

# Solving Stochastic Büchi Games on Infinite Arenas with a Finite Attractor\*

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We consider games played on an infinite probabilistic arena where the first player aims at satisfying generalized Büchi objectives almost surely, i.e., with probability one. We provide a fixpoint characterization of the winning sets and associated winning strategies in the case where the arena satisfies the finite-attractor property. From this we directly deduce the decidability of these games on probabilistic lossy channel systems.

## 1 Introduction

2-player stochastic games are games where two players, Alice and Bob, interact in a probabilistic environment. Given an objective formalized, e.g., as an  $\omega$ -regular condition, the goal for Alice is to maximize the probability to fulfill the condition, against any behaviour of her opponent. Qualitative questions ask whether Alice can win almost-surely (resp. positively) from a given initial configuration. Solving a stochastic game then amounts to deciding the latter question, as well as providing winning strategies for the players. In the case where the arena is finite, the literature offers several general results on the existence of optimal strategies, the determinacy of the games, and algorithmic methods for computing solutions, when the objectives range in complexity from simple reachability objectives to arbitrary Borel objectives [39, 25, 23].

For infinite arenas, general results are scarce and mostly concern purely mathematical, non-algorithmic, aspects, such as determinacy [31, 20]. An obvious explanation for the lack of algorithmic results is that infinite-state spaces usually lead to undecidable, even highly undecidable, problems. This already happens for the simplest objectives with a single player and no stochastic aspects.

Decidability can be regained for infinite arenas if it is known that they are generated in some specific way. The stochastic games on infinite arenas considered in the field of algorithmic verification originate from classical —i.e., non-stochastic and non-competitive— infinite-state models. Prominent examples with positive results are stochastic games on systems with recursion [26, 19], on one-counter automata [17, 18], and on lossy channel systems [8, 2]. In all these examples, the description of winning sets and winning strategies is specific to the underlying infinite-state model, and rely on ad-hoc techniques.

In this paper we follow a more generic approach, and study stochastic games on (finite-choice) infinite arenas where we only assume the finite-attractor property [6]. That is, we assume that some finite set of configurations is almost-surely visited infinitely often, independently of the behaviors of the players.

**Our contributions.** Our first contribution is a simple fixpoint characterization of the winning sets and associated winning strategies for generalized Büchi objectives with probability one. The characterization

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is not concerned with computability and applies to any finite-choice countable arena with a finite attractor. We use  $\mu$ -calculus notation to define, and reason about, the winning sets and winning strategies: one of our goals is to give a *fully detailed* generic correctness proof (we used the characterization without proof in [12, section 9.2]).

Our second contribution is an application of the above characterization to prove the computability of winning sets (for generalized Büchi objectives) in arenas generated by probabilistic lossy channel systems (PLCSs). Rather than using ad-hoc reasoning, we follow the approach advocated in [7, 12] and use a generic finite-time convergence theorem for well-structured transition systems (more generally: for fixpoints over the powersets of WQO's). This allows us to infer the computability and the regularity of the winning sets directly from the fact that their fixpoint characterization uses “upward-guarded” fixpoint terms built on regularity-preserving operators. The method easily accommodates arbitrary regular arena partitions, PLCSs extended with regular guards, and other kinds of unreliability.

**Related work on lossy channel systems.** An early positive result for stochastic games on probabilistic lossy channel systems is the decidability of single-player reachability or Büchi games with probability one (dually, safety or co-Büchi with positive probability) [8]. Abdulla *et al.* then proved the determinacy and decidability of two-player stochastic games on PLCSs for (single) Büchi objectives with probability one [2] and we gave a simplified and generalized proof in [12]. On PLCSs, these positive results cannot be extended much—in particular to parity objectives—since Büchi games with positive probability are undecidable, already in the case of a single player [8]. Attempts to extend the decidability beyond (generalized) Büchi must thus abandon some generality in other dimensions, e.g., by restricting to finite-memory strategies, as in the one-player case [8].

**Outline of the paper.** Section 2 introduces the necessary concepts and notations on turn-based stochastic games. Section 3 provides the characterization of winning configurations in the general case of arenas with a finite attractor. Section 4 focuses on stochastic games on lossy channel systems and explains how decidability is obtained.

## 2 Stochastic games with a finite attractor

We consider general 2-player stochastic turn-based games on finite-choice countable arenas. In such games, the two players choose moves in turns and the outcome of their choice is probabilistic.

**Definition 2.1.** A turn-based stochastic arena is a tuple  $\mathcal{G} = (\text{Conf}, \text{Moves}, \mathbf{P})$  such that  $\text{Conf}$  is a countable set of configurations partitioned into  $\text{Conf}_A \sqcup \text{Conf}_B$ ,  $\text{Moves}$  is a set of moves, and  $\mathbf{P} : \text{Conf} \times \text{Moves} \rightarrow \text{Dist}(\text{Conf})$  is a partial function whose values are probabilistic distribution of configurations. We say that move  $m$  is enabled in configuration  $c$  when  $\mathbf{P}(c, m)$  is defined.  $\mathcal{G}$  is eternal (also deadlock-free) if for all  $c$  there is some enabled  $m$ .

The set of possible configurations  $\text{Conf}$  of the game is partitioned into configurations “owned” by each of the players: in some  $c \in \text{Conf}_A$ , player A, or “Alice”, chooses the next move, while if  $c \in \text{Conf}_B$ , it is player B, “Bob”, who chooses. It is useful to consider informally that, beyond Alice and Bob, there is a third party called “the environment” who is responsible for the probabilistic behaviors. This is why the game is stochastic: after each move  $m$  of one of the players, the environment chooses the next configuration probabilistically according to  $\mathbf{P}(c, m)$ . For a configuration  $c$ , when move  $m \in \text{Moves}$  is selected, we write  $\text{Post}[m](c)$  for the set of possible configurations from  $c$  after  $m$ :  $\text{Post}[m](c) \stackrel{\text{def}}{=} \{c' \in$

$Conf \mid \mathbf{P}(c, m)(c') > 0\}$ , and, symmetrically,  $Pre[m](c) \stackrel{\text{def}}{=} \{c' \in Conf \mid \mathbf{P}(c', m)(c) > 0\}$  denotes the set of possible predecessors by  $m$ .

**Runs and strategies.** For simplification purposes, we assume in the rest of this paper that all arenas are eternal, aka deadlock-free. A *run* of  $\mathcal{G}$  is a (non-empty) sequence  $\rho \in Conf^* \cup Conf^\omega$ , finite or infinite, of configurations. A *strategy* for player  $A$  resolves all non-deterministic choices in  $Conf_A$  by mapping every run ending in an  $A$ -configuration (i.e., a configuration  $c \in Conf_A$ ) to a move enabled in  $c$ . Formally, a strategy  $\sigma$  for Alice (an  $A$ -strategy) is a mapping  $\sigma : Conf^* Conf_A \rightarrow Moves$  such that, for every history run  $\rho = c_0 c_1 \cdots c_n$  with  $c_n \in Conf_A$ ,  $\sigma(\rho)$  is enabled in  $c_n$ . Symmetrically, a strategy for Bob (a  $B$ -strategy) is a mapping  $\tau : Conf^* Conf_B \rightarrow Moves$  which assigns an enabled move with each history run ending in  $Conf_B$ . The pair of strategies  $(\sigma, \tau)$  is called a *strategy profile*. Note that in this paper we restrict to pure, also called deterministic, strategies. Allowing for mixed, aka randomized, strategies would not change the winning configurations [22].

Not all runs agree with a given strategy profile. We say that a finite or infinite run  $\rho = c_0 c_1 \cdots c_n \cdots$  is *compatible* with  $(\sigma, \tau)$  if for every prefix  $\rho_i = c_0 \cdots c_i$  of  $\rho$ ,  $c_i \in Conf_A$  implies  $\mathbf{P}(c_i, \sigma(\rho_i))(c_{i+1}) > 0$ , and  $c_i \in Conf_B$  implies  $\mathbf{P}(c_i, \tau(\rho_i))(c_{i+1}) > 0$ .

**Probabilistic semantics.** The behavior of  $\mathcal{G}$  under strategy profile  $(\sigma, \tau)$  is described by an infinite-state Markov chain  $\mathcal{G}_{\sigma, \tau}$  where the states are all the finite runs compatible with  $(\sigma, \tau)$ , and where there is a transition from  $\rho_i$  to  $\rho_{i+1} = \rho_i \cdot c_{i+1}$  with probability  $\mathbf{P}(c_i, \sigma(\rho_i))(c_{i+1})$  if  $c_i \in Conf_A$ , and  $\mathbf{P}(c_i, \tau(\rho_i))(c_{i+1})$  if  $c_i \in Conf_B$ . Standardly —see, e.g., [34] for details— with the Markov chain  $\mathcal{G}_{\sigma, \tau}$  and a starting configuration  $c_0$ , is associated a probability measure on the set of runs of  $\mathcal{G}$  starting with  $c_0$  and where behaviors are ruled by  $(\sigma, \tau)$ .

It is well-known that given  $\varphi$  an LTL formula where atomic propositions are arbitrary sets of configurations, the set of runs that satisfy  $\varphi$  is measurable. Below we write  $\mathbb{P}_{\sigma, \tau}(c_0 \models \varphi)$  for the measure of runs of  $\mathcal{G}_{\sigma, \tau}$  that start with  $c_0$  and satisfy  $\varphi$ , and use the standard “ $\square$ ”, “ $\diamond$ ” and “ $\bigcirc$ ” symbols for linear-time modalities “always”, “eventually” and “next”.

**Game objectives.** Given a stochastic arena  $\mathcal{G}$ , the objective of the game describes the goal Alice aims at achieving. In this paper we consider generalized Büchi objectives. Let  $R_1, \dots, R_r \subseteq Conf$  be  $r$  sets of configurations, with an associated generalized Büchi property  $\varphi = \bigwedge_{i=1}^r \square \diamond R_i$ . We consider the game on  $\mathcal{G}$  where Alice’s objective is to satisfy  $\varphi$  with probability one.

