, is surjective. The set $\{(A_i, p_i) \mid i \in I\}$ is a representation of A as sub-direct product of the algebras A_i, symetrically such a representation can be given as a family of surjective morphisms $p_i : A \to A_i$ jointly injective, that is to say such that the product $\langle p_i \rangle : A \to \prod_{i\in I} A_i$ is injective. An algebra A is say to be subdirectly irreducible if for every representation $\{(A_i, p_i) \mid i \in I\}$ of A as a sub-direct product of algebras A_i there exists at least one index i for which the corresponding morphism p_i is an isomorphism.

$M_\theta = [0]_\theta$ of A. Conversely we associate each such monoid M to the congruence $\theta_M = \{(a,b) \in A^2 \mid b \ominus a, a \ominus b \in M\}$. The correspondances $\theta \mapsto M_\theta$ et $M \mapsto \theta_M$ are inverses to each other and they establish an isomorphism between the lattice of congruences of A and the lattice of the convex sub-monoids of A. Moreover for every $a \in A$, the principal congruence generated by the equation $a = 0$ corresponds to the convex sub-monoid generated by a. A Petri algebra is then subdirectly irreducible if and only if it admits a least non trivial convex sub-monoid. Let us assume that A is a totally ordered Petri algebra. Let

$$M(x) = \{y \in A \mid \exists k \in \mathbb{N} \cdot y \sqsubseteq k \cdot x = \underbrace{x \oplus \cdots \oplus x}_{k \text{ times}}\}$$

denote the principal convex sub-monoid generated by $x \in A$. $M(x)$ is non-trivial if and only if $x \neq 0$. Now if x is some element of a convex sub-monoid of A one necessarily have $M(x) \subseteq M$; thus a minimal convex sub-monoid is principal and is generated by any of its non null elements. Since A is totally ordered and $x \le y \ \Rightarrow \ M(x) \subseteq M(y)$ we deduce that A admits at most one minimal non trivial sub-monoid. $M(x)$ is minimal if and only if $y \ll x \ \Rightarrow \ y = 0$ where relation $\ll$ is given by

$$y \ll x \quad \Leftrightarrow \quad \forall k \in \mathbb{N} \cdot k \cdot y \sqsubset x$$

Otherwise stated $y \ll x$ if and only if $y \sqsubset x$ and $M(y)$ is strictly included in $M(x)$. Therefore A has no non trivial minimal sub-monoid if and only if for every $x \in A \setminus \{0\}$ one can find some $y \in A \setminus \{0\}$ such that $y \ll x$. Under that condition one can form an infinite strictly decreasing chain thus proving that the order relation $\ll$ is not well-founded[3]. Conversely if this order is well-founded then any non empty subset of A, and thus in particular $A \setminus \{0\}$ if A is not trivial, admits a least element for this order which shows the existence of a minimal non trivial sub-monoid. We thus have established the following:

Theorem 3.3. *A Petri algebra is subdirectly irreducible if and only if it is a chain and the order relation* $y \ll x \ \Leftrightarrow \ \forall k \in \mathbb{N} \cdot k \cdot y \sqsubset x$ *is well-founded.*

3.2. **Lexicographic Petri nets.** The lexicographic product of two ordered groups $\mathbb{G}$ and $\mathbb{H}$ is the product group $\mathbb{G} \times \mathbb{H}$ equipped with the lexicographic order relation:

$$(x,y) \le_{\mathbb{G}\circ\mathbb{H}} (x',y') \quad \Leftrightarrow \quad x <_{\mathbb{G}} x' \text{ or } (x = x' \text{ and } y \le_{\mathbb{H}} y')$$

If $\mathbb{G}$ and $\mathbb{H}$ are simply ordered abelian groups then the same hold for their lexicographic product. This product is associative and we can defined inductively $L_n(\mathbb{G}) = (\mathbb{G}^n)^+$ for every simply ordered abelian group $\mathbb{G}$ and integer $n \in \mathbb{N}$ by letting $\mathbb{G}^0 = \{0\}$ be the trivial group and $\mathbb{G}^{n+1} = \mathbb{G}^n \circ \mathbb{G}$. The group $\mathbb{G}^n$ naturally embedds into $\mathbb{G}^m$ when $n \le m$; the projective limit of this sequence of embeddings is the group $\mathbb{G}^\omega$ whose elements are the

[3]We recall that an order relation is well-founded if it satisfies any of the following equivalent conditions:

(1) every decreasing infinite sequence is stationary,
(2) every stricly decreasing sequence is finite,
(3) every non empty subset admit a least element.

An ordinal, or well-ordered set, is a set with a total and well-founded order.

infinite sequences of elements in $\mathbb{G}$, with componentwise composition and the lexicographic order relation defined as follows: $u \leq_{lex} v \quad \Leftrightarrow \quad u <_{lex} v$ or $u = v$ where

$$u <_{lex} v \quad \Leftrightarrow \quad \exists n \in \mathbb{N}\, \forall m \leq n \;\; u_m = v_m \text{ and } u_n <_{\mathbb{G}} v_n$$

The inductive limit, or "union" $\bigcup_{n<\omega} \mathbb{G}^n$, is the subgroup of $\mathbb{G}^\omega$ consisting of the sequences u of finite support ($supp(u) = \sup\{k \in \mathbb{N} \mid u_k \neq 0\} < \omega$) with $\mathbb{G}^n$ identified with the subgroup of $u \in \mathbb{G}^\omega$ such that $supp(u) \leq n$.

