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Residuation of tropical series: rationality issues

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Abstract: Decidability of existence, rationality of delay controllers and robust delay controllers are investigated for systems with time weights in the tropical and interval semirings. Depending on the (max,+) or (min,+)-rationality of the series specifying the controlled system and the control objective, cases are identified where the controller series defined by residuation is rational, and when it is positive (i.e., when delay control is feasible). When the control objective is specified by a tolerance, i.e. by two bounding rational series, a nice case is identified in which the controller series is of the same rational type as the system specification series.

Key-words: Control theory, (max,plus) automata, residuation.

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Résiduation de séries tropicales : étude de la rationalité

Résumé : Les questions abordées dans ce rapport concernent l'existence et la rationalité d'un contrôleur (retardateur) pour des systèmes représentés par des automates à poids, dans des semi-anneaux tropicaux. Selon la (min,plus) ou (max,plus)-rationalité des séries spécifiant le système à contrôler et l'objectif du contrôle, on identifie des cas où le contrôleur défini par résiduation de séries est rationel et où il est positif (le système est alors contrôlable).

Quand l'objectif de contrôle est spécifié par un intervalle de tolérance (encadrement par deux séries), un cas est identifié pour lequel la série du contrôleur a le même type de rationalité, (max,plus) ou (min,plus), que la spécification du système.

Mots-clés : Théorie du contrôle, automates (max,plus), résiduation.

1 Introduction

Timed discrete-event systems are discrete-event systems whose behavior depends on timing constraints and not only on logical constraints such as the ordering of events. Such systems are often modeled by weighted automata [5], also called automata with multiplicities, where weights (multiplicities) may range over an arbitrary semiring. E.g., the (max,+)-automata proposed in [6] are weighted in $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ (the tropical semiring), while the (min,+)-automata are weighted in $\mathbb{R}_{\min} = (\mathbb{R} \cup \{\infty\}, \min, +)$. The latter are also called price automata, because the multiplicity of a transition often represents a cost.

In order to increase the expressive power, one may consider automata with weights taken from semirings of intervals, therefore called interval automata. Such automata were first introduced in [4] where they were defined as Büchi automata over an alphabet of pairs made of an event and a real time interval.

Our scope is not limited to deterministic (max,+) or interval automata, because their expressive power is too limited. However, nondeterministic weighted automata suffer from several drawbacks: there is no (finite state) determinization procedure, no general (state) minimization algorithm, and their behaviors (rational formal power series) can in general not be checked for equality by effective procedures.

In this paper, we aim at extending to abstract systems and specifications both given by pairs of lower and upper bound series the supervisory control approach proposed in [9] for (max,+) automata. In [9], the behavior of the closed-loop system is represented by the Hadamard product of the system and controller series, and the controller series is formally computed using residuation theory [3]. Namely, when the controller can delay the controllable transitions but it cannot prevent the firing of transitions, the residuation S_1/S_2 of the (specification) series S_1 by the (system) series S_2 amounts to the Hadamard product of S_1 with the series $-S_2$ with all coefficients multiplied by -1. Residuated series may have both positive and negative coefficients, hence they do not always define feasible delay controllers.

A major problem with the above recalled approach using Hadamard inversion is that the residuated series needs not be rational. Changing all coefficients to their opposite sends a (max,+)-rational series to a (min,+)-rational series and vice versa, but the multiplication of coefficients by -1 is neither a (max,+)-rational nor a (min,+)-rational operation. It was indeed shown by Lombardy and Mairesse [12] that the opposite of a (max,+)-rational series is (max,+)-rational iff it is *unambiguous*, *i.e.* there is at most one successful path in the (max,+) automaton labeled by w for every word w. It seems reasonable to assume that specification series are unambiguous, but it would be very restrictive to require also non-ambiguity from the system series.

In this paper we show that if the specification series is (min,+)-rational and the system series is (max,+)-rational, then the controller series defined by residuation is (min,+)-rational (and similarly for the opposite polarities), hence in particular one can decide whether this series is non-negative.

We shall try to extend residuation further to interval valued formal power series and to intervals of formal power series. Interval valued series can serve to model behaviors of systems whose transitions have uncertain costs or durations. Intervals of series may serve to the same effect, but with some added flexibility since the two bounding series of an interval are structurally independent. For series of intervals, we show that the controller series has generally not the same type or polarity as the system series, unless assuming that both the system and the controller series are sequential, an assumption which is even stronger than unambiguity. The situation turns out to be more favorable with intervals of series, *i.e.* when the expected behavior of the closed-loop system is specified by a tolerance made of a lower bound series and an upper bound series, and the behavior of the uncontrolled system is described similarly. We are interested in *robust* control, *i.e.* in finding bounds on controller (cost or) delay series such that the specified tolerance is met by the closed loop system for all possible behaviors of the uncontrolled system within its defining bounds. We identify a situation in which the controller series interval is guaranteed to be rational and of the same type as the specification series interval.

Deciding about non-emptiness of the residuated series interval is crucial for applications. Fortunately, this can be done since the inequality $S \leq S'$ can be decided for S (max,+)-rational and S' (min,+)-rational (unlike the opposite inequality).

2 (max,+) and (min,+) algebras

In this section, we recall elements of the theory of idempotent semirings, also called dioids (see [1]), a basic structure used throughout the paper.

2.1 Definition

A dioid is a set $\mathcal{D}$ equipped with two internal operations, denoted by $\oplus$ and $\otimes$, such that the addition $\oplus$ is commutative, associative, idempotent, and has a zero element ϵ, while the multiplication $\otimes$ is associative, has a unit element e, has the absorbing element ϵ, and distributes over $\oplus$. The addition $\oplus$ induces a natural order $\preceq$, namely $a \preceq b \Leftrightarrow a \oplus b = b$. Dioid operations may be extended to dioids of matrices as follows. Let $A, B \in \mathcal{D}^{m,n}$ and $C \in \mathcal{D}^{n,\ell}$. Then:

- $\forall i, j \in \{1, \dots, n\}$, $(A \oplus B)_{i,j} = A_{i,j} \oplus B_{i,j}$;
- $\forall i, k \in \{1, \dots, \ell\}$, $(A \otimes B)_{i,k} = \bigoplus_{j=1}^{n} A_{i,j} \otimes C_{j,k}$.

In the sequel, we use the extended notations $\mathcal{D}^{m,Q}$, $\mathcal{D}^{P,n}$ and $\mathcal{D}^{P,Q}$ for finite sets P and Q. Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$. The basic dioids used in this paper are $(\mathbb{R}_{\max}, \max, +)$ and $(\mathbb{R}_{\min}, \min, +)$. The unit element is 0 for both dioids. The zero element is $-\infty$ for $\mathbb{R}_{\max}$ and $+\infty$ for $\mathbb{R}_{\min}$. The order induced by $\mathbb{R}_{\max}$ is the usual order, whereas the order induced by $\mathbb{R}_{\min}$ is the reverse of the usual order. The completions of $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ w.r.t. the induced order relations are noted $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$, respectively. Thus, $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$ have the same carrier set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, but the supremum of $\overline{\mathbb{R}_{\max}}$ is $+\infty$ whereas the supremum of $\overline{\mathbb{R}_{\min}}$ is $-\infty$. In the sequel, $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ are called the (max,+)-dioid and the (min,+)-dioid, respectively. $\overline{\mathbb{R}_{\max}}$ is also a dioid, equipped with the product operation $\otimes_{\max}$ specified in Table 1 (see Section 3.1), and similarly for $\overline{\mathbb{R}_{\min}}$.

In order to represent intervals, we use products of semirings. The two product semirings that we consider are the following. For sake of concision and particularly when dealing with general properties of dioids, we will use the notations defined at the beginning of this section. The next properties depend only on the algebraic structure and it is straightforward to adapt them in the (max,plus) or (min,plus) case.

