# Analyse et Conception Formelles

## Lesson 1

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## Propositional logic
## First order logic



## Bibliography


- *Cours de logique, préparation à l'agrégation*, C. Paulin, https://www.lri.fr/~paulin/Agreg/predicat.pdf
- *Cours de logique LOG*, S. Pinchinat, https://people.irisa.fr/Sophie.Pinchinat/LOG.html
- *Logique et fondements de l'informatique* de Richard Lassaigne et Michel de Rougemont. Hermes 1993.

A selected bibliography on the Isabelle/HOL prover and Scala

- http://people.irisa.fr/Thomas.Genet/ACF/Bibliography/


The web page of the course

- http://people.irisa.fr/Thomas.Genet/ACF

Solutions of Isabelle/HOL exercises (uploaded after each lecture)

- http://people.irisa.fr/Thomas.Genet/ACFSol


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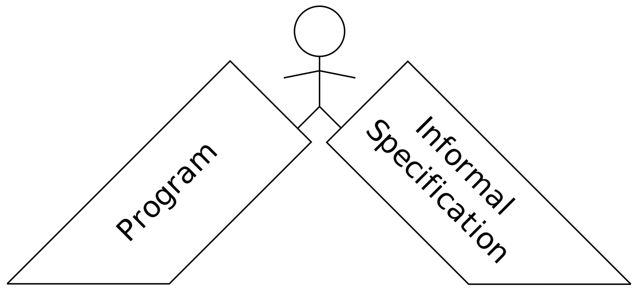
## Outline

- Why using logic for specifying/verifying programs?
- Propositional logic
  - Formula syntax
  - Interpretations and models
  - Isabelle/HOL commands
- First-order logic
  - Formula syntax
  - Interpretations and models
  - Isabelle/HOL commands



## Why using logic for specifying/verifying programs?



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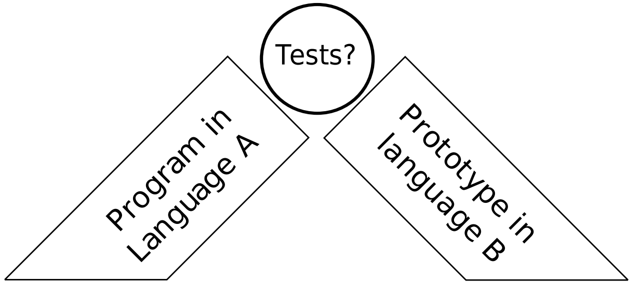




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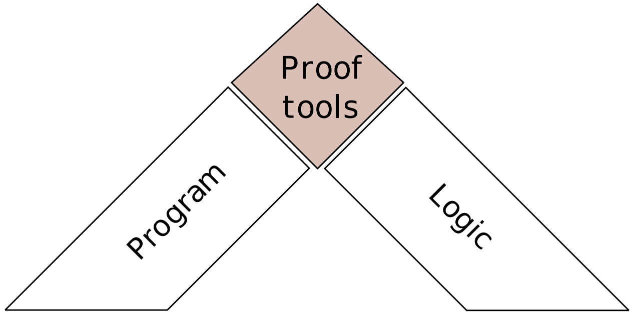
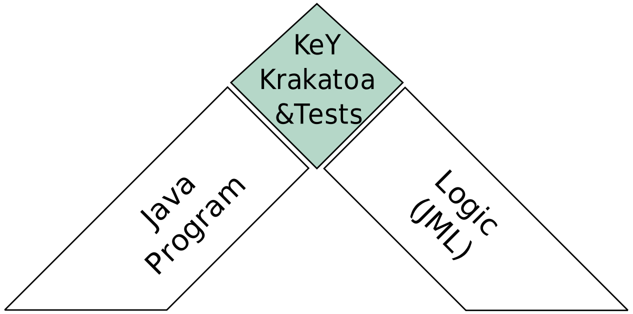





## Why using functional paradigm to program?



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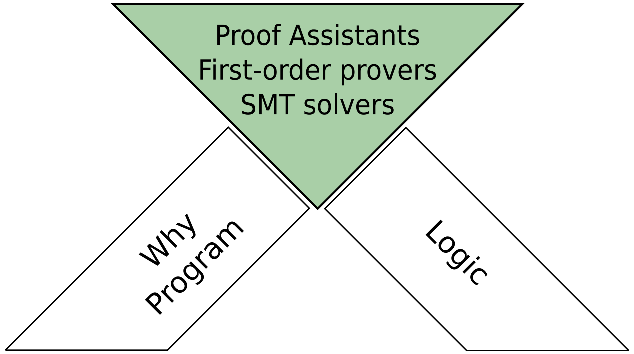