We say that an  $A$ -strategy  $\sigma$  is *almost-surely winning* from  $c_0$  for objective  $\varphi$  if for every  $B$ -strategy  $\tau$ ,  $\mathbb{P}_{\sigma, \tau}(c_0 \models \varphi) = 1$ . In this case, we say that configuration  $c_0$  is *winning* (for Alice). The set of winning configurations is denoted  $\langle\langle A \rangle\rangle^1 \varphi$ , using PATL-like notation [24, 9].

**Finite attractor.** In this paper, we focus on a subclass of stochastic arenas, namely those with a finite attractor, following a terminology introduced in [3]. We say that a subset  $F \subseteq Conf$  is a *finite attractor* for the arena  $\mathcal{G}$  if (1)  $F$  is finite, and (2) for every initial configuration  $c_0$  and for every strategy profile  $(\sigma, \tau)$ ,  $\mathbb{P}_{\sigma, \tau}(c_0 \models \square \diamond F) = 1$ . In words,  $F$  is almost surely visited infinitely often under all strategy profiles. Note that an attractor is not what is called a recurrent set in Markov chains, since —depending on  $c_0$  and  $(\sigma, \tau)$ — it does not necessarily hold that all configurations in  $F$  are visited infinitely often almost surely. An attractor is also not an absorbing set since the players may leave  $F$  after visiting it —but they will almost surely return to it. Note also that in game theory one sometimes uses the term “attractor” to denote a set from where one player can ensure to reach a given goal, something that we

call a winning set. The existence of a finite attractor is a powerful tool for reasoning about infinite runs in countable Markov chains, see examples in [3, 6, 35, 8, 2].

**Finite-choice hypothesis.** Beyond the finite attractor property, we also require that the adversary, Bob, only has finitely many choice: more precisely, we assume that in every configuration of  $Conf_B$ , the set of enabled moves is finite. Note that we do not assume a *uniform* bound on the number of moves enabled in Bob’s configurations, and also that the finite-choice hypothesis only applies to Bob, the adversarial player. These are called  $\diamond$ -*finitely-branching games* in [20], and are not a strong restriction in applications, unlike the finite-attractor assumption that is usually not satisfied in practice. We want to stress that we allow infinite arenas that are *infinitely branching* both for Alice —she may have countably many enabled moves in a given  $c$ — and for the environment — $Post[m](c)$  may be infinite for given  $c$  and  $m$ —, and thus are not *coarse*, i.e., non-zero probabilities are not bounded from below.

### 3 Solving generalized Büchi games

In this section we provide a simple fixpoint characterization of the set of winning configurations (and of the associated winning strategies) for games with a generalized Büchi objective that should be satisfied almost-surely. For this characterization and its proof of correctness, we use terms with fixpoints combining functions and constants over the complete lattice  $2^{Conf}$  of all sets of configurations.

#### 3.1 A $\mu$ -calculus for fixpoint terms

We assume familiarity with  $\mu$ -calculus notation and only recall the basic concepts and notations we use below. The reader is referred to [5, 16] for more details.

The set of subsets of configurations ordered by inclusion,  $(2^{Conf}, \subseteq)$ , is a complete Boolean lattice. We consider *monotonic operators*, i.e.,  $n$ -ary mappings  $f : (2^{Conf})^n \rightarrow (2^{Conf})$  such that  $f(U_1, \dots, U_n) \subseteq f(V_1, \dots, V_n)$  when  $U_i \subseteq V_i$  for all  $i = 1, \dots, n$ . (A *constant*  $U \subseteq Conf$  is a 0-ary monotonic operator.) Formally, the language  $L_\mu = \{\varphi, \psi, \dots\}$  of *terms with fixpoints* is given by the following abstract grammar

$$\varphi ::= f(\varphi_1, \dots, \varphi_n) \mid X \mid \mu X. \varphi \mid \nu X. \varphi$$

where  $f$  is any  $n$ -ary monotonic operator and  $X$  is any variable. Terms of the form  $\mu X. \varphi$  and  $\nu X. \varphi$  are least and greatest fixpoint expressions.

The complementation operator  $\neg$ , defined with  $\neg U = Conf \setminus U$ , may be used as a convenience when writing down  $L_\mu$  terms as long as any bound variable is under the scope of an even number of negations. Such terms can be rewritten in positive forms by using the *dual*  $\tilde{f}$  of any  $f$ , defined with  $\tilde{f}(U_1, \dots, U_n) \stackrel{\text{def}}{=} \neg f(\neg U_1, \dots, \neg U_n)$ . Note that  $\tilde{f}$  is monotonic since  $f$  is.

The semantics of  $L_\mu$  terms is as expected (see [12, 16]). Since we only use monotonic operators in our fixpoint terms, all the terms have a well-defined interpretation as a subset of  $Conf$  for closed terms, and more generally as a monotonic  $n$ -ary mapping over  $2^{Conf}$  for terms with  $n$ -free variables. We slightly abuse notation, letting e.g.  $\varphi(X_1, \dots, X_n)$  denote both a term in  $L_\mu$  and its denotation as an  $n$ -ary monotonic operator. Similarly,  $\varphi(\psi_1, \dots, \psi_n)$  is the term obtained by substituting  $\psi_1, \dots, \psi_n \in L_\mu$  for the (free occurrences of) the  $X_i$ ’s in  $\varphi$ . Finally, when  $U_1, \dots, U_n \subseteq Conf$  are constants,  $\varphi(U_1, \dots, U_n)$  also denotes the application of the operator defined by  $\varphi$  over the  $U_i$ ’s.

When reasoning on fixpoint terms, one often uses *unfoldings*, i.e., the following equalities stating that a least or greatest fixpoint is indeed a fixpoint:

$$\mu X.\varphi(X, \dots) = \varphi(\mu X.\varphi(X, \dots), \dots), \quad \nu X.\varphi(X, \dots) = \varphi(\nu X.\varphi(X, \dots), \dots).$$

Recall that the least (or greatest) fixpoint is the least *pre-fixpoint* (greatest *post-fixpoint*):

$$\varphi(U) \subseteq U \text{ implies } \mu X.\varphi(X) \subseteq U, \quad \varphi(U) \supseteq U \text{ implies } \nu X.\varphi(X) \supseteq U.$$

It is well-known (Kleene's fixpoint theorem) that when monotonic operators are  $\cup$ - and  $\cap$ -continuous, —i.e., satisfy  $f(\cup_i U_i) = \cup_i f(U_i)$  and  $f(\cap_i U_i) = \cap_i f(U_i)$ —, their least and greatest fixpoints are obtained as the limits of  $\omega$ -length sequences of approximants. Since we do not assume  $\cap/\cup$ -continuity in our setting —e.g., *Pre* is not  $\cap$ -continuous when finite-branching is not required—, fixpoints are obtained as the stationary limits of transfinite ordinal-indexed sequences of approximants, see [16]. For a set  $U = \mu X.\varphi(X)$  defined as a least fixpoint, the approximants  $(U_\alpha)_{\alpha \in \text{Ord}}$  are defined inductively with  $U_0 \stackrel{\text{def}}{=} \emptyset$ ,  $U_{\beta+1} \stackrel{\text{def}}{=} \varphi(U_\beta)$  for a successor ordinal, and  $U_\lambda \stackrel{\text{def}}{=} \cup_{\beta < \lambda} U_\beta$  for a limit ordinal  $\lambda$ . For a greatest fixpoint  $V = \nu X.\varphi(X)$ , they are given by  $V_0 \stackrel{\text{def}}{=} \text{Conf}$ ,  $V_{\beta+1} \stackrel{\text{def}}{=} \varphi(V_\beta)$  and  $V_\lambda \stackrel{\text{def}}{=} \cap_{\beta < \lambda} V_\beta$ .

### 3.2 A characterization of winning sets

We first introduce auxiliary operators that let us reason about strategies and characterize the winning sets. Let  $\text{Enabled}(c) \subseteq \text{Moves}$  denote the set of moves enabled in configuration  $c$  and for  $X, Y \subseteq \text{Conf}$  let

$$\begin{aligned} \text{Pre}^\exists(X, Y) &\stackrel{\text{def}}{=} \{c \in \text{Conf} \mid \exists m \in \text{Enabled}(c), \text{Post}[m](c) \subseteq X \text{ and } \text{Post}[m](c) \cap Y \neq \emptyset\}, \\ \text{Pre}^\forall(X, Y) &\stackrel{\text{def}}{=} \{c \in \text{Conf} \mid \forall m \in \text{Enabled}(c), \text{Post}[m](c) \subseteq X \text{ and } \text{Post}[m](c) \cap Y \neq \emptyset\}. \end{aligned}$$

One can see that  $\text{Pre}^\exists$  and  $\text{Pre}^\forall$  are monotonic in both arguments by reformulating their definitions in terms of the more familiar *Pre* operator (recall that  $c \in \widetilde{\text{Pre}}[m](\emptyset)$  iff  $m$  is not enabled in  $c$ ):

$$\begin{aligned} \text{Pre}^\exists(X, Y) &= \bigcup_{m \in \text{Moves}} [\widetilde{\text{Pre}}[m](X) \cap \text{Pre}[m](Y)], \\ \text{Pre}^\forall(X, Y) &= \bigcap_{m \in \text{Moves}} (\widetilde{\text{Pre}}[m](\emptyset) \cup [\widetilde{\text{Pre}}[m](X) \cap \text{Pre}[m](Y)]). \end{aligned}$$

We further define  $\text{Pre}_A^\otimes(X, Y) \stackrel{\text{def}}{=} (\text{Conf}_A \cap \text{Pre}^\exists(X, Y)) \cup (\text{Conf}_B \cap \text{Pre}^\forall(X, Y))$ . In other words,  $\text{Pre}_A^\otimes(X, Y)$  is exactly the set from where Alice can guarantee in one step to have  $X$  surely and  $Y$  with positive probability. This can be summarized as:

**Fact 3.1.** *Let  $X, Y \subseteq \text{Conf}$ .*

1. *If  $c \in \text{Pre}_A^\otimes(X, Y)$ , then,  $A$  has a memoryless strategy  $\sigma$  such that, for every strategy  $\tau$  for  $B$ :  $\mathbb{P}_{\sigma, \tau}(c \models \bigcirc X) = 1$  and  $\mathbb{P}_{\sigma, \tau}(c \models \bigcirc Y) > 0$ .*
2. *If  $c \notin \text{Pre}_A^\otimes(X, Y)$ , then,  $B$  has a memoryless strategy  $\tau$  such that, for every strategy  $\sigma$  for  $A$ :  $\mathbb{P}_{\sigma, \tau}(c \models \bigcirc X) < 1$  or  $\mathbb{P}_{\sigma, \tau}(c \models \bigcirc Y) = 0$ .*

Building on  $\text{Pre}_A^\otimes$ , we may define the following unary operators: for  $i = 1, \dots, r$ ,  $H_i$  is given by

$$H_i(X) \stackrel{\text{def}}{=} \mu Z.X \cap \text{Pre}_A^\otimes(X, R_i \cup Z). \tag{1}$$

The intuition is that, from  $H_i(X)$ , Alice has a strategy ensuring a positive probability of reaching  $R_i$  later—which would be characterized by “ $\mu Z.Pre_A^\otimes(Conf, R_i \cup Z)$ ”—*all the while staying surely in  $X$* , hence the amendments. See Lemma 3.6 for a precise statement. Unfolding its definition, we see that  $H_i(X) \subseteq X$ , i.e.,  $H_i$  is *contractive*.