Definition 2.1 (adapted from [11]). *Let $\mathcal{I}_{\max}^{\max}$ denote the idempotent semiring with carrier set $\{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \overline{\mathbb{R}_{\max}}\} \wedge \underline{x} \leq \overline{x}\}$ defined with:*

$$\begin{aligned}[\underline{x_1}, \overline{x_1}] \oplus [\underline{x_2}, \overline{x_2}] &= [\max(\underline{x_1}, \underline{x_2}), \max(\overline{x_1}, \overline{x_2})], \\ [\underline{x_1}, \overline{x_1}] \otimes [\underline{x_2}, \overline{x_2}] &= [\underline{x_1} + \underline{x_2}, \overline{x_1} + \overline{x_2}],\end{aligned}$$

$\varepsilon = [-\infty, -\infty]$ *(zero interval) and* $e = [0, 0]$ *(unit interval).*

Definition 2.2. *Let $\mathcal{I}_{\max}^{\min}$ denote the idempotent semiring with carrier set $\{[\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \overline{\mathbb{R}}\}\}$ defined with:*

$$\begin{aligned}[\underline{x_1}, \overline{x_1}] \oplus [\underline{x_2}, \overline{x_2}] &= [\max(\underline{x_1}, \underline{x_2}), \min(\overline{x_1}, \overline{x_2})], \\ [\underline{x_1}, \overline{x_1}] \otimes [\underline{x_2}, \overline{x_2}] &= [\underline{x_1} + \underline{x_2}, \overline{x_1} + \overline{x_2}],\end{aligned}$$

zero and unit are resp. $\varepsilon = [-\infty, +\infty]$ *and* $e = [0, 0]$.

In Definition 2.2, as opposed to Definition 2.1, we do not exclude imaginary intervals $[\underline{x}, \overline{x}]$ where the lower bound $\underline{x}$ is greater than the upper bound $\overline{x}$. For instance, $[1, 3] \oplus [2, 4] = [2, 3]$ is a well-formed interval but $[1, 3] \oplus [4, 5] = [4, 3]$ is an imaginary interval. Indeed, well formed intervals are naturally preserved by addition in $\mathcal{I}_{\max}^{\max}$ but they are not preserved by addition in $\mathcal{I}_{\max}^{\min}$. Algebraically, $\mathcal{I}_{\max}^{\min}$ is just the direct product of the (idempotent) semirings $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$ but $\mathcal{I}_{\max}^{\max}$ is not the direct product of $\overline{\mathbb{R}_{\max}}$ with itself.

Now we recall the notions of $\mathcal{D}$-series and $\mathcal{D}$-automaton over an arbitrary dioid $\mathcal{D}$ and the equivalence between the properties of recognizability by $\mathcal{D}$-automata and rationality for $\mathcal{D}$-series. We also recall the construction of $\mathcal{D}(\Sigma)$, the dioid of all $\mathcal{D}$-series over Σ.

Given a dioid $\mathcal{D}$ and a finite alphabet Σ, a $\mathcal{D}$-series over Σ is a function $S : \Sigma^* \to \mathcal{D}$ where Σ^* is the set of all finite words on Σ. We denote by $\mathcal{D}(\Sigma)$ the set of all $\mathcal{D}$-series over Σ. The support of the series S is the set $supp(S)$ of all words w such that $S(w) \neq \epsilon$. By convention, we write series as formal sums $S = \bigoplus_{w \in \Sigma^*} S(w)w$ or $S = \bigoplus_{w \in supp(S)} S(w)w$. Let Q be a finite set of states. A finite $\mathcal{D}$-automaton over Σ and Q is a triple $\mathcal{A} = (\alpha, \mu, \beta)$ where $\alpha \in \mathcal{D}^{1,Q}$, $\beta \in \mathcal{D}^{Q,1}$ and μ is a morphism of monoids from Σ^* to $\mathcal{D}^{Q,Q}$. The series recognized by $\mathcal{A}$ is defined as $\bigoplus_{w \in \Sigma^*}(\alpha \otimes \mu(w) \otimes \beta)w$. A famous Schützenberger's theorem [15] states that the series which are recognized by finite $\mathcal{D}$-automata coincide with the rational $\mathcal{D}$-series, *i.e.* $\mathcal{D}$-series generated from finite $\mathcal{D}$-series using the rational operations of sum, Cauchy product and iterated Cauchy product. Recall that the Cauchy product of two series $S, T \in \mathcal{D}(\Sigma)$ is defined as $S \otimes T = \bigoplus_{w \in \Sigma^*}(\oplus_{uv=w} S(u) \otimes T(v))w$. We denote by $\mathcal{D}Rat(\Sigma)$ the set of the rational $\mathcal{D}$-series over Σ. A rational $\mathcal{D}$-series S is *unambiguous* if it is recognized by a finite $\mathcal{D}$-automaton (α, μ, β) with set of states Q such that, for any word $w = \sigma_1 \dots \sigma_n \in supp(S)$, there exists a unique sequence of states $q_0, q_1, \dots, q_n q_{n+1}$ such that $\alpha(q_0)$, $\mu(\sigma_i)(q_i, q_{i+1})$ and $\beta(q_n)$ differ from ϵ for all $0 \leq i \leq n-1$. A rational $\mathcal{D}$-series is *sequential* if it is recognized by a $\mathcal{D}$-automaton (α, μ, β) such that the underlying automaton on Σ^* has a single initial state q_0 ($\alpha(q) = \epsilon$ for all $q \neq q_0$) and it has a deterministic transition relation (for all σ and q, $\mu(\sigma)(q, q') \neq \epsilon$ for at most one state q'). The following result due to Lombardy and Mairesse shows the interest of unambiguous series in the context of tropical semirings.

Theorem 2.3 ([12]). *A rational (max,+) series is a rational (min,+) series if and only if it is unambiguous.*

The sequential $\mathcal{D}$-series, *i.e.* the series recognized by $\mathcal{D}$-automata with underlying deterministic automata, are of course unambiguous. The set $\mathcal{D}(\Sigma)$ of all $\mathcal{D}$-series over Σ may be endowed with two operations so as to form a dioid. One way to obtain this is to use point-wise addition and the Cauchy product. The other way is to use point-wise addition and the Hadamard product. Therefore, for $S, T \in \mathcal{D}(\Sigma)$, we let:

- $S \oplus T = \bigoplus_{w\in\Sigma^*}(S(w) \oplus T(w))w$;
- $S \odot T = \bigoplus_{w\in\Sigma^*}(S(w) \otimes T(w))w$ (Hadamard product).

Since the above operations preserve the rationality of series, both $(\mathcal{D}(\Sigma), \oplus, \odot)$ and $(\mathcal{D}Rat(\Sigma), \oplus, \odot)$ are dioids. Note that $(\mathcal{D}(\Sigma), \oplus, \odot)$ is complete if $\mathcal{D}$ is complete, but this is not the case for $(\mathcal{D}Rat(\Sigma), \oplus, \odot)$.

3 Residuation of (max,+) and (min,+) series and rational series

In this section, we recall the definition of residuation in dioids and in particular in dioids of $\mathcal{D}$-series over Σ. Then, we focus on the residuation of (max,+) series. After reviewing the results obtained in [9], we examine to what extent they can be applied in the context of supervisory control and underline some drawbacks. We then turn to consider *hybrid* residuation operations of (max,+) series by (min,+) series and conversely. We observe that such operations preserve rationality, and that an easy decision of the control problem ensues when the specification series and the system series have different polarities. For the sake of simplicity, we assume that all transitions of the system are uncontrollable in the sense of this term defined by Ramadge and Wonham [13], *i.e.* that the transitions of the plant may be delayed but cannot be disabled otherwise.

3.1 Residuation of (max,+) series

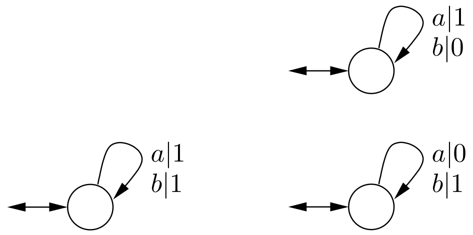
In any dioid $\mathcal{D}$, the (right) residue of an element b by an element a, denoted b/a, is the greatest solution of the inequality $a \otimes x \preceq b$ (where $\preceq$ is the order relation induced by the addition operation), if such a greatest solution exists. The existence of residues is guaranteed for all b and a in any *complete* dioid $\mathcal{D}$, *i.e.* a dioid in which arbitrary subsets have least upper bounds. Table 1 shows the residuation map b/a for the complete (max,+) dioid $\overline{\mathbb{R}_{\max}}$. Note that $\preceq$ coincides in $\overline{\mathbb{R}_{\max}}$ with the usual order relation $\leq$ whereas it coincides in $\overline{\mathbb{R}_{\min}}$ with the reverse order relation $\geq$.