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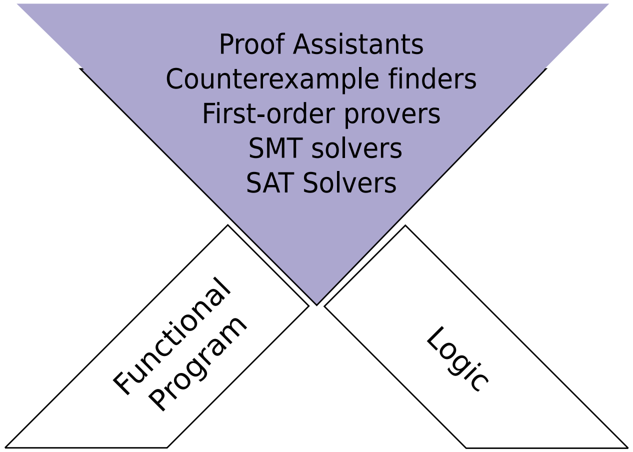




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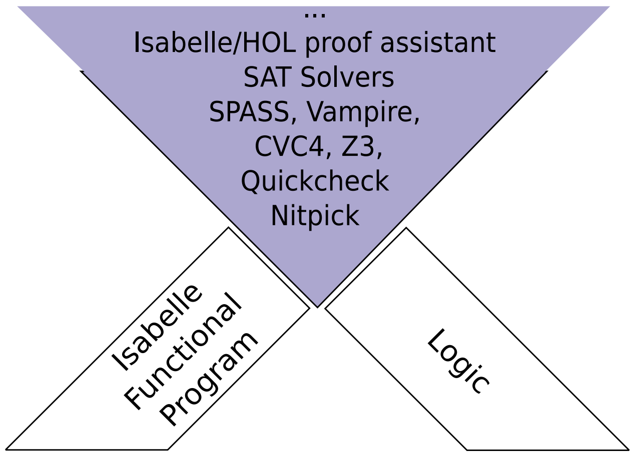




## Propositional logic: syntax and interpretations

**Definition 1 (Propositional formula)**

Let $P$ be a set of propositional variables. The set of propositional formula is defined by
$\phi ::= p \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid \phi_1 \longrightarrow \phi_2$ where $p \in P$

**Definition 2 (Propositional interpretation)**

An *interpretation* $I$ associates to variables of $P$ a value in $\{\texttt{True}, \texttt{False}\}$.

**Example 3**

Let $\phi = (p_1 \wedge p_2) \longrightarrow p_3$. Let $I$ be the interpretation such that $I[\![p_1]\!] = \texttt{True}$, $I[\![p_2]\!] = \texttt{True}$ and $I[\![p_3]\!] = \texttt{False}$.



## Propositional logic: syntax and interpretations (II)

We extend the domain of $I$ to formulas as follows:

$$I[\![\neg\phi]\!] = \begin{cases} \texttt{True iff } I[\![\phi]\!] = \texttt{False} \\ \texttt{False iff } I[\![\phi]\!] = \texttt{True} \end{cases}$$

$$I[\![\phi_1 \vee \phi_2]\!] = \texttt{True iff } I[\![\phi_1]\!] = \texttt{True or } I[\![\phi_2]\!] = \texttt{True}$$

$$I[\![\phi_1 \wedge \phi_2]\!] = \texttt{True iff } I[\![\phi_1]\!] = \texttt{True and } I[\![\phi_2]\!] = \texttt{True}$$

$$I[\![\phi_1 \longrightarrow \phi_2]\!] = \texttt{True iff } \begin{cases} I[\![\phi_1]\!] = \texttt{False or} \\ I[\![\phi_1]\!] = \texttt{True and } I[\![\phi_2]\!] = \texttt{True} \end{cases}$$

**Example 4**

Let $\phi = (p_1 \wedge p_2) \longrightarrow p_3$ and $I$ the interpretation such that $I[\![p_1]\!] = \texttt{True}$, $I[\![p_2]\!] = \texttt{True}$ and $I[\![p_3]\!] = \texttt{False}$.