Definition 3.4. The set $Lex\,(\mathbb{G})$ of lexicographic Petri net based on a simply ordered abelian group $\mathbb{G}$ is the set of (homogeneous) generalized Petri net of type $(\mathbb{G}^\omega)^+$. $Lex\,(\mathbb{G}, n) \subseteq Lex\,(\mathbb{G})$ is the set of n-dimensional lexicographic Petri nets with type $L_n\,(\mathbb{G}) = (\mathbb{G}^n)^+ \subseteq (\mathbb{G}^\omega)^+$, i.e. all flow arc inscriptions and initial place contents, and hence all place contents in every accessible marking, are element in $(\mathbb{G}^n)^+$.

If $\mathcal{K}$ and $\mathcal{L}$ are subclass of generalized Petri nets we let $\mathcal{K} \lesssim \mathcal{L}$ when every net in $\mathcal{K}$ is equivalent to some net in $\mathcal{L}$. This is a pre-order relation, we let $\approx$ denote its associated equivalence relation and $\lnapprox$ the corresponding strict relation: $\mathcal{K} \lnapprox \mathcal{L}$ when every net in $\mathcal{K}$ is equivalent to some net in $\mathcal{L}$ but there exists some net in $\mathcal{L}$ not equivalent to any net in $\mathcal{K}$. Notice that $Lex(\mathbb{G}, n) \lesssim Lex(\mathbb{H}, m)$ when $\mathbb{G} \subseteq \mathbb{H}$ and $n \leq m$; and that $Lex(\mathbb{Z}, 1)$ is the class of Petri nets.

Lemma 3.5. *Any finitely generated subdirectly irreducible Petri algebra A is isomorphic to a subalgebra of the positive cone of some finite power of the additive group of real numbers: $A \subseteq (\mathbb{R}^n)^+$.*

Proof. A is a chain and we can partition its set of generators into $G = G_1 \cup \cdots \cup G_n$ such that (i) $\forall i < j \leq n \;\, a_i \in G_i \wedge a_j \in G_j \Rightarrow a_i \ll a_j$ and (ii) all element in G_i are comparable : $a_i,\, a_i' \in G_i \Rightarrow \exists k \in \mathbb{N}\;\, a_i \sqsubseteq k \cdot a_i'$. Let A_i the sub algebra of A generated by G_i. A_i is a simply ordered archimedean ($a \ll b \Rightarrow a = 0$) divisibility monoid (Archimedean systems of Magnitudes) and thus is isomorphic to an ordered-submonoid of the additive monoid of positive real numbers ([4, 8]), hence $A_i \subseteq \mathbb{R}^+$. Now we check that A is a sub-algebra of the lexicographic product of the A_i's: $A \subseteq A_n \circ \cdots \circ A_1$. For each decomposition $a = \sum_{g \in G} \lambda_g \cdot g$ of an element $a \in A$ in terms of the generators ($\lambda_g \in \mathbb{N}$) we obtain a related decomposition $a = a_n \oplus \cdots \oplus a_1$ with $a_i \in A_i$ by letting $a_i = \sum_{g \in G_i} \lambda_g \cdot g$. Let us verify that such a decomposition is unique. We shall assume $n = 2$, the proof extends by induction for larger n. Thus we assume $a_1 \oplus a_2 = a_1' \oplus a_2'$ with $a_i,\, a_i' \in A_i$. Since the order relation is total and the addition is monotonic we may assume that $a_1 \sqsubseteq a_1'$ and $a_2' \sqsubseteq a_2$. Then

$$\begin{array}{rcll} a_1 \oplus a_2 = a_1' \oplus a_2' & \Leftrightarrow & a_1' = (a_1 \oplus a_2) \ominus a_2' & \text{By Lemma 2.10} \\ & \Leftrightarrow & a_1' = a_1 \oplus (a_2 \ominus a_2') & \text{By Lemma 2.12} \\ & \Leftrightarrow & a_1' \ominus a_1 = a_2 \ominus a_2' & \text{By Lemma 2.10} \end{array}$$

Now $a_1' \ominus a_1 \in A_1$ and $a_2 \ominus a_2' \in A_2$ and the only way that these elements can be comparable is when they are null, i.e. $a_1 = a_1'$ and $a_2 = a_2'$ as required. We let $\varphi_i : A \to A_i$ be the map such that $\varphi_i(a) = a_i$ where $a = a_n \oplus \cdots \oplus a_1$ with $a_i \in A_i$. These maps are monoid morphisms, we let $\varphi : A \to A_n \circ \cdots \circ A_1$ denote the monoid morphism obtained

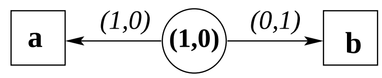

FIGURE 3.1. A two dimensional lexicographic Petri net

by collecting them : $\varphi(a) = (\varphi_n(a), \ldots, \varphi_1(a))$. By construction φ is an injection with left-inverse $\psi : A_n \circ \cdots \circ A_1 \to A$ given by $\psi(x_n, \ldots, x_1) = x_n \oplus \cdots \oplus x_1$. Thus φ induces an isomorphism between the monoid reduct of A and a submonoid of the monoid reduct of $A_n \circ \cdots \circ A_1$. Now by definition of the lexicographic order we have that this injection φ preserves and reflects the respective order relations. Since residuation in a Petri algebra is characterized by the monoid structure and the order relation, we deduce that the image of A by φ is a sub-algebra of $A_n \circ \cdots \circ A_1$. □

By Theorem 3.1 we deduce the following result.