$\otimes_{\max}$	$-\infty$	a	$+\infty$
$-\infty$	$-\infty$	$-\infty$	$-\infty$
b	$-\infty$	$a+b$	$+\infty$
$+\infty$	$-\infty$	$+\infty$	$+\infty$

$/_{\max}$	$-\infty$	a	$+\infty$
$-\infty$	$+\infty$	$-\infty$	$-\infty$
b	$+\infty$	$b-a$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	$+\infty$

Table 1: (max,plus) product and the corresponding residuation.

In a complete dioid $\mathcal{D}$, the operation of residuation in $\mathcal{D}$ extends pointwise to $(\mathcal{D}(\Sigma), \oplus, \odot)$, the dioid of $\mathcal{D}$-series with the Hadamard product. Namely, for any series S_1, S_2 and for any word $w \in \Sigma^*$, $(S_1 \not{\!\!/} S_2)(w) = S_1(w)/S_2(w)$. Based on this fact, it was proposed in [9] to use (max,+) residuation for computing delay controllers. Given a specification series S_1 and a system series S_2, the residuated (max,+) series gives for each word $w \in \Sigma^*$ in $supp(S_2)$ the maximum delay $(S_1 \oslash S_2)(w)$ that can be added to the worst-case duration $S_2(w)$ of the sequence of actions w in the plant without exceeding the specified upper bound $S_1(w)$. This proposal has the outcome that the behavior of the closed loop system may be defined as the product of two (max,+) series, namely $S_2 \otimes (S_1 \oslash S_2)$, but there are some drawbacks. First, it is not clear that one can decide whether the controller series $S_1 \oslash S_2$ is non-negative. Second, it is not always possible to represent $S_1 \oslash S_2$ with a (max,+)-automaton, as the following example shows, hence the controller obtained by residuation may have no finite representation.



(a) (b)

Figure 1: Two automata that recognize (a) the length of a word and (b) the maximum number of occcurences of a letter in a word.

Example 1. *Take $\Sigma = \{a, b\}$. Consider the two automata shown on the left resp. right in Figure 1. The automaton (a) recognizes the (max,+) series $S_1 = \bigoplus_{w \in \Sigma^*} |w|w$. The automaton (b) recognizes the series $S_2 = \bigoplus_{w \in \Sigma^*} \max(|w|_a, |w|_b)w$. Clearly, for any word w, $S_1(w)/_{\max} S_2(w) = |w| - \max(|w|_a, |w|_b) = \min(|w|_a, |w_b|)$. Therefore, $S_1 \oslash_{\max} S_2 = \bigoplus_{w \in \Sigma^*} \min(|w|_a, |w_b|)w$. This series can be recognized by the automaton (b) seen as a (min,+) automaton. Now, it has been shown in [8] that the (min,+) series recognized by the automaton (b) is ambiguous. Therefore $S_1 \oslash S_2$ is an ambiguous (min,plus) rational series and in view of Theorem 2.3, this series cannot be a (max,plus) rational series.*

3.2 Residuation of a (max,+) series by a (min,+) series

Example 1 has shown a case where residuating rational (max,+) series does not produce rational (max,+) series but rational (min,+) series. In that example, S_1 was clearly unambiguous, hence in fact it was also a rational (min,+) series. Generalizing over this example, we are going to study the residuation of (min,+) rational series by (max,+) rational series and symmetrically. We will show that when S_1 and S_2 are rational series with different polarities, then $S_1 \oslash S_2$ is a rational series with the same polarity as S_1. Now, there is a price to pay: if $S_1 \oslash S_2$ is interpreted as a controller enforcing the specification S_1 on the system S_2, then it is generally not possible to represent the closed loop system as a rational power series since S_2 (the system) and $S_1 \oslash S_2$ (the controller) live in different algebras.

In order to achieve the above goal, we need a hybrid product of elements from the (max,+) and (min,+) semirings, and a corresponding residuation. As $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$ have the same carrier set $\mathbb{R} \cup \{-\infty, +\infty\}$, the operations $b \otimes_{\max} a$ and $b/_{\max} a$ already defined for the (max,+) semiring (see Table 1) may as well be seen as operations with profile $\overline{\mathbb{R}_{\max}} \times \overline{\mathbb{R}_{\min}} \rightarrow \overline{\mathbb{R}_{\max}}$. It is worth noting that the operation $/_{\max} : \overline{\mathbb{R}_{\max}} \times \overline{\mathbb{R}_{\min}} \rightarrow \overline{\mathbb{R}_{\max}}$ restricts on the incomplete dioids $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ and co-restricts on the incomplete dioid $\mathbb{R}_{\max}$ (because it never produces the result $+\infty$). Dual operations $\otimes_{\min}$ and $/_{\min}$ can be defined similarly by exchanging $+\infty$ and $-\infty$ in Table 1.

We are now ready to extend the operations $\otimes_{\max}$ and $/_{\max}$ to (max,+) and (min,+) series. As our product of series is the Hadamard product $S \odot_{\max} T = \bigoplus_{w\in\Sigma^*}(S(w) \otimes_{\max} T(w))w$, the corresponding residuation operation $\oslash_{\max}$ on series is given by pointwise extension of the operation $/_{\max}$, thus $S \oslash_{\max} T = \bigoplus_{w\in\Sigma^*}(S(w)/_{\max}T(w))w$. A symmetric operation $\oslash_{\min}$ can be defined similarly. The following theorem shows that both residuation operations $\oslash$ preserve rationality, *i.e.* a finite automaton recognizing $S \oslash T$ may be constructed from finite automata recognizing S and T.

Theorem 3.1. *Let* $S \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$ *and* $T \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$. *Then*

- $S \oslash_{\max} T \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$;
- $T \oslash_{\min} S \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$.

Proof. We only prove the first assertion, as the other is similar (it suffices to exchange max and min). For alleviating notations, $/_{\max}$ and $\oslash_{\max}$ are replaced below with $/$ and $\oslash$. Let $\mathcal{A}_S = (\alpha_S, \mu_S, \beta_S)$ be a (min,+) automaton that recognizes the series S, with state set Q_S, and let $\mathcal{A}_T = (\alpha_T, \mu_T, \beta_T)$ be a (max,+) automaton that recognizes the series T, with state set Q_T. We construct a (min,+) automaton $\mathcal{A} = (\alpha, \mu, \beta)$ that recognizes S/T. The set of states of $\mathcal{A}$ is $Q = Q_S \times Q_T$ and we set the following:

- $\forall (p, q) \in Q_S \times Q_T$, $\alpha_{(p,q)} = (\alpha_S)_p/(\alpha_T)_q$;
- $\forall a \in \Sigma$, $\forall (p, q), (r, s) \in Q_S \times Q_T$, $\mu(a)_{(p,q),(r,s)} = \mu_S(a)_{(p,r)}/\mu_T(a)_{(q,s)}$;
- $\forall (p, q) \in Q_S \times Q_T$, $\beta_{(p,q)} = (\beta_S)_p/(\beta_T)_q$.

In view of Table 1, the support of $S \oslash T$ is the intersection of the supports of S and T, and it coincides by construction of $\mathcal{A}$ with the support of the series recognized by $\mathcal{A}$. Indeed, $\mu(a)_{(p,q),(r,s)}$ (resp. $\alpha_{p,q}$, resp. $\beta_{p,q}$) $= +\infty \Leftrightarrow \mu_S(a)_{p,r}$ (resp. $(\alpha_S)_p$, resp. $(\beta_S)_p$) $=$ $+\infty$ or $\mu_T(a)_{q,s}$ (resp. $(\alpha_T)_q$, resp. $(\beta_T)_q$) $= -\infty$. Then there is a path labelled with w in $\mathcal{A}$ iff there is a path in $\mathcal{A}_S$ and in $\mathcal{A}_T$ labelled with w. Now, for every word $w \in Supp(S \oslash T)$, $S \oslash T(w) = S(w) - T(w)$. Let p be an accepting path labelled by w in $\mathcal{A}$, and let p_S and p_T be the respective projections of p on $\mathcal{A}_S$ and $\mathcal{A}_T$. Then, the weight of p_S is more than $S(w)$ and the weight of p_T is less than $T(w)$. That is, the weight of p is more than $S(w)-T(w)$. Moreover, there exists a path in $\mathcal{A}_S$ (resp. $\mathcal{A}_T$) labelled by w with the exact weight $S(w)$ (resp. $T(w)$). By considering jointly the two paths in $\mathcal{A}_S$ and $\mathcal{A}_T$, one can see that $\alpha\mu(w)\beta \geq S(w) - T(w)$, which concludes the proof. □

It would also be natural to look at $T \oslash_{\max} S$. When dealing with non-complete diods, the same kind of results holds: if $S \in \mathbb{R}_{\min}Rat(\Sigma)$ and $T \in \mathbb{R}_{\max}Rat(\Sigma)$, $T \oslash_{\max} S \in \mathbb{R}_{\max}Rat(\Sigma)$. Unfortunately, some difficulties arise for complete diods (in particular with the support of the residuation). Moreover, this residuation of dioids with different sets does not correspond to any natural product.