We have $I[\![p_1 \wedge p_2]\!] = \texttt{True}$ and $I[\![(p_1 \wedge p_2) \longrightarrow p_3]\!] = \texttt{False}$.

## Propositional logic: syntax and interpretations (III)

The presentation using truth tables is generally preferred:

| $a$ | $\neg a$ |
|---|---|
| False | True |
| True | False |

| $a$ | $b$ | $a \vee b$ |
|---|---|---|
| False | False | False |
| True | False | True |
| False | True | True |
| True | True | True |

| $a$ | $b$ | $a \wedge b$ |
|---|---|---|
| False | False | False |
| True | False | False |
| False | True | False |
| True | True | True |

| $a$ | $b$ | $a \longrightarrow b$ |
|---|---|---|
| False | False | True |
| True | False | False |
| False | True | True |
| True | True | True |



## Propositional logic: models

**Definition 5 (Propositional model)**

$I$ is a *model* of $\phi$, denoted by $I \models \phi$, if $I[\![\phi]\!] = \texttt{True}$.

**Definition 6 (Valid formula/Tautology)**

A formula $\phi$ is *valid*, denoted by $\models \phi$, if for all $I$ we have $I \models \phi$.

**Example 7**

Let $\phi = (p_1 \wedge p_2) \longrightarrow p_3$ and $\phi' = (p_1 \wedge p_2) \longrightarrow p_1$. Let $I$ be the interpretation such that $I[\![p_1]\!] = \texttt{True}$, $I[\![p_2]\!] = \texttt{True}$ and $I[\![p_3]\!] = \texttt{False}$. We have $I \not\models \phi$, $I \models \phi'$, and $\models \phi'$.



## Propositional logic: decidability and tools in Isabelle/HOL

**Property 1**

*In propositional logic, given $\phi$, the following problems are decidable:*

- *Is $\models \phi$?*
- *Is there an interpretation $I$ such that $I \models \phi$?*
- *Is there an interpretation $I$ such that $I \not\models \phi$?*

- To automatically prove that $\models \phi$ .................... `apply auto`
  (if the formula is not valid, there remains some unsolved goals)
- To build $I$ such that $I \not\models \phi$ (or $I \models \neg\phi$) .................. `nitpick`
  (i.e. find a counterexample... may take some time on large formula)

Other useful commands

- To close the proof of a proven formula........................ `done`
- To abandon the proof of an unprovable formula .............. `oops`
- To abandon the proof of (potentially) provable formula ....... `sorry`



## Writing and proving propositional formulas in Isabelle/HOL

**Example 8 (Valid formula)**

```
lemma "(p1 /\ p2) --> p1"
apply auto
done
```

**Example 9 (Unprovable formula)**

```
lemma "(p1 /\ p2) --> p3"
nitpick
oops
```

## Isabelle/HOL: ASCII notations

| Symbol | ASCII notation |
|---|---|
| `True` | `True` |
| `False` | `False` |
| $\wedge$ | `/\` |
| $\vee$ | `\/` |
| $\neg$ | `~` |
| $\neq$ | `~=` |
| $\longrightarrow$ | `-->` |
| $\longleftrightarrow$ | `=` |
| $\forall$ | `ALL` |
| $\exists$ | `?` |
| $\lambda$ | `%` |

See the Isabelle/HOL's cheat sheet at the end of the document!



## Propositional logic: exercises in Isabelle/HOL

**Exercise 1**

*Using Isabelle/HOL, for each formula, say if it is valid or give a counterexample interpretation, otherwise.*

1. $A \vee B$
2. $(((A \wedge B) \longrightarrow \neg C) \vee (A \longrightarrow B)) \longrightarrow A \longrightarrow C$
3. *If it rains, Robert takes his umbrella. Robert does not have his umbrella hence it does not rain.*
4. $(A \longrightarrow B) \longleftrightarrow (\neg A \vee B)$



## First-order logic (FOL) / Predicate logic

1. Terms and Formulas
2. Interpretations
3. Models
4. Logic consequence and verification



## First-order logic: terms

**Definition 10 (Terms)**

Let $\mathcal{F}$ be a set of symbols and $ar$ a function such that $ar : \mathcal{F} \Rightarrow \mathbb{N}$ associating each symbol of $\mathcal{F}$ to its arity (the number of parameter). Let $\mathcal{X}$ be a variable set.