Theorem 3.6. *Every generalized Petri net is equivalent to some lexicographic Petri net, more precisely :*

$$GenPetri \approx Lex\,(\mathbb{R})$$

We have in fact proved a more precise result, namely that every generalized Petri net is equivalent to a finite dimensional lexicographic Petri net based on the additive group of real numbers, i.e. $GenPetri \approx \bigcup_{n<\omega} Lex\,(\mathbb{R}, n)$.

Proposition 3.7. $Petri = Lex(\mathbb{Z}, 1) \lnapprox GenPetri$

Proof. We provide an example showing that $Lex(\mathbb{Z}, 1) \lnapprox Lex(\mathbb{Z}, 2)$, i.e. that lexicographic Petri nets based on the group of integers of dimension 2 are already strictly more expressive than the class of Petri nets. Let us denote $n \cdot \omega + m$ the element $(n, m) \in \mathbb{Z} \circ \mathbb{Z}$ and let us consider the net of type

$$L_2 = (\mathbb{Z} \circ \mathbb{Z})^+ = \{(n, m) \mid (n = 0 \text{ and } m \geq 0) \text{ or } (n > 0 \text{ and } m \in \mathbb{Z})\}$$

depicted in Fig. 3.1. This net has a unique place $P = \{p\}$ and two transitions $T = \{a; b\}$ with the following flow relations $Pre(p, a) = \omega$, $Pre(p, b) = 1$ et $Post(p, a) = Post(p, b) = 0$. From the initial marking ω transition a can fire once $\omega\,[a\rangle\,0$ and transition b can fire an infinite number of time leading to the infinite firing sequence

$$\omega\,[b\rangle\,\omega - 1\,[b\rangle\,\omega - 2 \ldots [b\rangle\,\omega - n \ldots$$

and there are no other transitions in the marking graph of the net. Suppose there exists some Petri net with an isomorphic marking graph. Since transition b can fire an infinite number of time, and

$$M_0\,[b^n\rangle\,M_n \quad \Rightarrow \quad \forall p \in P\ M_n(p) = M_0(p) - n \times (Pre(p, b) - Post(p, b))$$

we deduce that for every place p it is the case that $Post(p, b) \sqsupseteq Pre(p, b)$ and thus $M_n(p) \sqsupseteq M_0(p)$. By monotony of the firing rule, any transition that can fire in the initial marking M_0

can also fire in any of the markings M_n obtained by firing b. Transition a is in contradiction with this property. □

By generalizing on the above we could establish that

$$n < m < \omega \;\Rightarrow\; Lex(\mathbb{Z}, n) \lnapprox Lex(\mathbb{Z}, m) \lnapprox Lex(\mathbb{Z})$$

it can also be shown that

$$Lex(\mathbb{Z}, n) \approx Lex(\mathbb{Q}, n) \lnapprox Lex(\mathbb{R}, n)$$

3.3. **Lexicographic Petri nets and Petri nets with inhibitor arcs.** It appears to be diffcult to obtain strict extensions of the class of Petri nets that preserve all of its decidable properties. Many of these extensions, like the class of Petri nets with inhibitor arcs, are indeed Turing-powerful.

Definition 3.8. A Petri net with inhibitor arcs is a structure $\mathcal{N} = (P, T, Pre, Post, M_0)$ where P, T, $Post$, and M_0(initial marking) are defined as for the usual Petri nets. Pre is a map from $P \times T$ to $\mathbb{N} \cup \{\varsigma\}$ where ζ is a special symbol. We say that we have an inbitor arc from place p to transition t when $Pre(p,t) = \varsigma$. Transition t is firable in marking $M \in \mathbb{N}^P$ when

$$\forall p \in P \quad [Pre(p,t) \in \mathbb{N} \;\wedge\; M(p) \sqsupseteq Pre(p,t)] \;\vee\; [Pre(p,t) = \varsigma \;\wedge\; M(p) = 0]$$

i.e. they are enough resources in the input places ${}^\bullet t = \{p \in P \mid Pre(p,t) > 0\}$ of transition t and all of its inhibiting places ${}^\circ t = \{p \in P \mid Pre(p,t) = \varsigma\}$ are empty. Then the firing of transition t leads to a new marking M' defined as follows:

$$\forall p \in P \qquad M'(p) = \begin{cases} M(p) - Pre(p,t) + Post(p,t) & \text{if} \quad Pre(p,t) \in \mathbb{N} \\ M(p) + Post(p,t) & \text{if} \quad Pre(p,t) = \varsigma \end{cases}$$

We let $M\,[t\rangle\,M'$ express the fact that transition t can fire in marking M and leads to marking M' when it fires. This relation can be extended inductively to sequences $u \in T^*$ of transitions by letting $M\,[\varepsilon\rangle\,M$ for every marking M and $M\,[t \cdot u\rangle\,M'$ if and only if there exists some marking M'' such that $M\,[t\rangle\,M''$ and $M''\,[u\rangle\,M'$ for every $t \in T$ and $u \in T^*$. The set of ***reachable markings*** is $Reach(\mathcal{N}) = \{M \mid \exists u \in T^*\; M_0\,[u\rangle\,M\}$, and the ***marking graph*** of a Petri net with inhibitor arcs $\mathcal{N} = (P, T, Pre, Post, M_0)$ is the labelled graph $\Gamma_{\mathcal{N}} = (V, \Lambda, v_0)$ whose set of vertices is given by the set $V = Reach(\mathcal{N})$ of reachable markings with $v_0 = M_0$ and whose set of arcs $\Lambda \subseteq V \times T \times V$ is the restriction of the transition relation to the set of reachable markings: $\Lambda = \{(M, t, M') \mid M, M' \in V \;\wedge\; M\,[t\rangle\,M'\}$.