Suppose that S is (min,+) rational and T is (max,+) rational. As before, the (min,+) series $S \oslash_{\max} T$ defines for each w the maximal delay that can be added to the actual duration $T(w)$ needed by the plant without exceeding the specified upper bound $S(w)$, but now one can decide whether $S \oslash_{\max} T(w) \geq 0$ for all w in $supp(S) \cap supp(T)$ in view of the following proposition, the proof of which is recalled for the sake of completeness.

Proposition 3.2. *Let $\mathcal{A} = (\alpha, \mu, \beta)$ be a weighted automaton. Let $S_{\min}$ and $S_{\max}$ be the respective (min,+) and (max,+) series recognized by $\mathcal{A}$. One can decide whether $S_{\min}(w) < 0$ (resp. $S_{\max}(w) > 0$) for some w in supp($S_{\min}$) (resp. in supp($S_{\max}$)).*

Proof. $S_{\min}(w) < 0$ for some w in the support of S if and only if the automaton contains some minimal path $q_0\, a_1\, q_1 \ldots a_n\, q_n$ such that $\alpha_{q_0} + \sum_{i=1}^{n} \mu(a_i)_{(q_{i-1},q_i)} + \beta_{q_n}$ is finite and this sum is negative or there exists a minimal cycle with finite negative weight through some state $q_j \in \{q_0, \ldots, q_n\}$. □

When S is (min,+) rational and T is (max,+) rational, Theorem 3.1 and Proposition 3.2 provide an effective procedure for deciding whether there exists a non-negative delay-controller series, namely the (min,+) rational series $S \not{\phi}_{\max} T$, and then constructing it. Note that, by a result of Krob presented in [10], any (min,+) automaton recognizing $S \not{\phi}_{\max} T$ can be transformed to an equivalent (min,+) automaton (α, μ, β) in which all entries of α, μ, and β are either non-negative or equal to $-\infty$ [10]. A (min,+) rational series of this type is less unlikely than an arbitrary series to represent a useful delay controller. However, a (min,+) automaton which fails to be sequential cannot easily be used for on-line control, whence the problem to construct a sequential (max,+) series K as large as possible such that $K(w) \leq (S \not{\phi}_{\max} T)(w)$ for all w. For such (non-optimal) controller series K, the closed-loop system $T \odot_{\max} K$ could in fact be represented by a (max,+) rational series, but this problem is open.

3.3 Additions in the spirit of DES control

For partially observed systems, the alphabet Σ of the specification series S may be strictly smaller than the alphabet Σ' of the system series T'. In this case, one may always transform T' into a series over Σ by abstracting from all unobservable transitions in $\Sigma' \setminus \Sigma$. Provided that the (max,+) automaton (α', μ', β') for T' is free from unobservable loops, it suffices for this purpose to construct a new (max,+) automaton (α, μ, β) with the same set of states, as follows. First introduce a new action symbol $\tau \notin \Sigma'$ and extend μ with $\mu(\tau) = \bigoplus_{\sigma' \in \Sigma' \setminus \Sigma} \mu(\sigma')$. Then let $\alpha = \alpha'$, $\mu(\sigma) = \mu(\tau)^* \otimes \mu(\sigma)$ for all $\sigma \in \Sigma$, and $\beta = \mu(\tau)^* \otimes \beta$ where * means Kleene's product of (max,+) matrices. Note that $\mu(\tau)^*$ can be replaced with a finite equivalent star-free expression using the assumption that there is no unobservable loop.

For partially controlled systems, where $\Sigma_{uc} \subseteq \Sigma$ represents uncontrollable actions, one may wish that the delay controller does not delay any action $\sigma \in \Sigma_{uc}$. Provided that the (max,+) automaton recognizing the system series T is free from uncontrollable loops, one can assume w.l.o.g. that the (min,+) automaton recognizing $S \not{\phi}_{\max} T$ has the same property. Indeed, one can always intersect the support of the residuated series $S \not{\phi}_{\max} T$ by the support of the system series T, thus yielding a (min,+) automaton with the same control effect as $S \not{\phi}_{\max} T$. Under the considered assumption, a (min,+) automaton (α, μ, β) recognizing the series $S \not{\phi}_{\max} T$ may be transformed to an admissible controller as follows. Let $\mu(\sigma)_{(q,q')} = z$ be abbreviated to $q \xrightarrow{(\sigma,z)} q'$. Add new states and transitions by applying inductively the following rules where $\tau \in \Sigma_{uc}$, $\sigma \in \Sigma \setminus \Sigma_{uc}$, and $z, z' \neq 0$.

$$\frac{q \xrightarrow{(\tau,z)} q'}{q \xrightarrow{(\tau,0)} (q', z)} \qquad \frac{q \xrightarrow{(\tau,z)} q'}{(q, z') \xrightarrow{(\tau,z)} (q', z' + z)} \qquad \frac{q \xrightarrow{(\sigma,z)} q'}{(q, z') \xrightarrow{(\sigma,z'+z)} q'}$$

After stabilization of the set of states, drop all transitions of the form $q \xrightarrow{(\tau,z)} q'$ with $\tau \in \Sigma_{uc}$ and $z \neq 0$, and finally extend α and β by setting $\alpha(q',z) = \epsilon$ and $\beta(q',z) = \beta(q') + z$.

4 Interval Weighted Automata

4.1 Background

Residuation of intervals in a dioid $\mathcal{D}$ has been studied in [11]. In that work, the set $\mathcal{I}(\mathcal{D})$ of all intervals $[\underline{x}, \overline{x}] = \{t \in \mathcal{D} \,|\, \underline{x} \preceq t \preceq \overline{x}\}$ is shown to be a dioid, with (left and right) residuation operations. The residuation operations serve to compute robust compensating controllers for Timed Event Graphs. Timed behaviors are defined by associating to each transition a formal power series in one variable γ over $\overline{\mathbb{Z}}_{\max}$, such that the coefficient of γ^k in this series is the date of the k-th firing of the transition. For specifications, intervals express tolerances on the desired behavior. For systems, intervals reflect an imprecise knowledge of the exact timed behavior, whence the need for *robust* controllers. Technically speaking, controllers are computed by residuation in $\mathcal{I}(\mathcal{D})$, where $\mathcal{D} = \overline{\mathbb{Z}}^{+}_{\max}[\![\gamma]\!]$ is the set of so-called *causal elements.*Residuation results in intervals $[\underline{x}, \overline{x}]$ whose bounds $\underline{x}$ and $\overline{x}$ are *realizable* series in $\overline{\mathbb{Z}}^{+}_{\max}[\![\gamma]\!]$, which is much stronger than rational series. Such bounds define indeed controllers that can be realized by Timed Event Graphs.

While intervals of (max,+) rational series are the basic setting used in [11], we will investigate here the alternative setting of rational series of intervals, first in $\mathcal{I}^{\max}_{\max}$ and then in $\mathcal{I}^{\min}_{\max}$. In both cases, we examine what residuation can afford, and end up with the conclusion that the results are not worth the effort. Note that, differently from [11], we consider formal power series on alphabets Σ with more than one symbol, and we do not care for the realizability of series by Timed Event Graphs but only for their rationality, and hopefully for non-ambiguity or sequentiality.

4.2 Residuation of $\mathcal{I}^{\max}_{\max}$ -series

In order to make the definition of $\mathcal{I}^{\max}_{\max}$ and of residuation in this dioid precise, let us recall definitions and results adapted from [11].