The set $\mathcal{T}(\mathcal{F}, \mathcal{X})$, the set of *terms* built on $\mathcal{F}$ and $\mathcal{X}$, is defined by: $\mathcal{T}(\mathcal{F}, \mathcal{X}) = \mathcal{X} \cup \{f(t_1, \ldots, t_n) \mid ar(f) = n \text{ and } t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})\}$.

**Example 11**

Let $\mathcal{F} = \{f : 1, g : 2, a : 0, b : 0\}$ and $\mathcal{X} = \{x, y, z\}$.

$f(x)$, $a$, $z$, $g(g(a, x), f(a))$, $g(x, x)$ are terms and belong to $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

$f$, $a(b)$, $f(a, b)$, $x(a)$, $f(a, f(b))$ do not belong to $\mathcal{T}(\mathcal{F}, \mathcal{X})$.

In term $f(a, f(b))$, terms $a$, $f(b)$, and $b$ are called **subterms** of $(a, f(b))$.

## First-order logic: formula syntax

**Definition 12 (Formulas)**

Let $P$ be a set of predicate symbols all having an arity, i.e. $ar : P \Rightarrow \mathbb{N}$.
The set of formulas defined on $\mathcal{F}, \mathcal{X}$ and $P$ is:

$$\phi ::= \neg\phi \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid \phi_1 \longrightarrow \phi_2 \mid \forall x.\phi \mid \exists x.\phi \mid p(t_1, \ldots, t_n)$$

where $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $x \in \mathcal{X}$, $p \in P$ and $ar(p) = n$.

**Example 13**

Let $P = \{p : 1, q : 2, \leq : 2\}$, $\mathcal{F} = \{f : 1, g : 2, a : 0\}$ and $\mathcal{X} = \{x, y, z\}$.
The following expressions are all formulas:

- $p(f(a))$
- $q(g(f(a), x), y)$
- $\forall x.\exists y.y \leq x$
- $\forall x.\forall y.\forall z.x \leq y \wedge y \leq z \longrightarrow x \leq z$



## First-order logic syntax: the quiz

**Quiz 1**

*Let* $P = \{p : 1, q : 2, \leq : 2\}$, $\mathcal{F} = \{f : 1, g : 2, a : 0\}$ *and* $\mathcal{X} = \{x, y, z\}$.

- $a$ *is a term* | V | True || R | False
- $x$ *is a term* | V | True || R | False
- $f(g(a))$ *is a term* | V | True || R | False
- $\forall x.\ x$ *is a term* | V | True || R | False
- $\forall x.\ x$ *is a formula* | V | True || R | False
- $p(f(g(a, x)))$ *is a formula* | V | True || R | False
- $\forall x.\ p(x) \wedge x \leq y$ *is a formula* | V | True || R | False



## Interlude: a touch of lambda-calculus

**We need to define *anonymous* functions**

- Classical notation for functions

  $f : \mathbb{N} \times \mathbb{N} \Rightarrow \mathbb{N}$ or, for short, $f : \mathbb{N}^2 \Rightarrow \mathbb{N}$
  $f(x, y) = x + y$ $f(x, y) = x + y$

- Lambda-notation of functions

  $f : \mathbb{N}^2 \Rightarrow \mathbb{N}$
  $f = \lambda(x, y).\ x + y$

**$\lambda x\ y.\ x + y$ is an anonymous function adding two naturals**

This corresponds to

- `fun x y -> x+y` in OCaml/Why3
- `(x: Int, y:Int) => x + y` in Scala



## Interlude: a touch of lambda-calculus (in Isabelle HOL)

**Isabelle/HOL also use function update using (:=) as in:**

- $(\lambda x.x)(0 := 1, 1 := 2)$ the identity function except for 0 that is mapped to 1 and 1 that is mapped to 2
- $(\lambda x.\_)(a := b)$ a function taking one parameter and whose result is unspecified except for value $a$ that is mapped to $b$

**Predicates in Isabelle/HOL**

- A predicate is a function mapping values to $\{\texttt{True}, \texttt{False}\}$

  For instance the predicate $p$ on $\{a, b\}$
  $p = (\lambda x.\_)(a := False, b := False)$

## First-order formulas: interpretations and valuations

**Definition 14 (First-order interpretation)**

Let $\phi$ be a formula and $D$ a domain. An *interpretation* $I$ of $\phi$ on the domain $D$ associates:

- a function $f_I : D^n \Rightarrow D$ to each symbol $f \in \mathcal{F}$ such that $ar(f) = n$,
- a function $p_I : D^n \Rightarrow \{\texttt{True}, \texttt{False}\}$ to each predicate symbol $p \in P$ such that $ar(p) = n$.