Letting  $H_{1,r}(X) \stackrel{\text{def}}{=} \bigcap_{i=1}^r H_i(X)$ , we finally define the following fixpoint terms:

$$W \stackrel{\text{def}}{=} \nu X. H_{1,r}(X) = \nu X. \bigcap_{i=1}^r \left[ \mu Z. X \cap Pre_A^\otimes(X, R_i \cup Z) \right], \quad (2)$$

$$W' \stackrel{\text{def}}{=} \nu X. Pre_A^\otimes(H_{1,r}(X), Conf) = \nu X. Pre_A^\otimes\left(\bigcap_{i=1}^r \left[ \mu Z. X \cap Pre_A^\otimes(X, R_i \cup Z) \right], Conf\right), \quad (3)$$

$$W_1 \stackrel{\text{def}}{=} \nu X. \mu Z. Pre_A^\otimes(X, R_1 \cup Z). \quad (4)$$

**Theorem 3.2** (Fixpoint characterization of winning sets). *We fix a stochastic arena with a finite attractor, and assume it is finite-choice for Bob. Then, for generalized Büchi objectives the winning set  $\langle\langle A \rangle\rangle^=1 \wedge_{i=1}^r \square \diamond R_i$  coincides with  $W$ . Moreover  $W = W'$  and from  $W$  Alice has an almost-surely winning strategy  $\sigma_W$  that is a finite-memory strategy.*

*In the case  $r = 1$  of simple Büchi objectives the winning set  $\langle\langle A \rangle\rangle^=1 \square \diamond R_1$  coincides with  $W_1$  and the winning strategy  $\sigma_W$  is even a memoryless strategy.*

Before proving Theorem 3.2, let us explain how, in the case where  $r = 1$ , one derives the correctness of  $W_1$  from the correctness of  $W$ . Setting  $r = 1$  in Eq. (2) yields  $W = \nu X. \mu Z. X \cap Pre_A^\otimes(X, R_1 \cup Z)$ . In this situation, we can use Eq. (†), a purely algebraic and lattice-theoretical equality that holds for any monotonic binary  $f$  (see Appendix for a proof):

$$\nu X. \mu Z. X \cap f(X, Z) = \nu X. \mu Z. f(X, Z). \quad (\dagger)$$

Applying Eq. (†) on  $W = \nu X. \mu Z. X \cap Pre_A^\otimes(X, R_1 \cup Z)$  yields  $W = \nu X. \mu Z. Pre_A^\otimes(X, R_1 \cup Z) = W_1$ .

Theorem 3.2 provides two different characterizations of the winning set  $\langle\langle A \rangle\rangle^=1 (\wedge_{i=1}^r \square \diamond R_i)$ . Let us now prove its validity, in the general context of finite-choice stochastic arenas with a finite attractor<sup>1</sup>. The proof is divided in two parts: correctness of  $W'$  in Proposition 3.5, completeness of  $W$  in Proposition 3.7, and some purely lattice-theoretical reasoning closing the loop in Lemma 3.8.

### 3.3 Correctness for $W'$

We prove that  $W'$  only contains winning configurations for Alice by exhibiting a strategy with which she ensures almost surely  $\wedge_i \square \diamond R_i$  when starting from some  $c \in W'$ . We first define  $r$  strategies  $(\sigma_i)_{1 \leq i \leq r}$ , one for each goal set  $R_1, \dots, R_r$ , and prove their relevant properties. It will then be easy to combine the  $\sigma_i$ 's in order to produce the required strategy.

For  $i = 1, \dots, r$ , unfolding Eq. (1) yields  $H_i(W') = W' \cap Pre_A^\otimes(W', R_i \cup H_i(W'))$ . We let  $\sigma_i$  be the memoryless  $A$ -strategy defined as follows: for  $c \in Conf_A \cap H_i(W')$ , Alice picks an enabled move  $m$  such that  $Post(c)[m] \subseteq W'$  and  $Post[m](c) \cap (R_i \cup H_i(W')) \neq \emptyset$ , which is possible by definition of  $Pre_A^\otimes$ , while for  $c \in Conf_A \cap W' \cap \neg H_i(W')$ , Alice picks an enabled move  $m$  with  $Post(c)[m] \subseteq H_{1,r}(W')$ , which is possible since  $W' = Pre_A^\otimes(H_{1,r}(W'), Conf)$  by Eq. (3).

$H_{1,r}$  is contractive since the  $H_i$ 's are, hence  $H_{1,r}(W') \subseteq W'$  and we deduce that “ $\sigma_i$  stays in  $W'$ ”:

$$\forall c \in W' : \forall \tau : \mathbb{P}_{\sigma_i, \tau}(c \models \square W') = 1. \quad (5)$$

<sup>1</sup>We show later that the characterizations of the winning sets is *not correct* if one does not assume the finite attractor property.

**Lemma 3.3.** *For all  $c \in W'$  there exists some  $\gamma_c > 0$  such that  $\mathbb{P}_{\sigma_i, \tau}(c \models \Diamond R_i) \geq \gamma_c$  for all  $B$ -strategies  $\tau$ .*

*Proof.* Here we use the finite-choice assumption. First consider the case where  $c \in H_i(W')$ . Writing  $(Z_\alpha)_{\alpha \in \text{Ord}}$  for the approximants of  $H_i(W') = \mu Z. W' \cap \text{Pre}_A^\otimes(W', R_i \cup Z)$ , we prove, by induction on  $\alpha$ , that  $\gamma_c > 0$  exists when  $c \in Z_\alpha$ . The base case  $\alpha = 0$  holds vacuously since  $Z_0 = \emptyset$ . For  $\alpha = \lambda$  (a limit ordinal),  $Z_\lambda = \bigcup_{\beta < \lambda} Z_\beta$  so each  $c \in Z_\lambda$  is in some  $Z_\beta$  and the induction hypothesis applies.

Now to the successor case  $\alpha = \beta + 1$ . Here  $Z_\alpha = W' \cap \text{Pre}_A^\otimes(W', R_i \cup Z_\beta)$  and, given  $\sigma_i$  and for any  $\tau$ , from  $c \in Z_\alpha$  Alice or Bob will pick a move  $m$  with  $\text{Post}[m](c) \cap (R_i \cup Z_\beta) \neq \emptyset$ . The probability that after probabilistic environment's move the play will be in  $R_i$  exactly at the next step is precisely  $\gamma = \sum_{d \in R_i} \mathbf{P}(c, m)(d)$  and  $\gamma > 0$  if  $\text{Post}[m](c) \cap R_i \neq \emptyset$  (and only then). If  $\gamma = 0$  then  $\text{Post}[m](c) \cap R_i = \emptyset$  so that  $\text{Post}[m](c) \cap Z_\beta \neq \emptyset$ . Then there is a positive probability  $\gamma' = \sum_{d \in Z_\beta} \mathbf{P}(c, m)(d)$  that after probabilistic decision the play will be in  $Z_\beta$  at the next step, hence (by induction hypothesis) a positive probability  $\gamma''$  that it will be in  $R_i$  later, with  $\gamma'' \geq \sum_{d \in Z_\beta} \gamma_d \cdot \mathbf{P}(c, m)(d)$ . Note that for  $d \in Z_\beta$ ,  $\gamma_d$  does not depend on  $\tau$  (by ind. hyp.) so that  $\gamma$  and the lower bound for  $\gamma''$  only slightly depend on  $\tau$ : they depend on what move  $m$  is chosen by Bob if  $c \in \text{Conf}_B$ . Now, since there are only finitely many moves enabled in  $c$ , we can pick a strictly positive value that is a lower bound for all the corresponding  $\max(\gamma, \gamma'')$ , proving the existence of  $\gamma_c > 0$  for  $c \in Z_\alpha$ .