Definition 4.1. *A (closed) interval in dioid $\mathcal{D}$ is a non-empty set of the form $\mathbf{x} = [\underline{x}, \overline{x}] = \{t \in \mathcal{D} \,|\, \underline{x} \preceq t \preceq \overline{x}\}$.*

Proposition 4.2. *The set of intervals, denoted $I(\mathcal{D})$, endowed with the coordinate-wise operations $[\underline{x}, \overline{x}] \oplus [\underline{y}, \overline{y}] = [\max(\underline{x}, \underline{y}), \max(\overline{x}, \overline{y})]$ and $[\underline{x}, \overline{x}] \otimes [\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$, is a dioid, where the intervals $\epsilon = [\epsilon, \epsilon]$ resp. $\mathbf{e} = [e, e]$ are the zero resp. the neutral element. If $\mathcal{D}$ is complete, then $I(\mathcal{D})$ is complete.*

Proposition 4.3. *For any interval $\mathbf{a} \in I(\mathcal{D})$, the right product by $\mathbf{a}$* i.e. *the operation $\mathbf{x} \to \mathbf{x} \otimes \mathbf{a}$, has a (right) adjoint residual operation $\mathbf{y} \to \mathbf{y} \not{/} \mathbf{a}$, given by $\mathbf{b} \not{/} \mathbf{a} = [\underline{b} \not{/} \underline{a} \wedge \overline{b} \not{/} \overline{a}, \overline{b} \not{/} \overline{a}]$ for any intervals $\mathbf{a} = [\underline{a}, \overline{a}]$ and $\mathbf{b} = [\underline{b}, \overline{b}]$.*

Take $\mathcal{D} = \overline{\mathbb{R}_{\max}}$ and let $\mathcal{D}' = I(\mathcal{D})$ (thus $\mathcal{D}' = \mathcal{I}^{\max}_{\max}$). In view of the above, $\mathcal{I}^{\max}_{\max}$ is a complete dioid with residuation. As residuation in $\mathcal{D}'$ extends to series in the dioid $\mathcal{D}'(\Sigma)$ equipped with Hadamard product, we get for free a residuation operation on $\mathcal{I}^{\max}_{\max}(\Sigma)$. Now, if we call *degenerated* those intervals $\mathbf{x} = [\underline{x}, \overline{x}]$ for which $\underline{x} = \overline{x}$, then the subset of the degenerated intervals induces a restriction of $\mathcal{I}_{\max}^{\max}$ which is isomorphic to $\overline{\mathbb{R}_{\max}}$. Thus, $\overline{\mathbb{R}_{\max}}(\Sigma)$ embeds isomorphically into a complete subdioid of $\mathcal{I}_{\max}^{\max}(\Sigma)$. It then follows from Example 1, after replacing numbers 0 and 1 with corresponding intervals $[0,0]$ and $[1,1]$, that residuation of series in $\mathcal{I}_{\max}^{\max}(\Sigma)$ does not preserve rationality, hence it does not enable an effective computation of compensating controllers.

4.3 Residuation of $\mathcal{I}_{\max}^{\min}$ -series

We consider now series of intervals in $\mathcal{I}_{\max}^{\min}(\Sigma)$. As $\mathcal{I}_{\max}^{\min}$ is isomorphic to the direct product of $\overline{\mathbb{R}_{\max}}$ and $\overline{\mathbb{R}_{\min}}$, residuation in $\mathcal{I}_{\max}^{\min}$ operates componentwise, *i.e.* $[\underline{x},\overline{x}] \phi [\underline{y},\overline{y}] = [\underline{x}/_{\max}\,\underline{y}, \overline{x}/_{\min}\,\overline{y}]$. The induced restriction of $\mathcal{I}_{\max}^{\min}(\Sigma)$ on intervals of the form $[\underline{x},+\infty]$ is a complete subdioid isomorphic to $\overline{\mathbb{R}_{\max}}$. It therefore follows from Example 1, where the numbers 0 and 1 are replaced with the intervals $[0,+\infty]$ and $[1,+\infty]$, that residuation in $\mathcal{I}_{\max}^{\min}(\Sigma)$ does not preserve rationality, hence it does not provide an effective computation of controllers.

5 Robust control of partially known systems against tolerance specifications

In this section, we consider intervals of formal power series over $\overline{\mathbb{R}}$, whose lower and upper bounds $\underline{S} \in \overline{\mathbb{R}_{\max}}\,Rat(\Sigma)$ and $\overline{S} \in \overline{\mathbb{R}_{\min}}\,Rat(\Sigma)$ have the same support and specify a tolerance $[\underline{S},\overline{S}]$ on the desired behavior of a plant. We assume that the behavior of the uncontrolled plant is described abstractly, hence imprecisely, by an interval $[\underline{T},\overline{T}]$ of formal power series over $\overline{\mathbb{R}}$, whose lower and upper bounds $\underline{T} \in \overline{\mathbb{R}_{\min}}\,Rat(\Sigma)$ and $\overline{T} \in \overline{\mathbb{R}_{\max}}\,Rat(\Sigma)$ have the same support as the tolerance $[\underline{S},\overline{S}]$. This is coherent with the general assumption that compensating controllers can delay the plant's actions but cannot otherwise prevent them from firing. Although no procedure is known for deciding whether $\underline{T}(w) \leq \overline{T}(w)$ for all $w \in \Sigma^*$, we do not consider this as a problem since the interval $[\underline{T},\overline{T}]$ is supposed to describe albeit imprecisely the behavior of a real system which naturally belongs to this interval. Our goal is to compute the largest interval $[\underline{K},\overline{K}]$ of compensating (or delay) controller series K over $\overline{\mathbb{R}}$ such that $(\overline{T} \odot_{\max} K)(w) \leq \overline{S}(w)$ and $(\underline{T} \odot_{\min} K)(w) \geq \underline{S}(w)$ for all $w \in supp(S)$ and for all formal power series $T \in [\underline{T},\overline{T}]$. Any such formal power series $K \in [\underline{K},\overline{K}]$ thus provides robust control enforcing the specified tolerance $[\underline{S},\overline{S}]$ on the plant.

For $T \in [\underline{T},\overline{T}]$, $w \in supp(\overline{S})$, $(T \odot_{\max} K)(w) \leq \overline{S}(w)$ iff $K(w) \leq \overline{S}(w)/_{\max}T(w)$, and therefore $(T \odot_{\max} K) \leq \overline{S}$ for all $T \in [\underline{T},\overline{T}]$ iff $K \leq \overline{S}\phi_{\max}\overline{T}$.

For $T \in [\underline{T},\overline{T}]$, $w \in supp(\underline{S})$, $(T \odot_{\min} K)(w) \geq \underline{S}(w)$ iff $K(w) \geq \underline{S}(w)/_{\min}T(w)$, and therefore $(T \odot_{\min} K) \geq \underline{S}$ for all $T \in [\underline{T},\overline{T}]$ iff $K \geq \underline{S}\phi_{\min}\underline{T}$.

For any $w \in \Sigma^*$, $\underline{S}(w) = -\infty$ iff $\underline{T}(w) = +\infty$, and $\overline{S}(w) = +\infty$ if $\overline{T}(w) = -\infty$, as we have assumed that all series $\underline{S}$, $\overline{S}$, $\underline{T}$ and $\overline{T}$ have the same support.

As a result, the interval of robust delay controller series $[\underline{K},\overline{K}]$ is given by $\underline{K} = \underline{S}\phi_{\min}\underline{T}$ and $\overline{K} = \overline{S}\phi_{\max}\overline{T}$. Now $\underline{S}$ is (max,+) rational and $\underline{T}$ is (min,+) rational, hence $\underline{K}$ is (max,+) rational. Similarly, $\overline{S}$ is (min,+) rational and $\overline{T}$ is (max,+) rational, hence $\overline{K}$ is (min,+) rational. Altogether, the controller $[\underline{K},\overline{K}]$ is therefore in the same format as the original specification $[\underline{S},\overline{S}]$.

If $\underline{K}(w) > \overline{K}(w)$ for some w, then the control problem has no solution, *i.e.* the interval of possible controller series K is empty. Seeing that $\underline{K} \in \overline{\mathbb{R}_{\max}}\,Rat(\Sigma)$ and $\overline{K} \in \overline{\mathbb{R}_{\min}}\,Rat(\Sigma)$, the series $\overline{K}\phi_{\max}\underline{K}$ is (min,+) rational, hence by Proposition 3.2, one can decide upon this property (the same technique may be used right at the beginning to check that $[\underline{S}, \overline{S}]$ is a well-formed interval). Note also that the controller series interval $[\underline{K}, \overline{K}]$ may in turn be considered as a specification to be enforced on a new plant component T' that runs concurrently with T.