**Example 15 (Some interpretations of $\phi = \forall x.ev(x) \longrightarrow od(s(x))$)**

- Let $I$ be the interpretation such that domain $D = \mathbb{N}$ and
  $s_I \equiv \lambda x.x + 1 \quad ev_I \equiv \lambda x.((x \bmod 2) = 0) \quad od_I \equiv \lambda x.((x \bmod 2) = 1)$
- Let $I'$ be the interpretation such that domain $D = \{a, b\}$ and
  $s_{I'} \equiv \lambda x.\mathit{if}\ x = a\ \mathit{then}\ b\ \mathit{else}\ a \quad ev_{I'} \equiv \lambda x.(x = a) \quad od_{I'} \equiv \lambda x.False$

**Definition 16 (Valuation)**

Let $D$ be a domain. A *valuation* $V$ is a function $V : \mathcal{X} \Rightarrow D$.



## First-order logic: interpretations and valuations (II)

**Definition 17**

The interpretation $I$ of a formula $\phi$ for a valuation $V$ is defined by:

- $(I, V)[\![x]\!] = V(x)$ if $x \in \mathcal{X}$
- $(I, V)[\![f(t_1, \ldots, t_n)]\!] = f_I((I, V)[\![t_1]\!], \ldots, (I, V)[\![t_n]\!])$ if $f \in \mathcal{F}$ and $ar(f) = n$
- $(I, V)[\![p(t_1, \ldots, t_n)]\!] = p_I((I, V)[\![t_1]\!], \ldots, (I, V)[\![t_n]\!])$ if $p \in P$ and $ar(p) = n$
- $(I, V)[\![\phi_1 \vee \phi_2]\!] = \texttt{True}$ iff $(I, V)[\![\phi_1]\!] = \texttt{True}$ or $(I, V)[\![\phi_2]\!] = \texttt{True}$
- etc...
- $(I, V)[\![\forall x.\phi]\!] = \bigwedge_{d \in D} (I, V + \{x \mapsto d\})[\![\phi]\!]$
- $(I, V)[\![\exists x.\phi]\!] = \bigvee_{d \in D} (I, V + \{x \mapsto d\})[\![\phi]\!]$

where $(V + \{x \mapsto d\})(x) = d$ and $(V + \{x \mapsto d\})(y) = V(y)$ if $x \neq y$.



## First-order logic: satisfiability, models, tautologies

**Definition 18 (Satisfiability)**

$I$ and $V$ *satisfy* $\phi$ (denoted by $(I, V) \models \phi$) if $(I, V)[\![\phi]\!] = \texttt{True}$.

**Definition 19 (First-order Model)**

An interpretation $I$ is a *model* of $\phi$, denoted by $I \models \phi$, if for all valuation $V$ we have $(I, V) \models \phi$.

**Definition 20 (First-order Tautology)**

A formula $\phi$ is a tautology if all its interpretations are models, i.e. $(I, V) \models \phi$ for all interpretations $I$ and all valuations $V$.

**Remark 1**

*Free variables are universally quantified (e.g. $P(x)$ equivalent to $\forall x.\ P(x)$)*



## First-order logic: decidability and tools in Isabelle/HOL

**Property 2**

*In first-order logic, given $\phi$, the following problems are undecidable:*

- *Is $\models \phi$?*
- *Is there an interpretation $I$ (and valuation $V$) such that $(I, V) \models \phi$?*
- *Is there an interpretation $I$ (and valuation $V$) such that $(I, V) \not\models \phi$?*

- Try to automatically prove $\models \phi$ ........................ `apply auto`
  Uses decision procedures (*e.g.* arithmetic) to try to prove the formula.
  If it does not succeed, it does not mean that the formula is unprovable!
- Try to build $I$ and $V$ such that $(I, V) \not\models \phi$ ................ `nitpick`
  If it does not succeed, it does not mean that there is no counterexample!