There remains the case where  $c \in W' \cap \neg H_i(W')$ : here  $\sigma_i$  ensures that the play will be in  $H_i(W')$  in the next step. If  $c \in \text{Conf}_B$ , we can let  $\gamma_c \stackrel{\text{def}}{=} \min_{m \in \text{Enabled}(c)} \sum_{d \in H_i(W')} \mathbf{P}(c, m)(d) \cdot \gamma_d$ , which ensures  $\gamma_c > 0$  by the finite-choice assumption. In case  $c \in \text{Conf}_A$ , we simply define  $\gamma_c \stackrel{\text{def}}{=} \sum_{d \in H_i(W')} \mathbf{P}(c, m)(d) \cdot \gamma_d$  where  $m$  is the move given by  $\sigma_i$  when in configuration  $c$ . In both cases, we thus have  $\gamma_c > 0$ , which concludes the proof.  $\square$

**Remark** (On the finite-choice assumption for  $B$ ). *Clearly enough, Lemma 3.3 does not hold if we relax the assumption that in every configuration of  $\text{Conf}_B$ , the set of enabled moves is finite. Indeed, consider a simple arena with three configurations  $c$ ,  $r$  and  $s$ , all belonging to player  $B$ , where  $r$  and  $s$  are sinks, and from  $c$  there are countably many enabled moves  $m_1, m_2, \dots$  whose respective effect is defined by  $\mathbf{P}(c, m_k)(r) = 1/2^k$  and  $\mathbf{P}(c, m_k)(s) = 1 - 1/2^k$ . Letting  $R = \{r\}$  and for the single Büchi objective  $\square \Diamond R$ , we obtain  $W' = \text{Conf}$ , and in particular  $c \in W'$ . Yet, there is no uniform lower bound  $\gamma_c$  with  $\mathbb{P}_{\sigma_i, \tau}(c \models \Diamond R_i) \geq \gamma_c$  for all  $B$ -strategies  $\tau$ .*

**Lemma 3.4.**  $\mathbb{P}_{\sigma_i, \tau}(c \models \square W' \wedge \square \Diamond R_i) = 1$  for all  $c \in W'$  and all  $B$ -strategies  $\tau$ .

*Proof.* This is where we use the finite-attractor property: there is a finite set  $F \subseteq \text{Conf}$  such that  $\mathbb{P}_{\sigma, \tau}(c \models \square \Diamond F) = 1$  for any  $c \in \text{Conf}$  and strategies  $\sigma$  and  $\tau$ . In particular, for  $\sigma_i$  and using Eq. (5), we deduce  $\mathbb{P}_{\sigma_i, \tau}(c \models \square W' \wedge \square \Diamond F) = 1$  for any  $c \in W'$  and any strategy  $\tau$  (entailing  $F \cap W' \neq \emptyset$ ). Let now  $\gamma \stackrel{\text{def}}{=} \min\{\gamma_f \mid f \in F \cap W'\}$  so that for any  $f \in F \cap W'$  and any  $B$ -strategy  $\tau$ , Lemma 3.3 gives  $\mathbb{P}_{\sigma_i, \tau}(f \models \Diamond R_i) \geq \gamma$ . Note that  $\gamma > 0$  since  $F \cap W'$  is finite. Since from  $F \cap W'$  and applying  $\sigma_i$ , the probability to eventually reach  $R_i$  is lower bounded by  $\gamma$ , and since  $\mathbb{P}_{\sigma_i, \tau}(c \models \square \Diamond (F \cap W')) = 1$ , we deduce that  $\mathbb{P}_{\sigma_i, \tau}(c \models \square \Diamond R_i) = 1$  by standard reasoning on recurrent sets.  $\square$

**Remark** (On the finite-attractor assumption). *Lemma 3.4 crucially relies on the finite-attractor property. Indeed, consider the random walk on the set of naturals where from any state  $n > 0$  the probability is  $\frac{3}{4}$  to move to  $n+1$  and  $\frac{1}{4}$  to move to  $n-1$  (and where state 0 is a sink where one stays forever). It is a well-known result that, starting from any  $n > 0$ , the probability is strictly less than 1 to visit state 0—and, in fact, any finite set of states—infinately often. This random walk however can be seen as a stochastic game (with a single player and a single move in each state) for which, and taking  $R_1 = \{0\}$ ,*

$W_1$  consists of the whole states set (indeed,  $W_1 = \text{Pre}^*(R_1)$  for single-player single-choice arenas). This provides a simple example showing that the finite-attractor property is required for Lemma 3.4, and for Theorem 3.2, to hold.  $\square$

**Proposition 3.5** (Correctness of  $W'$ ).  $W' \subseteq \langle\langle A \rangle\rangle^{\perp 1} (\bigwedge_{i=1}^r \square \diamond R_i)$ .

*Proof.* By combining the strategies  $\sigma_i$ 's, we define a finite-memory strategy  $\sigma_W$  that guarantees  $\mathbb{P}_{\sigma_W, \tau}(c \models \bigwedge_i \square \diamond R_i) = 1$  for any  $c \in W'$  and against any  $B$ -strategy  $\tau$ .

More precisely,  $\sigma_W$  has  $r$  modes:  $1, 2, \dots, r$ . In mode  $i$ ,  $\sigma_W$  behaves like  $\sigma_i$  until  $R_i$  is reached, which is bound to eventually happen with probability 1 by Lemma 3.4. Note that the play remains constantly in  $W'$ . Once  $R_i$  has been reached,  $\sigma_W$  switches to mode  $i + 1 \pmod{r}$ , playing at least one move. This is repeated in a neverending cycle, ensuring  $\square \diamond R_i$  with probability 1.  $\square$

**Remark** (On randomized memoryless strategies). *It is known that, if one considers mixed, aka randomized, strategies, generalized Büchi objectives on  $\mathcal{G}$  admit memoryless, aka deterministic, winning strategies [20]. Note that our  $\sigma_W$  is finite-memory (and not randomized). It is a natural question whether a simple randomized memoryless strategy like “at each step, choose randomly and uniformly between following  $\sigma_1, \dots, \sigma_r$ ” is almost-surely winning for  $\bigwedge_{i=1}^r \square \diamond R_i$ .*

### 3.4 Completeness of $W$

In order to prove that  $W$  contains the winning set for Alice, we show that  $\langle\langle A \rangle\rangle^{\perp 1} \bigwedge_i \square \diamond R_i$  is a post-fixpoint of  $H_{1,r}$ , thus necessarily included in its greatest fixpoint  $W$ . We start with the following lemma:

**Lemma 3.6.**  $H_i(X) \supseteq \{c \mid \exists \sigma \forall \tau, \mathbb{P}_{\sigma, \tau}(c \models \square X) = 1 \text{ and } \mathbb{P}_{\sigma, \tau}(c \models \bigcirc \diamond R_i) > 0\}$ .

*Proof.* We actually prove a stronger claim: we show that there exists a memoryless  $B$ -strategy  $\tau$  such that, for every  $c \notin H_i(X)$  and every  $A$ -strategy  $\sigma$ , either  $\mathbb{P}_{\sigma, \tau}(c \models \diamond \neg X) > 0$ , or  $\mathbb{P}_{\sigma, \tau}(c \models \bigcirc \square \neg R_i) = 1$ .

Let  $c \notin H_i(X)$ . By definition  $\neg H_i(X) = \neg X \cup \neg \text{Pre}_A^\otimes(X, R_i \cup H_i(X))$ . If  $c \notin X$ , then trivially  $\mathbb{P}_{\sigma, \tau}(c \models \diamond \neg X) > 0$  for any  $(\sigma, \tau)$  so we do not care how  $\tau$  is defined here. Consider now  $c \notin \text{Pre}_A^\otimes(X, R_i \cup H_i(X))$ . By Fact 3.1, Bob has a (memoryless) strategy  $\tau_c$  such that against any  $A$ -strategy  $\sigma$ ,  $\mathbb{P}_{\sigma, \tau_c}(c \models \bigcirc X) < 1$  or  $\mathbb{P}_{\sigma, \tau_c}(c \models \bigcirc (R_i \cup H_i(X))) = 0$ , which can be reformulated as  $\mathbb{P}_{\sigma, \tau_c}(c \models \bigcirc \neg X) > 0$  or  $\mathbb{P}_{\sigma, \tau_c}(c \models \bigcirc (\neg R_i \cap \neg H_i(X))) = 1$ . For  $c \in \text{Conf}_B$ , we define  $\tau(c)$  as the move given by  $\tau_c(c)$ . The resulting strategy  $\tau$  guarantees, starting from  $\neg H_i(X)$ , that the game will either always stay in  $\neg R_i \cap \neg H_i(X)$  (after the 1st step) or has a positive probability of visiting  $\neg X$  eventually.  $\square$

**Proposition 3.7** (Completeness of  $W$ ).  $\langle\langle A \rangle\rangle^{\perp 1} (\bigwedge_{i=1}^r \square \diamond R_i) \subseteq W$ .

*Proof.* Let  $c \in \langle\langle A \rangle\rangle^{\perp 1} \bigwedge_i \square \diamond R_i$ , and  $\sigma$  be a strategy ensuring  $\bigwedge_i \square \diamond R_i$  with probability 1 from  $c$ . Consider  $E = \{d \in \text{Conf} \mid \exists \tau : \mathbb{P}_{\sigma, \tau}(c \models \diamond d) > 0\}$ , i.e., the set of configurations that can be visited under strategy  $\sigma$ . Obviously  $c \in E$ . Furthermore, for any  $d \in E$  and any  $B$ -strategy  $\tau$ ,  $\mathbb{P}_{\sigma', \tau}(d \models \square E) = 1$  holds, where  $\sigma'$  is a “suffix strategy” of  $\sigma$  after  $d$  is visited, that is,  $\sigma'$  behaves from  $d$  like  $\sigma$  would after some prefix ending in  $d$ . Since furthermore  $\mathbb{P}_{\sigma', \tau}(d \models \bigwedge_i \square \diamond R_i) = 1$  by assumption, we deduce in particular  $\mathbb{P}_{\sigma', \tau}(d \models \bigcirc \diamond R_i) = 1$  for any  $i = 1, \dots, r$ . Hence  $E \subseteq H_i(E)$  for any  $i$  by Lemma 3.6, and thus  $E \subseteq H_{1,r}(E)$ . Finally  $E$  is a post-fixpoint of  $H_{1,r}$ , and is thus included in its greatest fixpoint. We conclude that  $c \in \nu X. H_{1,r}(X) = W$ .  $\square$

The loop is closed, and Theorem 3.2 proven, with the following lattice-theoretical reasoning:

**Lemma 3.8.**  $W \subseteq W'$ .



*Proof.* Recall that  $W = \nu X.H_{1,r}(X)$ , so that  $W = H_{1,r}(W)$ . Similarly, using Eq. (1), we deduce  $H_{1,r}(W) = \bigcap_i H_i(W) = \bigcap_i (W \cap \text{Pre}_A^\otimes(W, R_i \cup H_i(W)))$ , hence  $H_{1,r}(W) \subseteq \text{Pre}_A^\otimes(W, \text{Conf})$  by monotonicity of  $\text{Pre}_A^\otimes$  in its second argument. Combining these two points gives  $W = H_{1,r}(W) \subseteq \text{Pre}_A^\otimes(H_{1,r}(W), \text{Conf})$ , hence  $W$  is a post-fixpoint of  $X \mapsto \text{Pre}_A^\otimes(H_{1,r}(X), \text{Conf})$  and is included in its greatest fixpoint  $W'$ .  $\square$

## 4 Stochastic games on lossy channel systems

Theorem 3.2 entails the decidability of generalized Büchi games on channel systems with probabilistic message losses, or PLCSS. This is obtained by applying a generic and powerful “finite-time convergence theorem” for fixpoints defined on WQO’s.