Before the above results and constructions may be applied to practical control problems, one needs to solve the open problem of finding an unambiguous rational controller series K in $[\underline{K}, \overline{K}]$. Even better, one should search in this interval for a sequential controller series K that is moreover increasing, *i.e.* such that $K(wt) \geq K(w)$ for all $w \in \Sigma^*$ and $t \in \Sigma$. At present, we do not know whether one can decide upon the existence of these two types of controller series. However, the next Proposition 5.1 may help to find an unambiguous rational controller series K in $[\underline{K}, \overline{K}]$ (when both relations $\underline{K} \leq S_{\min}$ and $S_{\max} \leq \overline{K}$ are satisfied for the series $S_{\min}$ and $S_{\max}$ defined in the proposition - note that if $S_{\min} = S_{\max}$ then the considered series is in fact non-ambiguous).

Proposition 5.1. *Let $\mathcal{A} = (\alpha, \mu, \beta)$ be a weighted automaton, and $S_{\max}$ and $S_{\min}$ be the respective (max,+) and (min,+) series recognized by this automaton. Then there exists a non-ambiguous series S such that $\forall w \in Supp(S_{\max})$, $S_{\min}(w) \leq S(w) \leq S_{\max}(w)$.*

Proof. The result is straightforward when every state is final and the automaton is complete (*i.e.*, every letter labels at least one output arc from every state): it suffices then to choose for every state and letter one output arc labelled with this letter, to delete all other arcs and to keep only one initial state. The trimmed automaton remains complete, and every state stays final. Therefore, all words are recognized, and every path has the weight of a path labelled by the same word in the original automaton. As the trim automaton is deterministic, the recognized series is non ambiguous, hence the result.

In the general case, things are more complex since the support of S may be strictly included in the language recognized by $\mathcal{A}$. Following [8], we are going to construct the Schützenberger covering of $\mathcal{A}$ and then to eliminate competitions from this automaton, thus yielding at the end an unambiguous series recognizer.

Let $\mathcal{A}_{det}$ be the deterministic automaton obtained from $\mathcal{A}$ (seen as a classical unweighted automaton) by applying the usual subset construction. Then let $\mathcal{B} = (\gamma, \nu, \delta)$ be the weighted automaton defined as follows:

- $Q_{\mathcal{B}} = Q_{\mathcal{A}} \times Q_{\mathcal{A}_{det}}$;
- $\forall (p, q) \in Q_{\mathcal{A}} \times Q_{\mathcal{A}_{det}}$, $\gamma_{(p,q)} = \alpha_p$;
- $\forall a \in \Sigma$, $\forall (p, q), (r, s) \in Q_{\mathcal{A}} \times Q_{\mathcal{A}_{det}}$, $\nu(a)_{(p,q),(r,s)} = \mu(a)_{(p,r)}$;
- $\forall (p, q) \in Q_{\mathcal{A}} \times Q_{\mathcal{A}_{det}}$, $\delta_{(p,q)} = \beta_p$.

A competition in $\mathcal{B}$ is either a pair of final states (p, q) and (p', q) or a pair of transitions $((p, q), a, (r, s))$ and $((p', q), a, (r, s))$ such that $p \neq p'$. A non-ambiguous automaton can be obtained by removing one item in each pair of competing transitions or turning one of the two cmpeting final states into a non-final state. In the resulting automaton, each accepted word w labels a unique path and the weight of this path is equal to the weight of some path labelled with w in the original automaton $\mathcal{A}$. The recognized series S is therefore non ambiguous. □

Example 2. *Figure 2 gives an example of this construction. On the left is the weighted automaton, on the top its (unweighted) deterninized one. The Schützenberger covering of this automaton is depicted in the center, where the dashed transition are the one removed to obtain a non-ambiguous (max,plus) series.*

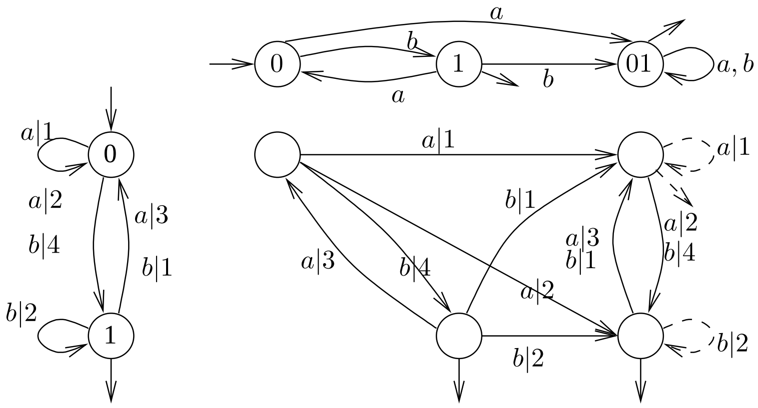


Figure 2: A non-ambiguous automaton S such that $S_{\min} \leq S \leq S_{\max}$.

6 Residuation w.r.t. Cauchy product

In this section, we propose an adaptation of the constructions developped in Section 3 to the case when the Hadamard product of formal power series is replaced with the Cauchy product. We propose also an adaptation of the elements suggested in Section 5 for reasoning on intervals of rational series. We finally discuss possible applications to contracts encountered in theories of software components.

Definition 6.1. *Given a dioid $\mathcal{D}$ and a finite alphabet Σ, the Cauchy product of two series $S, T \in \mathcal{D}(\Sigma)$ is defined by: $S \bigotimes T = \bigoplus_{w\in\Sigma^*}(\bigoplus_{uv=w}(S(u) \otimes T(v)))w$.*

It is well known that the Cauchy product preserves rationality of (max,+) or (min,+) series.

Indeed, if S and T are given by automata $\mathcal{A}_S = (\alpha_S, \mu_S, \beta_S)$ and $\mathcal{A}_T = (\alpha_T, \mu_T, \beta_T)$ with respective disjoint sets of states Q_S and Q_T, then $S \bigotimes T$ is recognized by the automaton (α, μ, β) with the set of states $Q \cup Q'$, where

- $\alpha = (\alpha_S, \alpha_S \otimes \beta_S \otimes \alpha_T)$;
- $\mu(a) = \begin{bmatrix} \mu_S(a) & \mu_S(a) \otimes \beta_S \otimes \alpha_T \\ \epsilon & \mu_T(a) \end{bmatrix}$;
- $\beta = (\epsilon, \beta_T)$.

μ and μ' are extended respectively with $\mu(a)_{(q_1,(q_2,q_1'))} = \mu(a)_{(q_1,q_2)} + \alpha'_{q_1'}$ and $\mu'(a')_{((q_2,q_1'),q_2')} = \beta(q_2) + \mu'(a')_{(q_1',q_2')}$.

In the sequel, we let $\mathcal{D} = \overline{\mathbb{R}_{\max}}$. The Cauchy product of two series is thus $S \bigotimes_{\max} T = \bigoplus_{w\in\Sigma^*}(\bigoplus_{w=uv}(S(u) \otimes_{\max} T(v)))w$ where $\otimes_{\max}$ is the residuation operator on the (max,+)

semiring (see Table 1). Both $(\mathcal{D}(\Sigma), \oplus, \otimes_{\max})$ and $(\mathcal{D}Rat(\Sigma), \oplus, \otimes_{\max})$ are dioids. The Cauchy product, as opposed to the Hadamard product, is not commutative, hence one must distinguish left residuals $S \oslash_{\max} T$ and right residuals $S \obackslash_{\max} T$ of formal power series w.r.t. $\bigotimes_{\max}$. The two residual operations $(- \oslash_{\max} T)$ and $(- \obackslash_{\max} T)$ are (left) adjoint to the operations $(- \otimes_{\max} T)$ and $(T \otimes_{\max} -)$ respectively, thus: $S \oslash_{\max} T = \bigvee\{X \mid X \otimes_{\max} T \leq S\}$, $S \obackslash_{\max} T = \bigvee\{X \mid T \otimes_{\max} X \leq S\}$.

Theorem 6.2. *Let* $S \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$ *and* $T \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$. *Then*

- $S \oslash_{\max} T \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$, *and* $S \obackslash_{\max} T \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$;
- $T \oslash_{\min} S \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$, *and* $T \obackslash_{\min} S \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$.