Figure (3.2) displays a Petri net with inhibitor arcs and its marking graph. Inhibitor arcs restrict the firing of transitions but have no effect on the content of places when transitions are allowed to occur. This means that the marking graph of a Petri net with inhibitor arcs $\mathcal{N}$ is a subgraph of the marking graph of the induced Petri net $\mathcal{N}^\circ$ obtained by removing the inhibitor arcs.

Theorem 3.9. *Lexicographic Petri nets are a strict extension of the class of Petri nets with inhibitor arcs.*

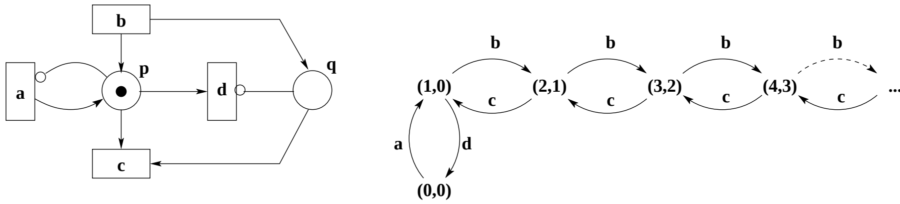

FIGURE 3.2. A Petri net with inhibitor arcs and its marking graph

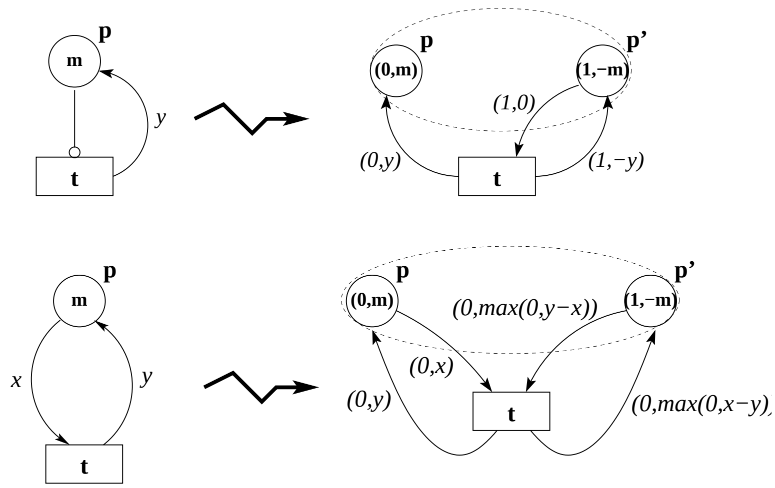

FIGURE 3.3. A translation from Petri nets with inhibitor arcs into lexicographic Petri nets

Proof. The translation of a Petri net with inhibitor arcs $\mathcal{N}$ into an equivalent lexicographic Petri net $\overline{\mathcal{N}}$, illustrated in Fig. (3.3), consists in splitting every place p with initial marking $m \in \mathbb{N}$ of the original net into two places denoted p and p' with initial markings $(0, m) \in (\mathbb{Z} \circ \mathbb{Z})^+$ and $(1, -m) \in (\mathbb{Z} \circ \mathbb{Z})^+$respectively. The restriction of this lexicographic Petri net to set of places p of the first family is isomorphic to $\mathcal{N}^\circ$, i.e. the original net in which the inhibitor arcs have been removed. We thus have a graph morphism $\Gamma_{\overline{\mathcal{N}}} \to \Gamma_{\mathcal{N}^\circ}$ sending a marking of the lexicographic Petri net to its restriction to the set of places of the original net $\mathcal{N}$. Now the set of reachable markings of $\overline{\mathcal{N}}$ satisfy the following property:

$$\mathcal{K}(M) \quad \equiv \quad \forall p \in P \quad \exists m_p \in \mathbb{N} \quad M(p) = (0,\, m_p) \;\wedge\; M(p') = (1, -m_p)$$

This property holds for the initial marking by definition and is preserved by the firing of a transition due to the fact that

$$x - y = \max(0,\, x - y) - \max(0,\, y - x)$$

Therefore this graph morphism is injective and one has a pair of graphs monomorphisms :

$$\Gamma_{\mathcal{N}} \overset{\iota}{\hookrightarrow} \Gamma_{\mathcal{N}^\circ} \overset{\iota'}{\hookleftarrow} \Gamma_{\overline{\mathcal{N}}}$$

The inhibitor arcs in $\mathcal{N}$ and the places of the second family in $\overline{\mathcal{N}}$ (i.e. places of the form p' for $p \in P$) serve the same purpose: cutting off a subset of transitions in $\Gamma_{\mathcal{N}^\circ}$. It just remain to shows that they cut off exactly the same transitions. Place p' inhibits the firing of transition t in some marking M satisfying property $\mathcal{K}$ if and only if $(1, -m_p) \sqsubset Pre(p', t)$; this is possible only if $Pre(p', t) = (1, 0)$ in $\overline{\mathcal{N}}$ i.e. $Pre(p, t) = \varsigma$ in $\mathcal{N}$ (there is an inhibitor arc from p to t) and $m_p \neq 0$ and these conditions are also clearly sufficient conditions:

$$(1, -m_p) \sqsubseteq Pre(p', t) \quad \Leftrightarrow \quad Pre(p, t) = \varsigma \;\wedge\; m_p \neq 0$$

which means there exists for the associated marking in $\mathcal{N}$ an inhibitor arc from a non-empty place to that transition. Figure (3.4) shows the lexicographic Petri net associated with the Petri net with inhibitor arcs of Fig. (3.2).