### 4.1 Channel systems with guards

A *channel system* is a tuple  $S = (Q, C, M, \Delta)$  consisting of a finite set  $Q = \{q, q', \dots\}$  of *locations*, a finite set  $C = \{\text{ch}_1, \dots, \text{ch}_d\}$  of *channels*, a finite *message alphabet*  $M = \{a, b, \dots\}$  and a finite set  $\Delta = \{\delta, \dots\}$  of *transition rules*. Each transition rule has the form  $(q, g, \text{op}, q')$ , written  $q \xrightarrow{g, \text{op}} q'$ , where  $g$  is a *guard* (see below), and  $\text{op}$  is an *operation* of one of the following three forms:  $\text{ch}!a$  (sending message  $a \in M$  along channel  $\text{ch} \in C$ ),  $\text{ch}?a$  (receiving message  $a$  from channel  $\text{ch}$ ), or  $\surd$  (an internal action with no I/O-operation).

Let  $S$  be a channel system as above. A *configuration* of  $S$  is a pair  $c = (q, w)$  where  $q$  is a location of  $S$  and  $w : C \rightarrow M^*$  is a mapping, that describes the current channel contents: we let  $\text{Conf}_S \stackrel{\text{def}}{=} Q \times M^{*C}$ .

A *guard* is a predicate on channel contents used to constrain the firability of rules. In this paper, a guard is a tuple  $g = (L_1, \dots, L_d) \in \mathbf{Reg}(M)^{|C|}$  of regular languages, one for each channel. For a configuration  $c = (q, w_1, \dots, w_d)$ , we write  $c \models g$ , and say that  $c$  *respects*  $g$ , when  $w_i \in L_i$  for all  $i = 1, \dots, d$ .

Rules give rise to transitions in the operational semantics. Let  $\delta = (q_1, g, \text{op}, q_2)$  be a rule in  $\Delta$  and let  $c = (q, w)$ ,  $c' = (q', w')$  be two configurations of  $S$ . We write  $c \xrightarrow{\delta} c'$ , and say that  $\delta$  is *enabled* in  $c$ , if  $q = q_1$ ,  $q' = q_2$ ,  $c \models g$ , and  $w'$  is the valuation obtained from  $w$  by applying  $\text{op}$ . Formally  $w' = w$  if  $\text{op} = \surd$ , and otherwise if  $\text{op} = \text{ch}_i!a$  (resp. if  $\text{op} = \text{ch}_i?a$ ) then  $w'_i = w_i.a$  (resp.  $a.w'_i = w_i$ ) and  $w'_j = w_j$  for all  $j \neq i$ .

For simplicity, we assume in the rest of the paper that  $S$  denotes a fixed channel system  $S = (Q, C, M, \Delta)$  that has no deadlock configurations, i.e., every  $c \in \text{Conf}_S$  has an enabled rule: this is no loss of generality since it is easy —*when guards are allowed*— to add rules going to a new sink location exactly in configurations where none of the original rules is enabled.

**Remark** (About guards in channel systems). *Allowing guards in transition rules is useful (e.g., for expressing priorities) but departs from the standard models of channel systems [32]. Indeed, testing the whole contents of a fifo channel is not a realistic feature when modeling distributed asynchronous systems. However, (unreliable) channel systems are now seen more broadly as a fundamental computational model closely related to Post’s tag systems and with algorithmic applications beyond distributed protocols: see, e.g., [29, 4, 11, 15]. In such settings, simple guards have been considered and proved useful: see, e.g., [15, 14, 27].*

*Using additional control states and messages, it is sometimes possible to simulate guards in (lossy) channel systems. We note that the known simulations preserve nondeterministic reachability but usually not game-theoretical properties in stochastic environments.*  $\square$

## 4.2 Probabilistic message losses

PLCSs are channel systems where messages can be lost (following some probabilistic model) while they are in the channels [33, 10, 3, 1, 35]. In this paper, we consider two kinds of unreliability caused by a stochastic environment: message losses on one hand, and combinations of message losses and duplications on the other hand.

Message losses are traditionally modeled via the subword relation: given two words  $u, v \in M^*$ , we write  $u \sqsubseteq v$  when  $u$  is a *subword*, i.e., a scattered subsequence, of  $v$ . For two configurations  $c = (q, w)$  and  $c' = (q', w')$ , we let  $c \sqsubseteq c' \stackrel{\text{def}}{\iff} (q = q' \text{ and } w_i \sqsubseteq w'_i \text{ for all } i = 1, \dots, d)$ . In other words,  $c \sqsubseteq c'$  when  $c$  is the result of removing some messages (possibly none) at arbitrary places in the channel contents for  $c'$ .

Message duplications are modeled by a rational transduction  $\mathcal{T}_{\text{dup}} \subseteq M^* \times M^*$  over sequences of messages, where every single message  $a \in M$  is replaced by either  $a$  or  $aa$ . We write  $u \preceq_{\text{dup}} v$  when  $(u, v) \in \mathcal{T}_{\text{dup}}$  (e.g.  $ab \preceq_{\text{dup}} aab$ ) and we extend to configurations with  $(q, w) \preceq_{\text{dup}} (q', w') \stackrel{\text{def}}{\iff} (q = q' \text{ and } w_i \preceq_{\text{dup}} w'_i \text{ for all } i = 1, \dots, d)$ .

For PLCs with only message losses, we write  $c \rightsquigarrow c'$  when  $c \sqsupseteq c' \stackrel{\text{def}}{\iff} c' \sqsubseteq c$ . For PLCs with losses and duplications,  $c \rightsquigarrow c'$  means that  $c \preceq_{\text{dup}} c'' \sqsupseteq c'$  for some  $c''$ .

In PLCs, message perturbations are probabilistic events. Formally, we associate a distribution  $D_{\text{env}}(c) \in \text{Dist}(\text{Conf}_S)$  with every configuration  $c \in \text{Conf}_S$  and we say that “ $D_{\text{env}}(c, c')$  is the probability that  $c$  becomes  $c'$  by message losses and duplications (in one step)”. Given  $D_{\text{env}}$  and a partition  $\text{Conf}_S = \text{Conf}_A \sqcup \text{Conf}_B$ , the channel system  $S$  with probabilistic losses defines a stochastic arena  $\mathcal{G}_S = (\text{Conf}_S, \Delta, \mathbf{P})$  where the moves available to the players are exactly the rules of  $S$ —thus  $\mathcal{G}_S$  is finite-choice—, and the probabilistic transition function  $\mathbf{P}$  is formalized by: for every  $c \in \text{Conf}_S$  and  $\delta$  enabled in  $c$ ,  $\mathbf{P}(c, \delta) \stackrel{\text{def}}{=} D_{\text{env}}(c')$  where  $c \xrightarrow{\delta} c'$ .

The qualitative properties that we are interested in do not depend on the exact choices made for  $D_{\text{env}}$ . In this paper, we only require that  $D_{\text{env}}$  is *well-behaved*, i.e., satisfies the following two properties:

**Compatibility with nondeterministic semantics:**  $D_{\text{env}}(c)(c') > 0$  iff  $c \rightsquigarrow c'$ .

**Finite attractor:** Some finite set  $F \subseteq \text{Conf}_S$  is visited infinitely often with probability one.

A now standard choice for  $D_{\text{env}}$  in PLCs models message losses (and duplications) as independent events. One assumes that at every step, each individual message can be lost with a fixed probability  $\lambda \in (0, 1)$ , duplicated with a fixed probability  $\lambda' \in [0, 1)$  (and remains unperturbed with probability  $1 - \lambda - \lambda'$ ). This is the so-called *local-fault* model from [3, 35, 38], and it gives rise to a well-behaved  $D_{\text{env}}$  when only message losses are considered, i.e., when  $\lambda' = 0$ , or when losses are more probable than duplications, i.e., when  $0 < \lambda' < \lambda$ . In particular, the set  $F_0 \stackrel{\text{def}}{=} \{(q, \varepsilon, \dots, \varepsilon) \mid q \in Q\}$  of configurations with empty channels is a finite attractor in  $\mathcal{G}_S$ . The interested reader can find in [3, sections 5&6] some detailed computations of  $D_{\text{env}}(c)(c')$  in the local-fault model, but s/he must be warned that the qualitative outcomes on PLCs do not depend on these values as long as  $D_{\text{env}}$  is well-behaved.

## 4.3 Regular model-checking of channel systems

Regular model-checking [13, 28] is a symbolic verification technique where one computes infinite but regular sets of configurations using representations from automata theory or from constraint solving.

**Definition 4.1.** A (regular) region of  $S$  is a set  $R \subseteq \text{Conf}_S$  of configurations that can be written under the form  $R = \bigcup_{i \in I} \{q_i\} \times L_i^1 \times \dots \times L_i^d$  with a finite index set  $I$ , and where, for  $i \in I$ ,  $q_i$  is some location  $\in Q$ , and each  $L_i^j$  for  $j = 1, \dots, d$  is a regular language  $\in \mathbf{Reg}(M)$ .

Let  $\mathcal{R} \subseteq 2^{\text{Conf}_S}$  denote the set of all regions of  $S$ . A monotonic operator  $f$  is *regularity-preserving*, if  $f(R_1, \dots, R_n) \in \mathcal{R}$  when  $R_1, \dots, R_n \in \mathcal{R}$ . A regularity-preserving  $f$  is *effective* if a representation for  $f(R_1, \dots, R_n)$  can be computed uniformly from representations for the  $R_i$ 's (and  $S$ ). For example, the set-theoretical  $\cap, \cup$  are regularity-preserving and effective. While not a monotonic operator, complementation is regularity-preserving and effective. Hence the dual  $\tilde{f}$  of any  $f$  is regularity-preserving and effective when  $f$  is.