Proof. We establish only the first statement, as all other statements are similar. By definition, $X \otimes_{\max} T \leq S$ if and only if $X(u) \otimes_{\max} T(v) \leq S(uv)$ for all u, v in Σ^*. The latter condition on u and v may be expressed equivalently as $X(u) \leq S(uv)/_{\max}T(v)$, where $/_{\max}$ is the residuation operator on the (max,+) semiring displayed in Table 1. As $S \oslash_{\max} T$ is the least upper bound of the series X satisfying the above condition for all u and v, one has necessarily for all $u \in \Sigma^*$:

$$(S \oslash_{\max} T)(u) = \bigwedge_{v \in \Sigma^*} S(uv) \,/_{\max}\, T(v).$$

Let $\mathcal{A}_S = (\alpha_S, \mu_S, \beta_S)$ and $\mathcal{A}_T = (\alpha_T, \mu_T, \beta_T)$ be automata, with disjoint sets of states Q_S and Q_T, recognizing the (min,+) series S and the (max,+) series T, respectively. Define $\mathcal{A} = (\alpha, \mu, \beta)$, with set of states $Q = Q_S \cup (Q_S \times Q_T)$, over the alphabet $\Sigma \cup \{\tau\}$ (where $\tau \notin \Sigma$), as follows. For all $a \in \Sigma$, $p_i \in Q$ and $q_i \in Q'$, let:

- $\alpha_{p_1} = \alpha_{S p_1}$;
- $\alpha_{p_1,q_1} = \alpha_{S p_1} - \alpha_{T q_1}$;
- $\mu(a)_{p_1,p_2} = \mu_S(a)_{p_1,p_2}$;
- $\mu(a)_{p_1,(p_2,q_2)} = \mu_S(a)_{p_1,p_2} - \alpha_{T q_2}$;
- $\mu(\tau)_{(p_1,q_1),(p_2,q_2)} = \bigwedge_{a \in \Sigma} \mu_S(a)_{p_1,p_2} - \mu_T(a)_{q_1,q_2}$;
- $\beta_{p_2,q_2} = \beta_{S p_2} - \beta_{T q_2}$.

In order to obtain an automaton recognizing the (min,+) series $(S \oslash_{\max} T)$, it suffices now to abstract from the τ-transitions of $\mathcal{A}$. This can be done as follows. For each pair of states $(p_1, q_1), (p_2, q_2)$ such that there exists some τ-path from (p_1, q_1) to (p_2, q_2), let $w((p_1, q_1), (p_2, q_2)) = -\infty$ if some such path contains a state (q, q') that appears on a minimal τ-cycle with negative weight, otherwise let $w((p_1, q_1), (p_2, q_2))$ be the minimal weight of a minimal τ-path from (p_1, q_1) to (p_2, q_2). For any $a \in \Sigma$, and for any $p_i \in Q$ and $q_i \in Q'$, one redefines:

- $\alpha_{p_2,q_2'} = \bigwedge_{(p_1,q_1)} \alpha_{p_1,q_1} + w((p_1, q_1), (p_2, q_2))$;
- $\mu(a)_{(p_3,q_3),(p_2,q_2)} = \bigwedge_{(p_1,q_1)} \mu(a)_{(p_3,q_3),(p_1,q_1)} + w((p_1, q_1), (p_2, q_2))$.

Finally, one removes τ from the alphabet. ☐

Consider a tolerance $[\underline{S},\overline{S}]$, specifying the desired behavior of the sequential composition $K \bigotimes_{\max} T$ or $T \bigotimes_{\max} K$ of two component systems K and T, where T is given but K is missing. Suppose it is known that the behavior of component T lies between two bounds $\underline{T} \in \overline{\mathbb{R}_{\min}}\, Rat(\Sigma)$ and $\overline{T} \in \overline{\mathbb{R}_{\max}}\, Rat(\Sigma)$. Assuming that all the series $\underline{S}$, $\overline{S}$, $\underline{T}$ and $\overline{T}$ have the same support, we want to compute from $[\underline{S},\overline{S}]$ and $[\underline{T},\overline{T}]$ the largest interval $[\underline{K},\overline{K}]$ such that $K \bigotimes_{\max} T$ (resp. $T \bigotimes_{\max} K$) lies in $[\underline{S},\overline{S}]$ for all possible components T.

Consider the sequential composition $T \bigotimes_{\max} K$. Thus, we require that $(T \bigotimes_{\max} K)(w) \leq \overline{S}(w)$ and $(T \bigotimes_{\min} K)(w) \geq \underline{S}(w)$ for all $T \in [\underline{T},\overline{T}]$ and for all words $w \in \Sigma^*$. The first requirement holds iff $T(u) \otimes_{\max} K(v) \leq \overline{S}(uv)$ for all $T \in [\underline{T},\overline{T}]$ and for all $u, v \in \Sigma^*$, iff $\overline{T}(u) \otimes_{\max} K(v) \leq \overline{S}(uv)$ for all u and v, iff $K(v) \leq \overline{S}(uv)/_{\max}\overline{T}(u)$ for all u and v, iff $K(v) \leq \bigwedge_{u\in\Sigma^*} \overline{S}(uv)/_{\max}\overline{T}(u)$ for all u, iff $K \leq (\overline{S} \obslash_{\max} \overline{T})$. The second requirement holds iff $T(u) \otimes_{\min} K(v) \geq \underline{S}(uv)$ for all $T \in [\underline{T},\overline{T}]$ and for all $u, v \in \Sigma^*$, iff $\underline{T}(u) \otimes_{\min} K(v) \geq \underline{S}(uv)$ for all u and v, iff $K(v) \geq \underline{S}(uv)/_{\min}\underline{T}(u)$ for all u and v, iff $K(v) \geq \bigvee_{u\in\Sigma^*} \underline{S}(uv)/_{\min}\underline{T}(u)$ for all u, iff $K \geq (\underline{S} \obslash_{\min} \underline{T})$. Finally, $\underline{K} = \underline{S} \obslash_{\min} \underline{T}$ and $\overline{K} = \overline{S} \obslash_{\max} \overline{T}$. By Theorem 7.2, $\underline{K}$ is a (max,+) series and $\overline{K}$ is a (min,+) series.

Consider now the sequential composition $K \bigotimes_{\max} T$. By similar reasoning, one obtains $\underline{K} = \underline{S} \oslash_{\min} \underline{T}$ and $\overline{K} = \overline{S} \oslash_{\max} \overline{T}$. So, the interval $[\underline{S} \oslash_{\min} \underline{T}, \overline{S} \oslash_{\max} \overline{T}]$ characterizes exactly the set of all components which fulfil the *contract* $[\underline{S},\overline{S}] \,/\, [\underline{T},\overline{T}]$, meaning that whenever they are composed sequentially on the right with a component T satisfying the *assumption* $[\underline{T},\overline{T}]$, the result of the composition satisfies the *guarantee* $[\underline{S},\overline{S}]$.

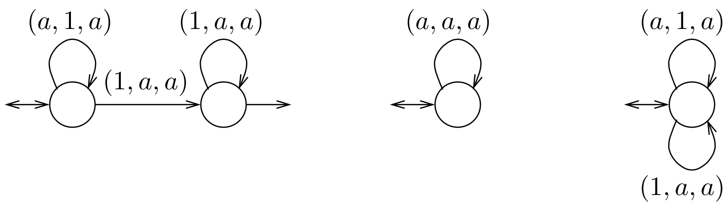
7 Residuation for general rational (max,plus) and (min,plus) products

The aim of this section is to generalise our results to any rational product.

Let Σ_S, Σ_T and Σ_U be three finite alphabets. A product is a relation $\pi \subseteq \Sigma_S^* \times \Sigma_T^* \times \Sigma_U^*$. The product of u and v is $\{w \mid (u, v, w) \in \pi\} = \pi(u, v)$.

We say that a product π is rational [14] iff its graph $\hat{\pi}$ is a rational set of $\Sigma_S^* \times \Sigma_T^* \times \Sigma_U^*$ (*i.e.* it is recognised by a finite automaton over the alphabet $\{(a, 1, 1), (1, b, 1), (1, 1, c) \mid a \in \Sigma_S,\ b \in \Sigma_T,\ c \in \Sigma_U\}$.

Example 3. *Cauchy, Hadamard, shuffle are rational and given by the following automata of Figure 3. For sake of concision, we label the transitions by words of* $\Sigma_S^* \times \Sigma_T^* \times \Sigma_U^*$.



(a) (b) (c)

Figure 3: Examples of rational products: (a) Cauchy product; (b) Hadamard product; (c) shuffle product. The behaviour is the same for any letter of the alphabets.