## First-order logic: exercises in Isabelle/HOL

### Exercise 2

*Using Isabelle/HOL, for each formula, say if it is valid or give a counterexample interpretation and valuation otherwise.*

1. $\forall x.\ p(x) \longrightarrow \exists x. p(x)$
2. $\exists x.\ p(x) \longrightarrow \forall x. p(x)$
3. $\forall x.\ ev(x) \longrightarrow od(s(x))$
4. $\forall x\ y.\ x > y \longrightarrow x+1 > y+1$
5. $x > y \longrightarrow x+1 > y+1$
6. $\forall m\ n.\ (\neg(m < n) \wedge m < n+1) \longrightarrow m = n$
7. $\forall x.\ \exists y.\ x + y = 0$
8. $\forall y.\ (\neg p(f(y))) \longleftrightarrow p(f(y))$
9. $\forall y.\ (p(f(y)) \longrightarrow p(f(y+1)))$



## Isabelle/HOL notations: priority, associativity, shorthands

- Here are the logical operators in decreasing order of priority:
  - $=, \neg, \wedge, \vee, \longrightarrow, \forall, \exists$
  - «a prioritary operator first chooses its operands»
- For instance
  - $\neg\neg P = P$ means $\neg\neg(P = P)$ !
  - $A \wedge B = B \wedge A$ means $A \wedge (B = B) \wedge A$!
  - $P \wedge \forall x. Q(x)$ will be parsed as $(P \wedge \forall)x.Q(x)$ !

    Hence, write $P \wedge (\forall x. Q(x))$ instead!
- All binary operators are associative to the right, for instance $A \longrightarrow B \longrightarrow C$ is equivalent to $A \longrightarrow (B \longrightarrow C)$
- Nested quantifications $\forall x. \forall y. \forall z.\ P$ can be abbreviated into $\forall x\ y\ z.\ P$
- Free variables are universally quantified, i.e. $P(x)$ is equiv. to $\forall x.\ P(x)$

  All Isabelle/HOL tools will prefer $P(x)$ to $\forall x.\ P(x)$



## First-order logic: satisfiability and models

### Definition 21 (Satisfiable formula)

A formula $\phi$ is *satisfiable* if there exists an interpretation $I$ and a valuation $V$ such that $(I, V) \models \phi$.

### Example 22

Let $\phi = p(f(y))$ with $\mathcal{F} = \{f : 1\}$, $P = \{p : 1\}$, $\mathcal{X} = \{y\}$.
The formula $\phi$ is satisfiable (there exists $(I, V)$ such that $(I, V) \models \phi$)

- Let $I$ be the interp. s.t. $D = \{0, 1\}$, $p_I \equiv \lambda x.(x = 0)$, $f_I = \lambda x.x$
- Let $V$ be the valuation such that $V(y) = 0$

We have $(I, V) \models \phi$. With $V'(y) = 1$, $(I, V') \not\models \phi$. Hence, $I$ is not a model of $\phi$.

- Let $I'$ be the interp. s.t. $D = \{0, 1\}$, $p_{I'} \equiv \lambda x.(x = 0)$, $f_{I'} = \lambda x.0$

We have $(I', V) \models \phi$ for all valuation $V$, hence $I'$ is a model of $\phi$.



## Satisfiability – the quiz

### Quiz 2

*Let $P = \{p : 1\}$, $\mathcal{F} = \{f : 1, a : 0, b : 0\}$ and $\mathcal{X} = \{x\}$.*

- $f(a)$ *is satisfiable* | V | True || R | False
- $p(f(a))$ *is satisfiable* | V | True || R | False
- $p(f(x))$ *is satisfiable* | V | True || R | False
- $p(f(x))$ *is a tautology* | V | True || R | False
- $\neg p(f(x))$ *is satisfiable* | V | True || R | False
- $\neg p(f(x)) \wedge p(f(x))$ *is satisfiable* | V | True || R | False
- $p(f(a)) \longrightarrow p(f(b))$ *is satisfiable* | V | True || R | False

# First-order logic: contradictions

**Definition 23 (Contradiction)**

A formula is *contradictory* (or *unsatisfiable*) if it cannot be satisfied, i.e. $(I, V) \not\models \phi$ for all interpretation $I$ and all valuation $V$.

**Property 3**

*A formula $\phi$ is contradictory iff $\neg\phi$ is a tautology.*

**Example 24 (See in Isabelle `cm1.thy` file)**

Let $\phi = (\forall y.\, \neg p(f(y))) \longleftrightarrow (\forall y.\, p(f(y)))$. The formula $\phi$ is contradictory and $\neg\phi$ is a tautology.