For the verification of (lossy) channel systems in general, and the resolution of games in particular, some useful operators are the unary pre-images  $\text{Pre}_S[\delta]$  for  $\delta \in \Delta$ , and the upward- and downward-closures  $C_\uparrow$  and  $C_\downarrow$ , defined with

$$\begin{aligned} \text{Pre}_S[\delta](U) &\stackrel{\text{def}}{=} \{c \in \text{Conf}_S \mid \exists c' \in U : c \xrightarrow{\delta} c'\}, & C_\uparrow(U) &\stackrel{\text{def}}{=} \{c \in \text{Conf}_S \mid \exists c' \in U : c' \sqsubseteq c\}, \\ \text{Pre}_S(U) &\stackrel{\text{def}}{=} \bigcup_{\delta \in \Delta} \text{Pre}_S[\delta](U), & C_\downarrow(U) &\stackrel{\text{def}}{=} \{c \in \text{Conf}_S \mid \exists c' \in U : c \sqsubseteq c'\}. \end{aligned}$$

Observe that  $\text{Pre}_S[\delta]$  and  $\text{Pre}_S$  are pre-images for steps of channel systems without/before message perturbations, while  $C_\uparrow$  and  $C_\downarrow$  are pre- and post-images for the message-losing relation.  $C_\uparrow$  and  $C_\downarrow$  are closure operators. Their duals are *interior operators*:  $K_\downarrow(U) \stackrel{\text{def}}{=} \widetilde{C_\uparrow}(U)$  and  $K_\uparrow(U) \stackrel{\text{def}}{=} \widetilde{C_\downarrow}(U)$  are the largest downward-closed and, resp., upward-closed, subsets of  $U$ . Finally, we are also interested in pre-images for  $\preceq_{\text{dup}}$ : we write  $\mathcal{T}_{\text{dup}}^{-1}(U)$  for  $\{c \mid \exists c' \in U : c \preceq_{\text{dup}} c'\}$ . We remark that  $\mathcal{T}_{\text{dup}}^{-1}(\text{Conf}_S) = \text{Conf}_S$ , and that  $\mathcal{T}_{\text{dup}}^{-1}(C_\uparrow U) = C_\uparrow(\mathcal{T}_{\text{dup}}^{-1}(C_\uparrow U)) = C_\uparrow(\mathcal{T}_{\text{dup}}^{-1}(U))$ , i.e., the definition of  $c \rightsquigarrow c'$  is not sensitive to the order of perturbations.

**Fact 4.2.**  *$\text{Pre}_S[\delta], \text{Pre}_S, C_\uparrow, C_\downarrow, \mathcal{T}_{\text{dup}}^{-1}$  and their duals are regularity-preserving and effective (monotonic) operators.*

When using effective regularity-preserving operators, one can evaluate any closed  $L_\mu$  term that does not include fixpoints. For a closed term  $U = \mu X.\varphi(X)$ , or  $V = \nu X.\varphi(X)$ , with a single fixpoint, any approximant  $U_k$  and  $V_k$  for a finite  $k \in \mathbb{N}$  can be evaluated but there is no guarantee that the fixpoint is reached in finite time, or that the fixpoint is a regular region. However, for fixpoints over a WQO like  $\text{Conf}_S$ , there exists a generic finite-time convergence theorem.

**Definition 4.3** (Guarded  $L_\mu$  terms). *1. A variable  $Z$  is upward-guarded in an  $L_\mu$  term  $\varphi$  if every occurrence of  $Z$  in  $\varphi$  is under the scope of an upward-closure  $C_\uparrow$  or upward-interior  $K_\uparrow$  operator.  
2. It is downward-guarded in  $\varphi$  if all its occurrences in  $\varphi$  are under the scope of a downward-closure  $C_\downarrow$  or downward-interior  $K_\downarrow$  operator.  
3. A term  $\varphi$  is guarded if every least fixpoint subterm  $\mu Z.\psi$  of  $\varphi$  has  $Z$  upward-guarded in  $\psi$ , and every greatest fixpoint subterm  $\nu Z.\psi$  has  $Z$  downward-guarded in  $\psi$ .*

**Theorem 4.4** (Effective & regularity-preserving fixpoints). *Any guarded  $L_\mu$  term  $\varphi(X_1, \dots, X_n)$  built with regularity-preserving and effective operators denotes a regularity-preserving and effective  $n$ -ary operator. Furthermore the denotation of a closed term can be evaluated by computing its approximants which are guaranteed to converge after finitely many steps.*

Theorem 4.4 is a special case of the main result of [12] (see also [30]) where it is stated for arbitrary well-quasi-ordered sets (WQO's) and a generic notion of “effective regions”. We recall that, by Higman's lemma,  $(\text{Conf}_S, \sqsubseteq)$  is a *well-quasi-ordered set*, i.e., a quasi-ordered set — $\sqsubseteq$  is reflexive and transitive— such that every infinite sequence  $c_0, c_1, c_2, \dots$  contains an increasing subsequence  $c_i \sqsubseteq c_j$  (with  $i < j$ ).

#### 4.4 Stochastic games on lossy channel systems

In the context of section 4.2 and the stochastic arena  $\mathcal{G}_S$ , we can reformulate the  $Pre$  operator used in Section 3 as a regularity-preserving and effective operator.

When we only consider message losses,  $Pre[\delta](X) = Pre_S[\delta](C_\uparrow X)$  (thanks to the assumption that  $D_{env}$  is *compatible* with the nondeterministic semantics). If also duplications are considered, then  $Pre[\delta](X) = Pre_S[\delta](\mathcal{T}_{dup}^{-1}(C_\uparrow X))$ . In order to deal uniformly with the two cases we shall let  $\mathcal{T}_{dup}$  be the identity relation when duplications are not considered. By duality  $\widetilde{Pre}[\delta](X) = \widetilde{Pre}_S[\delta](K_\downarrow \widetilde{\mathcal{T}}_{dup}^{-1}(X))$  and the derived operators satisfy  $Pre^\exists(X, Y) = Pre^\exists(K_\downarrow X, C_\uparrow Y)$ ,  $Pre^\forall(X, Y) = Pre^\forall(K_\downarrow X, C_\uparrow Y)$  and  $Pre_A^\otimes(X, Y) = Pre_A^\otimes(K_\downarrow X, C_\uparrow Y)$ . Thus Theorem 3.2 rewrites:

$$\langle\langle A \rangle\rangle^{\exists} = \bigwedge_{i=1}^r \square \diamond R_i = \nu X. \bigcap_{i=1}^r H_i(X) = \nu X. Pre_A^\otimes \left( K_\downarrow \bigcap_{i=1}^r H_i(X), Conf_S \right), \quad (6)$$

with  $H_i(X) \stackrel{\text{def}}{=} \mu Z. X \cap Pre_A^\otimes(K_\downarrow X, C_\uparrow(R_i \cup Z))$ . Observe how the closure properties of  $Pre_A^\otimes$  let us easily rewrite  $W'$  into a guarded term. The same technique does not apply to the simpler term  $W$  and this explains why we developed two characterizations of the winning set in Section 3. However, in the case where  $r = 1$ , the characterization with  $W$  can be simplified in  $W_1$  and Theorem 3.2 yields the following *guarded term* for stochastic Büchi games on lossy channel systems:  $\langle\langle A \rangle\rangle^{\exists} \square \diamond R_1 = \nu X. \mu Z. Pre_A^\otimes(K_\downarrow X, C_\uparrow(R_1 \cup Z))$ .

Since  $H_i(X)$  and  $\langle\langle A \rangle\rangle^{\exists} \bigwedge_{i=1}^r \square \diamond R_i$  have guarded  $L_\mu$  expressions, the following decidability result is an immediate application of Theorem 4.4 to Eq. (6).

**Theorem 4.5** (Decidability of Generalized Büchi games with probability 1). *In stochastic games on lossy channel system  $S$  with regular arena partition  $Conf_S = Conf_A \sqcup Conf_B$  and for regular goal regions  $R_1, \dots, R_r$ , the winning set  $\langle\langle A \rangle\rangle^{\exists} \bigwedge_{i=1}^r \square \diamond R_i$  is a regular region that can be computed uniformly from  $S$  and  $R_1, \dots, R_r$ .*

Furthermore, the winning strategies have simple finite representations. One first computes the regular region  $W (= W')$ . Then for each rule  $\delta \in \Delta$ , and each  $i = 1, \dots, r$ , one computes  $V_i^\delta \stackrel{\text{def}}{=} Conf_A \cap Pre_S[\delta](K_\downarrow W \cap C_\uparrow(R_i \cup H_i(W)))$ , these are again regular regions. The strategy  $\sigma_i$  for Alice is then “when in  $V_i^\delta$ , choose  $\delta$ ” and the strategy  $\sigma_W$  is just a combination of the  $\sigma_i$ ’s using finite memory and testing when we are in the  $R_i$ ’s.

**On complexity.** Theorem 4.4 does not only show that  $W = \langle\langle A \rangle\rangle^{\exists} \bigwedge_{i=1}^r \square \diamond R_i$  is computable from  $S$  and  $R_1, \dots, R_r$ . It also shows that  $W$  is obtained by computing the sequence of approximants  $(W_k)_{k \in \mathbb{N}}$  — given by  $W_0 = Conf_S$  and  $W_{k+1} = Pre_A^\otimes(K_\downarrow H_{1,r}(W_k), Conf_S)$  — until the sequence stabilizes, which is guaranteed to eventually occur. Furthermore, computing  $H_{1,r}(W_k)$ , i.e.,  $\bigcap_{i=1}^r H_i(W_k)$ , involves  $r$  fixpoint computations that can use the same technique: sequences of approximants guaranteed to converge in finite time by Theorem 4.4.