In order to show that lexicographic Petri nets are strictly more expressive than Petri nets with inhibitor arcs, it suffices to observe that there exists no Petri net with inhibitor arcs whose language is the language of the lexicographic Petri net of Fig. (3.5) i.e. the prefix closure of $b^* + ab$. Suppose on the contrary that such a Petri net with inhibitor arcs exists. Since transition b can fire an arbitrary number of time in the initial marking it should be the case that for every place p either there exists an inbitor arc from p to b and no output arc from b to p or $Post(p, b) \sqsupseteq Pre(p, b)$. Thus if follows that b can fire an arbitrary number of time in every marking at which it can fire once ; which contradict the fact that after the firing of transition a transition b can fire only once. □

Corollary 3.10. *Reachability, Coverability, Place-boundedness, Boundedness, Deadlock and Liveness are undecidable for the class of lexicographic Petri nets.*

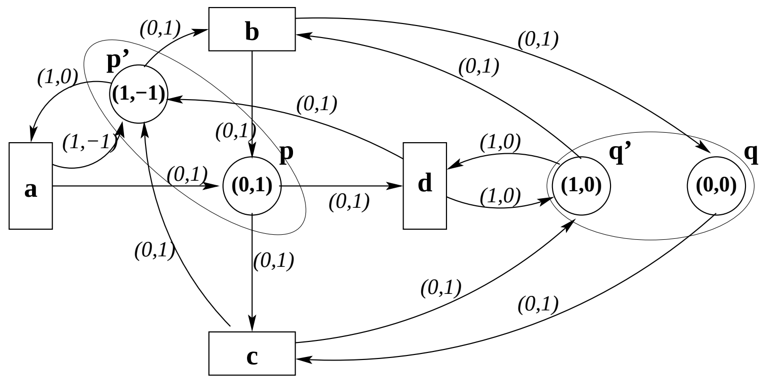

FIGURE 3.4. A lexicographic Petri net equivalent to the Petri net with inhibitor arcs of Fig.3.2

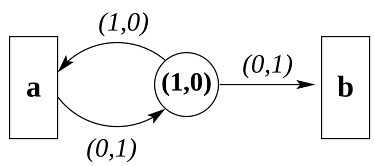

FIGURE 3.5. A lexicographic Petri net with no equivalent Petri net with inhibitor arcs

4. Bounded nets

4.1. **Bounded nets associated with MV-algebras.** A net is bounded if we can find an upper bound on the possible values of places in any accessible marking. Let us start our study on the algebraization of the dynamic of bounded nets by the following observation.

Proposition 4.1. *Any non trivial commutative Petri algebra is an unbounded lattice.*

Proof. Suppose a bounded and non trivial commutative Petri algebra, let $\infty \neq 0$ stand for its greatest element. One has

1. $\infty \ominus \infty = 0$ because more generally from $x \ominus x \sqsubseteq 0 \;\Leftrightarrow\; x \sqsubseteq 0 \oplus x = x$ it follows $x \ominus x = 0$.
2. $\infty \ominus 0 = \infty$ because $\infty \ominus 0 \sqsubseteq x \;\Leftrightarrow\; \infty \sqsubseteq 0 \oplus x = x$.
3. $\infty \oplus x = \infty$ because $\infty \oplus x \sqsupseteq y \;\Leftrightarrow\; \infty \sqsupseteq y \ominus x$.
4. $x \ominus \infty = 0$ because $x \ominus \infty \sqsubseteq y \;\Leftrightarrow\; x \sqsubseteq y \oplus \infty = \infty$.

Then the identity $(a \oplus b) \ominus b = a$ is not satisfied for $a = b = \infty$ since the left-hand side evaluates to 0. □

However we can enforce boundedness by modifying the rule of the token game. Let us consider first the case of the usual Petri nets: assume that each place $p \in P$ is associated with a capacity $k_p \in \mathbb{N}$ and that we want to enforce that the value of a place p of a Petri net be bounded by above by its capacity k_p. For that purpose we modify the firing rule as follows (where all computations are performed in $\mathbb{Z}$)

$$M\,[t\rangle\,M' \;\Leftrightarrow\; \forall p \in P \quad \begin{cases} M(p) \sqsupseteq Pre(p,t) \;\wedge\; (M(p) - Pre(p,t)) + Post(p,t) \sqsubseteq k_p \\ M'(p) = (M(p) - Pre(p,t)) + Post(p,t) \end{cases}$$

this rule can be reformulated as:

$$M\,[t\rangle\,M' \;\Leftrightarrow\; \forall p \in P \quad \begin{cases} Pre(p,t) \sqsubseteq M(p) \sqsubseteq (k_p + Pre(p,t)) - Post(p,t) \\ M'(p) = (M(p) - Pre(p,t)) + Post(p,t) \end{cases}$$

Petri algebras are the positive cones G^+ of lattice-ordered abelian groups $G = (G, +, 0, \sqcup, \sqcap)$. It is an algebra with the following operations:

- ❖ **the sum:** restriction of the group operation $x \oplus y = x + y$
- ❖ **the truncated difference:** $x \ominus y = (x - y) \sqcup 0$

Let $k \sqsupseteq 0$ be some element of this positive cone ; we suppose that it is a *strong unit* in the sense that $\forall g \in G\; \exists n \in \mathbb{N} \;\cdot\; n \cdot k \sqsupseteq g$. We can by modifying the firing rule enforce that the values of places stay within the interval $I = [0, k] = \{g \in G \mid 0 \sqsubseteq g \sqsubseteq k\}$. This interval with induced order is a bounded lattice that can be equipped with the following operations:

- ❖ **the truncated sum:** $x \boxplus y = (x + y) \sqcap k$
- ❖ **the truncated difference:** $x \ominus y = (x - y) \sqcup 0$
- ❖ **the product:** $x \bullet y = ((x + y) - k) \sqcup 0$
- ❖ **the implication:** $x \to y = ((y + k) - x) \sqcap k$
- ❖ **the negation:** $\neg x = (x \to 0) = k \ominus y$

The following properties are met:

(1) It is a bounded lattice: $0 \sqsubseteq x \sqsubseteq k$
(2) $(I, \boxplus, 0)$ is an abelian monoid whose unit is an absorbing element for the product: $0 \bullet x = 0$.
(3) $(I, \bullet, k)$ is an abelian monoid whose unit k is an absorbing element for the sum: $k \boxplus x = k$.
(4) Negation is an involution ($\neg\neg x = x$) such that $k = \neg 0$ (hence $0 = \neg k$) and $x \bullet y = \neg(\neg x \boxplus \neg y)$ (hence $x \boxplus y = \neg(\neg x \bullet \neg y)$)
(5) $x \sqsubseteq y \;\Leftrightarrow\; (x \to y) = k \;\Leftrightarrow\; (x \ominus y) = 0$
(6) $\bullet$ is a residuated monoid: $x \bullet y \sqsubseteq z \;\Leftrightarrow\; x \sqsubseteq y \to z$
(7) $\boxplus$ is a co-residuated monoid: $x \boxplus y \sqsupseteq z \;\Leftrightarrow\; x \sqsupseteq\; z \ominus y$
(8) $(y \to z) = z \boxplus \neg y$ and $z \ominus y = z \bullet \neg y$
(9) $x \sqcup y \;=\; (y \to x) \to x \;=\; (x \to y) \to y \;=\; x \boxplus (y \ominus x) \;=\; y \boxplus (x \ominus y)$
(10) $x \sqcap y \;=\; x \bullet (x \to y) \;=\; y \bullet (y \to x) \;=\; x \ominus (x \ominus y) \;=\; y \ominus (y \ominus x)$

Such a structure is called an MV-algebra. MV-algebras are generalizations of boolean algebras used in the algebraic analysis of Lukasiewcz infinite-valued propositional logic and this class of algebras admits several equivalent definitions [7, 12]. For instance an MV-algebra is a bounded commutative residuated lattice ($x \bullet y \sqsubseteq z \;\Leftrightarrow\; x \sqsubseteq y \to z$) such that $x \sqcup y \;=\; (y \to x) \to x$. An MV-algebra is also an abelian monoid $(A, \bullet, 1)$ together with an involution $\neg : A \to A$ such that if we let $x \boxplus y = \neg(\neg x \bullet \neg y)$, $0 = \neg 1$ and $(x \to y) = \neg x \boxplus y$ then $x \bullet 0 = 0$ and $(y \to x) \to x \;=\; (x \to y) \to y$.

Definition 4.2. A bounded net is a structure $\mathcal{N} = (P, T, Pre, Post, M_0)$ where P is a finite set of places with an MV-algebra A_p associated with each place $p \in P$ (the unit k_p of A_p is called the capacity of place p), T is a finite set of transitions disjoint from P, and $Pre, Post : P \times T \to \bigsqcup_{p \in P} A_p$, the flow relations, are such that $\forall p \in P \; \forall t \in T \quad Pre(p,t), Post(p,t) \in A_p$. A marking is a map $M : P \to \bigsqcup_{p \in P} A_p$ that associates with each place $p \in P$ the local value of the current configuration $M(p) \in A_p$ in this place. M_0 is some fixed marking, called the *initial marking*. The transition relation $M\,[t\rangle\,M'$ stating that transition t can fire in marking M and leads, when it is fired, to the new marking M' is given by:

$$M\,[t\rangle\,M' \;\Leftrightarrow\; \forall p \in P \begin{cases} Pre(p,t) \sqsubseteq M \sqsubseteq Post(p,t) \to Pre(p,t) \\ M' = (M \ominus Pre(p,t)) \boxplus Post(p,t) \end{cases} \tag{4.1}$$

This relation can be extended inductively to sequences $u \in T^*$ of transitions by letting $M\,[\varepsilon\rangle\,M$ for every marking M and $M\,[t \cdot u\rangle\,M'$ if and only if there exists some marking M'' such that $M\,[t\rangle\,M''$ and $M''\,[u\rangle\,M'$ for every $t \in T$ and $u \in T^*$. The set of *reachable markings* is $Reach(\mathcal{N}) = \{M \mid \exists u \in T^* \; M_0\,[u\rangle\,M\}$, and the *marking graph* of a bounded net $\mathcal{N} = (P, T, Pre, Post, M_0)$ is the labelled graph $\Gamma_{\mathcal{N}} = (V, \Lambda, v_0)$ whose set of vertices is given by the set $V = Reach(\mathcal{N})$ of reachable markings with $v_0 = M_0$ and whose set of arcs $\Lambda \subseteq V \times T \times V$ is the restriction of the transition relation to the set of reachable markings: $\Lambda = \{(M, t, M') \mid M, M' \in V \;\wedge\; M\,[t\rangle\,M'\}$.