7.1 Lifting the product to (max,plus) algebra

Let $\boxtimes$ be a rational product. Let S and T be two series and define the *(max,plus) product $\boxtimes_{\max}$ as:

$$S \boxtimes_{\max} T(w) = \sup\{S(u) + T(v) \mid u, v \text{ and } w \in u \boxtimes v\}.$$

Theorem 7.1. *Let $\Sigma_S, \Sigma_T, \Sigma_U$ be three alphabets and $\boxtimes$ be a rational product over $\Sigma_S^* \times \Sigma_T^* \times \Sigma_U^*$. If $S \in \overline{\mathbb{R}_{\max}}Rat(\Sigma_S)$ and $T \in \overline{\mathbb{R}_{\max}}Rat(\Sigma_T)$, then $S \boxtimes_{\max} T \in \overline{\mathbb{R}_{\max}}Rat(\Sigma_U)$.*

Proof. The proof is in three steps:

- compute a (max,plus) automaton for shuffle product of the two series;
- intersect with the product automaton;
- erase the labels in $\Sigma_S \cup \Sigma_T$.

Let $\mathcal{A}_S = (\alpha_S, \mu_S, \beta_S)$ and $\mathcal{A}_T = (\alpha_T, \mu_T, \beta_T)$ be the (max,plus) automata that respectively recognise S and T. The shuffle product of those two automata is $\mathcal{A}_{shuf} = (\alpha_U, \mu_U, \beta_U)$ such that:

- $\forall (p, q) \in Q_S \times Q_T$, $(\alpha_U)_{pq} = (\alpha_S)_p + (\alpha_T)_q$;
- $\forall a \in \Sigma_S,\ \forall p, r \in Q_S,\ \forall q \in Q_T,\ \mu_U(a)_{(p,q),(r,q)} = \mu_S(a)_{p,r}$;
- $\forall a \in \Sigma_T,\ \forall p \in Q_S,\ \forall q, s \in Q_T,\ \mu_U(a)_{(p,q),(p,s)} = \mu_S(a)_{q,s}$;
- $\forall (p, q) \in Q_S \times Q_T$, $(\beta_U)_{pq} = (\beta_S)_p + (\beta_T)_q$;

In order to intersect this automaton with the product automaton, we modify the automaton: the labels $(p, q) \xrightarrow{a,w} (r, q)$ are changed to $(p, q) \xrightarrow{(a,1,1),w} (r, q)$, the labels $(p, q) \xrightarrow{a,w} (p, s)$ are changed to $(p, q) \xrightarrow{(1,a,1),w} (p, s)$, and for every $a \in \Sigma_U$ for every (p, q), the transitions $(p, q) \xrightarrow{(1,1,a),0} (p, q)$ are added. We denote by $\mathcal{A}_{shuf}$ this automaton. For each $u \in \Sigma_S^*$ and each $v \in \Sigma_T^*$, if p_S is an accepting path in $\mathcal{A}_S$ labelled by u with maximum weight and p_T is an accepting path in $\mathcal{A}_T$ labelled by v with maximum weight, then for each $w \in \Sigma_U^*$ and for every sequence of atoms $(a, 1, 1)$ or $(1, a, 1)$ or $(1, 1, a)$ that evaluates to (u, v, w) under componentwise concatenation, there exists a path with weight $S(u) + S(v)$, and there is no accepting path labelled like this with a strictly greater weight.

The second step is to make the intersection with the product automaton, so that there is an accepting path labelled by (u, v, w) if and only if $(u, v, w) \in \boxtimes$. Moreover, from the first step, the maximum weight of such a path is $S(u) + T(v)$.

The third step only consists in forgetting about the labels of the first two coordinates in order to get a (max,plus) automaton over Σ_U, which is the automaton recognising $S \boxtimes_{\max} T$. □

Example 4. *Consider the automata of Figure 4(a). The shuffle $\mathcal{A}_{shuf}$ is given by Figure 4(b) and the Hadamard by Figure 4(c).*

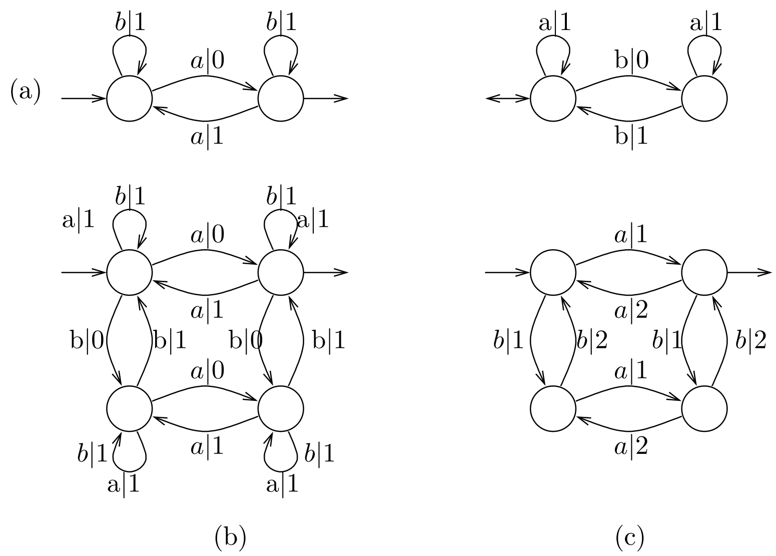


Figure 4: Example of product of (max,plus) automata. (a)two (max,plus) automata; (b) the shuffle product of those two automata (we differentiate the provenance of the letters using different fonts); (c) Hadamard product of those automata (after the erasing of the ε-transitions).

7.2 Residuation w.r.t a rational product

Let $\boxtimes$ be a rational product. This product is not necessary commutative, hence one must distinguish left residuals $S \boxslash_{\max} T$ and right residuals $S \boxbslash_{\max} T$ of formal power series w.r.t. $\boxtimes_{\max}$. The two residual operations $(- \boxslash_{\max} T)$ and $(- \boxbslash_{\max} T)$ are (left) adjoint to the operations $(- \boxtimes_{\max} T)$ and $(T \boxtimes_{\max} -)$ respectively, thus: $S \boxslash_{\max} T = \bigvee\{X \mid X \boxtimes_{\max} T \leq S\}$, $S \boxbslash_{\max} T = \bigvee\{X \mid T \boxtimes_{\max} X \leq S\}$.

Theorem 7.2. *Let $S \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$ and $T \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$. Then*

- $S \boxslash_{\max} T \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$, *and* $S \boxbslash_{\max} T \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$
- $T \boxslash_{\min} S \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$, *and* $T \boxbslash_{\min} S \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$

Proof. We establish only the first statement, as all other statements are similar. By definition, $X \boxtimes_{\max} T \leq S$ if and only if $X(u) \boxtimes_{\max} T(v) \leq S(w)$ for all u, v, w such that $w \in u \boxtimes v$. This is equivalent to $\forall u, v, w$ such that $w \in u \boxtimes v$, $X(u) \geq S(w)/_{\max}T(v)$ and to $X(u) \geq \inf S(w)/_{\max}T(v)$. As the residuation $S \boxslash_{\max} T$ is the least upper bound of the series X satisfying the above condition for all u and v, one has necessarily for all $u \in \Sigma^*$:

$$X(u) = \inf\{S(w)/_{\max}T(v) \mid w \in u \boxtimes v\}.$$

Now, it suffices to notice that the notion of rational product is completely symmetric. If $\boxtimes$ is a rational product, then $\boxslash$ is also a rational product having the same automaton, up to the renaming of the letters. Moreover, if $T \in \overline{\mathbb{R}_{\max}}Rat(\Sigma)$, then $-T \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$ and from Table 1, it is easy to see that $\forall m, n \in \overline{\mathbb{R}_{\max}}$, $m/_{\max}n = m \otimes_{\min} -n$. Then, using Theorem 7.1, $X \in \overline{\mathbb{R}_{\min}}Rat(\Sigma)$. ☐

8 Conclusion

In this paper, we have shown that residuation preserves rationality of formal power series and intervals thereof whenever the two operands of the residuation have opposite polarities (*e.g.* when one computes the residue S/T of a (max,+) rational series S by a (min,+) rational series T). This approach is not totally orthodox, and it adds nothing to the theory, but it might help solving some practical control or design problems, if a few central but open questions can be solved, such that deciding whether exists and constructing a sequential series between a (max,+) and a (min,+) rational series.

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