There now exist generic upper bounds on the convergence time of such sequences, see [36, 37]. In our case, they entail that the above symbolic algorithm computing the regular region  $\langle\langle A \rangle\rangle^{\exists} (\bigwedge_{i=1}^r \square \diamond R_i)$  is in  $\mathbf{F}_{\omega^\omega}$ , the first level in the Fast-Growing Complexity hierarchy that is not multiply-recursive, hence has “Hyper-Ackermannian” complexity.

This bound is optimal: deciding whether  $c \in \langle\langle A \rangle\rangle^{\exists} (\bigwedge_{i=1}^r \square \diamond R_i)$  is  $\mathbf{F}_{\omega^\omega}$ -hard since this generalizes reachability questions (on lossy channel systems) that are  $\mathbf{F}_{\omega^\omega}$ -hard [21, 37].

**Corollary 4.6.** *Deciding whether  $c \in \langle\langle A \rangle\rangle^{\exists} (\bigwedge_{i=1}^r \square \diamond R_i)$  for given  $S, c, R_1, \dots, R_r$  is  $\mathbf{F}_{\omega^\omega}$ -complete.*

## 5 Concluding remarks

We gave a simple fixpoint characterization of winning sets and winning strategies for 2-player stochastic games where a generalized Büchi objective should be satisfied almost-surely. The characterization is correct for any countable arena with a finite attractor and satisfying the finite-choice assumption for Bob.

Such fixpoint characterizations lead to symbolic model-checking and symbolic strategy-synthesizing algorithms for infinite-state systems and programs. The main issue here is the finite-time convergence of the fixpoint computations. For well-quasi-ordered sets, one can use generic results showing the finite-time convergence of so-called “guarded” fixpoint expressions as we demonstrated by showing the decidability of generalized Büchi games on probabilistic lossy channel systems, a well-quasi-ordered model that comes naturally equipped with a finite attractor.

We believe that Theorem 4.4 has more general applications for games, stochastic or not, on well-quasi-ordered infinite-state systems. We would like to mention quantitative objectives as an interesting direction for future works (see [35, 40]).

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## References

- [1] P. A. Abdulla, C. Baier, S. Purushothaman Iyer & B. Jonsson (2005): *Simulating Perfect Channels with Probabilistic Lossy Channels*. *Information and Computation* 197(1–2), pp. 22–40, doi:10.1016/j.ic.2004.12.001.
- [2] P. A. Abdulla, N. Ben Henda, L. de Alfaro, R. Mayr & S. Sandberg (2008): *Stochastic Games with Lossy Channels*. In: *Proc. 11th Int. Conf. Foundations of Software Science and Computational Structures (FOSACS 2008)*, *Lecture Notes in Computer Science* 4962, Springer, pp. 35–49, doi:10.1007/978-3-540-78499-9\_4.
- [3] P. A. Abdulla, N. Bertrand, A. Rabinovich & Ph. Schnoebelen (2005): *Verification of Probabilistic Systems with Faulty Communication*. *Information and Computation* 202(2), pp. 141–165, doi:10.1016/j.ic.2005.05.008.
- [4] P. A. Abdulla, J. Deneux, J. Ouaknine & J. Worrell (2005): *Decidability and complexity results for timed automata via channel machines*. In: *Proc. 32nd Int. Coll. Automata, Languages, and Programming (ICALP 2005)*, *Lecture Notes in Computer Science* 3580, Springer, pp. 1089–1101, doi:10.1007/11523468\_88.
- [5] A. Arnold & D. Niwiński (2001): *Rudiments of  $\mu$ -Calculus*. *Studies in Logic and the Foundations of Mathematics* 146, Elsevier Science.
- [6] C. Baier, N. Bertrand & Ph. Schnoebelen (2006): *A note on the attractor-property of infinite-state Markov chains*. *Information Processing Letters* 97(2), pp. 58–63, doi:10.1016/j.ipl.2005.09.011.
- [7] C. Baier, N. Bertrand & Ph. Schnoebelen (2006): *On computing fixpoints in well-structured regular model checking, with applications to lossy channel systems*. In: *Proc. 13th Int. Conf. on Logic for Programming and Artificial Intelligence, and Reasoning (LPAR 2006)*, *Lecture Notes in Computer Science* 4246, Springer, pp. 347–361, doi:10.1007/11916277\_24.
- [8] C. Baier, N. Bertrand & Ph. Schnoebelen (2007): *Verifying nondeterministic probabilistic channel systems against  $\omega$ -regular linear-time properties*. *ACM Transactions on Computational Logic* 9(1), doi:10.1145/1297658.1297663.
- [9] C. Baier, T. Brázdil, M. Größer & A. Kucera (2012): *Stochastic game logic*. *Acta Informatica* 49(4), pp. 203–224, doi:10.1007/s00236-012-0156-0.

- [10] C. Baier & B. Engelen (1999): *Establishing Qualitative Properties for Probabilistic Lossy Channel Systems: An Algorithmic Approach*. In: *Proc. 5th AMAST Workshop Formal Methods for Real-Time and Probabilistic Systems (ARTS 1999)*, *Lecture Notes in Computer Science* 1601, Springer, pp. 34–52, doi:10.1007/3-540-48778-6\_3.
- [11] P. Barceló, D. Figueira & L. Libkin (2012): *Graph Logics with Rational Relations and the Generalized Intersection Problem*. In: *Proc. 27th IEEE Symp. Logic in Computer Science (LICS 2012)*, IEEE Comp. Soc. Press, pp. 115–124, doi:10.1109/LICS.2012.23.
- [12] N. Bertrand & Ph. Schnoebelen (2012): *Computable fixpoints in well-structured symbolic model checking*. *Formal Methods in System Design*, doi:10.1007/s10703-012-0168-y. To appear. Long version of [7].
- [13] A. Bouajjani, B. Jonsson, M. Nilsson & T. Touili (2000): *Regular Model Checking*. In: *Proc. 12th Int. Conf. Computer Aided Verification (CAV 2000)*, *Lecture Notes in Computer Science* 1855, Springer, pp. 403–418, doi:10.1007/10722167\_31.
- [14] P. Bouyer, N. Markey, J. Ouaknine, Ph. Schnoebelen & J. Worrell (2008): *On Termination for Faulty Channel Machines*. In: *Proc. 25th Ann. Symp. Theoretical Aspects of Computer Science (STACS 2008)*, *Leibniz Int. Proc. in Informatics* 1, Leibniz-Zentrum für Informatik, pp. 121–132, doi:10.4230/LIPIcs.STACS.2008.1339.
- [15] P. Bouyer, N. Markey & P.-A. Reynier (2008): *Robust Analysis of Timed Automata via Channel Machines*. In: *Proc. 11th Int. Conf. Foundations of Software Science and Computational Structures (FOSSACS 2008)*, *Lecture Notes in Computer Science* 4962, Springer, pp. 157–171, doi:10.1007/978-3-540-78499-9\_12.
- [16] J. Bradfield & C. Stirling (2007): *Modal mu-calculi*. In P. Blackburn, J. Van Benthem & F. Wolter, editors: *Handbook of Modal Logic*, chapter 12, *Studies in Logic and Practical Reasoning* 3, Elsevier Science, pp. 721–756, doi:10.1016/S1570-2464(07)80015-2.
- [17] T. Brázdil, V. Brozek & K. Etessami (2010): *One-Counter Stochastic Games*. In: *Proc. 30th Conf. Found. of Software Technology and Theor. Comp. Sci. (FST&TCS 2010)*, *Leibniz Int. Proc. in Informatics* 8, Leibniz-Zentrum für Informatik, pp. 108–119, doi:10.4230/LIPIcs.FSTTCS.2010.108.
- [18] T. Brázdil, V. Brozek, K. Etessami & A. Kučera (2011): *Approximating the Termination Value of One-Counter MDPs and Stochastic Games*. In: *Proc. 38th Int. Coll. Automata, Languages and Programming (ICALP 2011)*, *Lecture Notes in Computer Science* 6756, Springer, pp. 332–343, doi:10.1007/978-3-642-22012-8\_26.
- [19] T. Brázdil, V. Brozek, A. Kučera & J. Obdržálek (2011): *Qualitative reachability in stochastic BPA games*. *Information & Computation* 209(8), pp. 1160–1183, doi:10.1016/j.ic.2011.02.002.
- [20] T. Brázdil, A. Kučera & P. Novotný (2013): *Determinacy in Stochastic Games with Unbounded Payoff Functions*. In: *Proc. 8th Int. Workshop Mathematical and Engineering Methods in Computer Science (MEMICS 2012)*, *Lecture Notes in Computer Science* 7721, Springer, pp. 94–105, doi:10.1007/978-3-642-36046-6\_10.
- [21] P. Chambart & Ph. Schnoebelen (2008): *The Ordinal Recursive Complexity of Lossy Channel Systems*. In: *Proc. 23rd IEEE Symp. Logic in Computer Science (LICS 2008)*, IEEE Comp. Soc. Press, pp. 205–216, doi:10.1109/LICS.2008.47.
- [22] K. Chatterjee, L. de Alfaro & T. A. Henzinger (2004): *Trading Memory for Randomness*. In: *Proc. 1st Int. Conf. Quantitative Evaluation of Systems (QEST 2004)*, IEEE Comp. Soc. Press, pp. 206–217, doi:10.1109/QEST.2004.10051.
- [23] K. Chatterjee, M. Jurdzinski & T. A. Henzinger (2003): *Simple Stochastic Parity Games*. In: *Proc. 17th Int. Workshop Computer Science Logic (CSL 2003) and 8th Kurt Gödel Coll. (KGL 2003)*, *Lecture Notes in Computer Science* 2803, Springer, pp. 100–113, doi:10.1007/978-3-540-45220-1\_11.
- [24] Taolue Chen & Jian Lu (2007): *Probabilistic Alternating-time Temporal Logic and Model Checking Algorithm*. In: *Proc. 4th Int. Conf. Fuzzy Systems and Knowledge Discovery (FSKD 2007)*, Aug. 2007, Haikou, Hainan, China, IEEE Comp. Soc. Press, pp. 35–39, doi:10.1109/FSKD.2007.458.
- [25] A. Condon (1992): *The Complexity of Stochastic Games*. *Information and Computation* 96(2), pp. 203–224, doi:10.1016/0890-5401(92)90048-K.