The boolean algebra $2 = \{0, 1\}$ is an MV-algebra where $x \boxplus y = x \sqcup y$ and $x \bullet y = x \sqcap y$; Let P be the set of places of a net and $B = 2^P = \wp(P)$ the corresponding product structure, the preceding firing rule can be reformulated as:

$$M\,[a\rangle\,M' \;\Leftrightarrow\; M \supseteq Pre(a) \;\wedge\; M \cap Post(a) \subseteq Pre(a) \;\wedge\; M' = (M \setminus Pre(a)) \cup Post(a)$$

which is the usual firing rule of one safe nets. If we replace the boolean algebra $2 = \{0,1\}$ by the interval $[0,1]$ of the additive group of real numbers, i.e. with $x \boxplus y = min(1, x+y)$ and $x \bullet y = max(0, x+y-1)$ then we obtain the firing rule of some kind of one safe "fuzzy" nets :

$$M\,[a\rangle\,M' \;\Leftrightarrow\; \forall p \in P \left\{ \begin{array}{l} Pre(a,p) \sqsubseteq M(p) \sqsubseteq Pre(a,p) + 1 - Post(a,p) \\ M'(p) = min(1, M(p) + Post(a,p) - Pre(a,p)) \end{array} \right.$$

4.2. **Complementary places.** Mundici [12] proved that MV-algebras coincide with $[0,k]$ intervals of abelian lattice-ordered groups where k is a strong unit. More precisely if $(G = G, +, 0)$ is an abelian lattice-ordered group with strong unit k then $\Gamma\,(G,k) = (A, \boxplus, 0, \bullet, k, \neg)$ where $G = [0,k] = \{x \in G \mid 0 \sqsubseteq x \sqsubseteq k\}$, $x \boxplus y = (x+y) \sqcap k$, $x \bullet y = ((x+y)-k) \sqcup 0$, and $\neg x = k - x$ is an MV-algebra such that the restriction of the order relation of the group on the unit interval $[0,k]$ coincides with the order relation of the MV-algebra: $x \sqsubseteq y \Leftrightarrow x \ominus y = 0$ (where $x \ominus y = x \bullet \neg y = (x-y) \sqcup 0$). Moreover Γ extends into an equivalence between the respective categories of abelian lattice-ordered groups with strong unit and MV-algebras, i.e. Γ induces a bijective correspondence between isomorphism classes of abelian lattice-ordered groups with strong unit and isomorphism classes of MV-algebras. Now we have seen that Petri algebras corresponds bijectively, up to isomorphism, to the positive cones of abelian lattice-ordered groups, we thus have a Petri algebra canonically associated with each MV-algebra. The following result shows that, by using complementary places, we can simulate a bounded Petri net by a generalized Petri net defined on the associated Petri algebras.

Theorem 4.3. *Any bounded net can be simulated by a generalized Petri net.*

Lemma 4.4. *For every elements a, b, and c in an MV-algebra one has*

$$b \sqsubseteq \neg a \sqsubseteq b \boxplus c \quad \Rightarrow \quad (a \boxplus b) \bullet c = (a \bullet (c \boxplus b)) \boxplus (b \bullet c) \tag{4.2}$$

Proof. Since every MV-algebra is a subdirect product of totally ordered MV-algebras we can assume without loss of generality that the given MV-algebra is totally ordered. Condition $b \sqsubseteq \neg a$ is equivalent to $a \bullet b = 0$ i.e. $a + b \sqsubseteq k$ in the underlying group. Condition $\neg a \sqsubseteq b \boxplus c$ is equivalent to $a \boxplus b \boxplus c = k$, i.e. $a + b + c \sqsupseteq k$ in the underlying group. On the one hand

$$(a \boxplus b) \bullet c = ((a+b) \sqcap k) \bullet c = (a+b) \bullet c = (a+b+c-k) \sqcup 0 = a+b+c-k$$

On the other hand

$$\begin{array}{rcl} a \bullet (c \boxplus b) & = & a \bullet [(c+b) \sqcap k] \\ & = & (a + [(c+b) \sqcap k] - k) \sqcup 0 \\ & = & \left\{ \begin{array}{ll} a & \text{if} \;\; c+b \sqsupseteq k \\ a+c+b-k & \text{if} \;\; c+b \sqsubseteq k \end{array} \right. \end{array}$$

and

$$b \bullet c = (b+c-k) \sqcup 0 = \left\{ \begin{array}{ll} b+c-k & \text{if} \;\; b+c \sqsupseteq k \\ 0 & \text{if} \;\; b+c \sqsubseteq k \end{array} \right.$$

and thus $(a \boxplus b) \bullet c = (a \bullet (c \boxplus b)) \boxplus (b \bullet c)$ as required. □

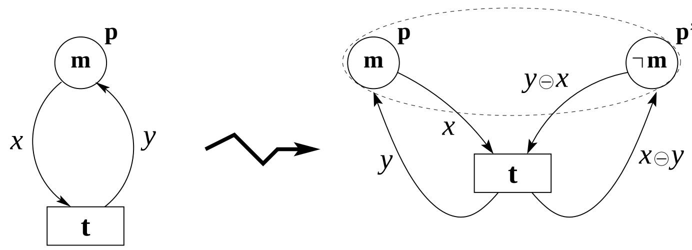

FIGURE 4.1. Translation from bounded nets to generalized Petri nets

The above property may be rephrased as :

$$b \sqsubseteq \neg a \sqsubseteq b \boxplus \neg c \; (= c \to b) \quad \Rightarrow \quad (a \boxplus b) \ominus c = (a \ominus (c \ominus b)) \boxplus (b \ominus c) \tag{4.3}$$

and thus appears as a weakening of equation (2.8) which ensures the associativity of the firing rule of generalized Petri nets.