- [26] K. Etessami & M. Yannakakis (2008): *Recursive Concurrent Stochastic Games*. *Logical Methods in Computer Science* 4(4), doi:10.2168/LMCS-4(4:7)2008.
- [27] P. Jančar, P. Karandikar & Ph. Schnoebelen (2012): *Unidirectional channel systems can be tested*. In: *Proc. 7th IFIP Int. Conf. on Theoretical Computer Science (IFIP TCS 2012)*, *Lecture Notes in Computer Science* 7604, Springer, pp. 149–163, doi:10.1007/978-3-642-33475-7\_11.
- [28] Y. Kesten, O. Maler, M. Marcus, A. Pnueli & E. Shahar (2001): *Symbolic model checking with rich assertional languages*. *Theoretical Computer Science* 256(1–2), pp. 93–112, doi:10.1016/S0304-3975(00)00103-1.
- [29] B. Konev, F. Wolter & M. Zakharyashev (2005): *Temporal logics over transitive states*. In: *Proc. 20th Conf. Automated Deduction (CADE 2005)*, *Lecture Notes in Computer Science* 3632, Springer, pp. 182–203, doi:10.1007/11532231\_14.
- [30] E. V. Kouzmin, N. V. Shilov & V. A. Sokolov (2004): *Model Checking mu-Calculus in Well-Structured Transition Systems*. In: *Proc. 11th Int. Symp. Temporal Representation and Reasoning (TIME 2004)*, IEEE Comp. Soc. Press, pp. 152–155, doi:10.1109/TIME.2004.1314433.
- [31] D. A. Martin (1998): *The determinacy of Blackwell Games*. *The Journal of Symbolic Logic* 63(4), pp. 1565–1581.
- [32] A. Muscholl (2010): *Analysis of Communicating Automata*. In: *Proc. 4th Int. Conf. Language and Automata Theory and Applications (LATA 2010)*, *Lecture Notes in Computer Science* 6031, Springer, pp. 50–57, doi:10.1007/978-3-642-13089-2\_4.
- [33] S. Purushothaman Iyer & M. Narasimha (1997): *Probabilistic Lossy Channel Systems*. In: *Proc. 7th Int. Joint Conf. Theory and Practice of Software Development (TAPSOFT '97)*, *Lecture Notes in Computer Science* 1214, Springer, pp. 667–681, doi:10.1007/BFb0030633.
- [34] M. L. Puterman (1994): *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons.
- [35] A. Rabinovich (2006): *Quantitative analysis of probabilistic lossy channel systems*. *Information and Computation* 204(5), pp. 713–740, doi:10.1016/j.ic.2005.11.004.
- [36] S. Schmitz & Ph. Schnoebelen (2011): *Multiply-Recursive Upper Bounds with Higman's Lemma*. In: *Proc. 38th Int. Coll. Automata, Languages, and Programming (ICALP 2011)*, *Lecture Notes in Computer Science* 6756, Springer, pp. 441–452, doi:10.1007/978-3-642-22012-8\_35.
- [37] S. Schmitz & Ph. Schnoebelen (2012): *Algorithmic Aspects of WQO Theory*. Lecture notes. Available at <http://cel.archives-ouvertes.fr/cel-00727025>.
- [38] Ph. Schnoebelen (2004): *The Verification of Probabilistic Lossy Channel Systems*. In: *Validation of Stochastic Systems: A Guide to Current Research*, *Lecture Notes in Computer Science* 2925, Springer, pp. 445–465, doi:10.1007/978-3-540-24611-4\_13.
- [39] L. S. Shapley (1953): *Stochastic games*. *Proc. Natl. Acad. Sci. USA* 39(10), pp. 1095–1100, doi:10.1073/pnas.39.10.1095.
- [40] W. Zielonka (2010): *Playing in stochastic environment: from multi-armed bandits to two-player games*. In: *Proc. 30th Conf. Found. of Software Technology and Theor. Comp. Sci. (FST&TCS 2010)*, *Leibniz Int. Proc. in Informatics* 8, Leibniz-Zentrum für Informatik, pp. 65–72, doi:10.4230/LIPIcs.FSTTCS.2010.65.

## A Proof of Equation (†) page 6

Section 3 relies on the following Lemma for simplifying the characterization of winning sets for simple Büchi objectives:

**Lemma A.1** (Contractive  $\mathbf{v}\text{-}\boldsymbol{\mu}$  fixpoint). *For any binary (monotonic) operator  $f$ ,  $\mathbf{v}X.\boldsymbol{\mu}Y.X \cap f(X, Y) = \mathbf{v}X.\boldsymbol{\mu}Y.f(X, Y)$ .*

This is a purely algebraic and lattice-theoretical result that is not specific to stochastic games or channel systems. We include its proof here for the sake of completeness.

We start with a simpler lemma: let  $h$  be a unary (monotonic) operator.

**Lemma A.2.** *Assume  $U = \mu Y.h(Y)$  and  $V \supseteq U$ . Then  $\mu Y.V \cap h(Y) = U$ .*

*Proof.* Write  $W$  for  $\mu Y.V \cap h(Y)$ . Now  $V \cap h(Y) \subseteq h(Y)$  entails  $\mu Y.V \cap h(Y) \subseteq \mu Y.h(Y)$ , i.e.,  $W \subseteq U$ , by monotonicity.

For the other inclusion, we consider the approximants  $(U_\alpha)_{\alpha \in Ord}$  of  $U$  and show, by induction over  $\alpha$ , that  $U_\alpha \subseteq W$  for all  $\alpha$ , which is sufficient since  $U = \bigcup_\alpha U_\alpha$ .

The base case  $\alpha = 0$  is clear since  $U_0 = \emptyset$ . For the inductive case  $\alpha = \beta + 1$ , one has  $U_\alpha \stackrel{\text{def}}{=} h(U_\beta)$ . From  $U_\beta \subseteq W$  (the ind. hyp.) we deduce  $h(U_\beta) \subseteq h(W)$ . From  $U_\beta \subseteq U$  and  $h(U) = U$ , we deduce  $h(U_\beta) \subseteq h(U) = U \subseteq V$ . Thus  $U_\alpha \subseteq V \cap h(W) = W$ . Now for a limit  $U_\lambda$ , we obtain  $U_\lambda \subseteq W$  from  $U_\lambda = \bigcup_{\beta < \lambda} U_\beta$  and the ind. hyp.  $\square$

We may now prove Lemma A.1. Write  $g(X, Y)$  for  $X \cap f(X, Y)$  and let  $U \stackrel{\text{def}}{=} \nu X.\mu Y.f(X, Y)$  and  $V \stackrel{\text{def}}{=} \nu X.\mu Y.g(X, Y)$ . From  $g(X, Y) \subseteq f(X, Y)$  we derive  $V \subseteq U$  by monotonicity.

For the reverse inclusion, let  $(V_\alpha)_{\alpha \in Ord}$  be the approximants of  $V$ . We claim that they satisfy the following inclusions and equalities:

$$\mu Y.f(V_\alpha, Y) \subseteq V_\alpha, \quad \mu Y.f(V_\alpha, Y) = V_{\alpha+1}, \quad U \subseteq V_\alpha, \quad (P_\alpha, P'_\alpha, P''_\alpha)$$

Note that  $(P'_\alpha)$  entails  $(P_\alpha)$  since  $V_{\alpha_1} \subseteq V_{\alpha_2}$  when  $\alpha_1 \geq \alpha_2$ . Reciprocally  $(P_\alpha)$  entails  $(P'_\alpha)$  since assuming  $(P_\alpha)$  and applying Lemma A.2 on  $h(Y) \stackrel{\text{def}}{=} f(V_\alpha, Y)$  gives  $\mu Y.f(V_\alpha, Y) = \mu Y.V_\alpha \cap f(V_\alpha, Y) = \mu Y.g(V_\alpha, Y)$ , which is the definition of  $V_{\alpha+1}$ . Therefore it is sufficient to prove  $(P_\alpha)$  and  $(P''_\alpha)$ , which we do by induction over  $\alpha$ .

For the base case,  $(P_0)$  and  $(P''_0)$  are clear since  $V_0 \stackrel{\text{def}}{=} Conf$ .

For the successor case  $\alpha = \beta + 1$ , we start with  $\mu Y.f(V_\alpha, Y) \subseteq \mu Y.f(V_\beta, Y)$  —by monotonicity, since  $V_\alpha \subseteq V_\beta$ — and combine with the ind. hyp.  $(P'_\beta)$ , i.e.,  $\mu Y.f(V_\beta, Y) = V_\alpha$ , to obtain  $(P_\alpha)$ . For  $(P''_\alpha)$ , we use the ind. hyp.  $U \subseteq V_\beta$  from which we deduce  $\mu Y.f(U, Y) \subseteq \mu Y.f(V_\beta, Y)$ , i.e.,  $U \subseteq V_\alpha$ , since  $U = \mu Y.f(U, Y)$  by definition of  $U$ , and  $V_\alpha = \mu Y.f(V_\beta, Y)$  is the ind. hyp.  $(P'_\beta)$ .

For the limit case  $\alpha = \lambda$ , one obtains  $(P''_\lambda)$  directly from the ind. hyp. and the definition  $V_\lambda = \bigcap_{\beta < \lambda} V_\beta$ . For  $(P_\lambda)$ , we know  $\mu Y.f(V_\lambda, Y) \subseteq \mu Y.f(V_\beta, Y)$  for all  $\beta < \lambda$  since  $V_\lambda \subseteq V_\beta$ . Hence  $\mu Y.f(V_\lambda, Y) \subseteq \bigcap_{\beta < \lambda} \mu Y.f(V_\beta, Y) \subseteq \bigcap_{\beta < \lambda} V_\beta$  (by ind. hyp.) =  $V_\lambda$ .

Finally, since  $(P''_\alpha)$  holds for all  $\alpha$  and since  $V = \bigcap_\alpha V_\alpha$ , we deduce  $U \subseteq V$ .