Proof. (of Theorem 4.3) The translation of a bounded net $\mathcal{N}$ into an equivalent generalized Petri net $\overline{\mathcal{N}}$, illustrated in Fig. (4.1), consists in splitting every place p with initial marking $m \in A_p$ of the bounded net into two places denoted p and p' with initial markings $m \in (\mathbb{G}_p)^+$ and $k_p - m \in (\mathbb{G}_p)^+$respectively where $(\mathbb{G}_p, k_p)$ is the lattice-ordered abelian group with strong unit associated with the MV-algebra $A_p \simeq \Gamma(\mathbb{G}_p, k_p)$. I.e. one has

$$\begin{array}{rcl} \overline{Pre}(p,t) & = & Pre(p,t) \\ \overline{Post}(p,t) & = & Post(p,t) \\ \overline{Pre}(p',t) & = & Post(p,t) \ominus Pre(p,t) \\ \overline{Post}(p',t) & = & Pre(p,t) \ominus Post(p,t) \end{array}$$

First one has $x \sqsubseteq m \sqsubseteq y \to x = \neg y \boxplus x$ if and only if $m \sqsupseteq x$ and $\neg m \sqsupseteq y \bullet \neg x = y \ominus x$; hence transition t can fire in marking M in $\mathcal{N}$ if and only if the homonymous transition t can fire in $\overline{\mathcal{N}}$ in the marking $\overline{M}$ such that $\overline{M}(p) = M(p)$ and $\overline{M}(p') = \neg M(p) = k_p - M(p)$. Finally $M\,[t\rangle\, M'$ in $\mathcal{N}$ if and only if $\overline{M}\,[t\rangle\, \overline{M'}$ in $\overline{\mathcal{N}}$ due to the fact that

$$x \sqsubseteq m \sqsubseteq y \to x \quad \Rightarrow \quad \neg\,[(m \ominus x) \boxplus y] = [\neg m \ominus (y \ominus x)] \boxplus (x \ominus y)$$

which follows from (4.3) since $\neg\,[(m \ominus x) \boxplus y] = (\neg m \boxplus x) \bullet \neg y = (\neg m \boxplus x) \ominus y$. □

5. Conclusion

In this paper we have put forward an axiomatisation of the token game of Petri nets by identifying a class of commutative residuated monoids, called Petri algebras, for which one can generalize the rule of token game of Petri nets to define the behaviour of generalized Petri net whose flow relation and place contents are valued in such algebras. This rule is given by

$$M\,[t\rangle\, M' \Leftrightarrow M \sqsupseteq M \;\wedge\; M' = (M \ominus Pre(t)) \oplus Post(t)$$

where the sum and its associated residuation capture respectively how resources within places are produced and consumed through the firing of a transition. The above rule extends to sequences of transitions by letting

$$\begin{array}{rcl} Pre(uv) & = & Pre(u) \oplus (Pre(v) \ominus Post(u)) \\ Post(uv) & = & (Post(u) \ominus Pre(v)) \oplus Post(v) \end{array}$$

subject to the condition that the associated binary operation

$$(x, y) \otimes (x', y') = (x \oplus (x' \ominus y), (y \ominus x') \oplus y')$$

be associative. We prove this latter condition to be equivalent to the cancellability of the sum which is also characterized by the identity:

$$(b \oplus c) \ominus a = (b \ominus (a \ominus c)) \oplus (c \ominus a)$$

The firing relation can then be expressed as

$$M\,[u\rangle\,M' \;\Leftrightarrow\; (0, M') = (0, M) \otimes (Pre(u), Post(u))$$

from which the reversibility of the firing rule follows, namely the three following statements are equivalent:

1. $M \sqsupseteq Pre(u)$ and $M' = (M \ominus Pre(u)) \oplus Post(u)$
2. $M' \sqsupseteq Post(u)$ and $M = (M' \ominus Post(u)) \oplus Pre(u)$
3. $M \sqsupseteq Pre(u)$; $M' \sqsupseteq Post(u)$ and $M \ominus Pre(u) = M' \ominus Post(u)$

We have proved that Petri algebras coincide with the positive cones of lattice-ordered abelian groups. We know that lattice-ordered abelian groups constitute the subvariety of lattice-ordered groups generated by the group $\mathbb{Z}$ of integer, and that their positive cones (i.e. Petri algebras) is the subvariety of residuated lattices generated by $\mathbb{N}$. The class of (usual) Petri nets is thus associated with the generator of the variety of Petri algebras which shows that these generalized Petri nets share all the algebraic properties of Petri nets, in particular they have the same equational and inequational theory. We however exhibit a Petri algebra whose corresponding class of nets is strictly more expressive than the class of Petri nets, i.e. their class of marking graphs is strictly larger. More precisely, we have introduced a class of nets, termed lexicographic Petri nets, that are associated with the positive cones of the lexicographic powers of the additive group of real numbers. This class of nets is proved to be universal in the sense that any generalized Petri net can be simulated by a lexicographic Petri net. All the classical decidable properties of Petri nets however (termination, covering, boundedness, structural boundedness, accessibility, deadlock, liveness ...) are proved to be undecidable on the class of lexicographic Petri nets, even if restricted to integer coefficients, by showing that this class is a strict extension of the class of Petri net with inhibitor arcs.

Finally we have turned our attention to bounded nets associated with commutative Petri algebra and have shown that their dynamic can be reformulated in term of MV-algebras and that any bounded net can be simulated by some generalized Petri net.

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