Decidable Approximations of Sets of Descendants and Sets of Normal Forms

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Abstract. We present here decidable approximations of sets of descendants and sets of normal forms of Term Rewriting Systems, based on specific tree automata techniques. In the context of rewriting logic, a Term Rewriting System is a program, and a normal form is a result of the program. Thus, approximations of sets of descendants and sets of normal forms provide tools for analysing a few properties of programs: we show how to compute a superset of results, to prove the sufficient completeness property, or to find a criterion for proving termination under a specific strategy, the sequential reduction strategy. The main technical contribution of the paper is the construction of an approximation automaton which recognises a superset of the set of normal forms of terms in a set E, w.r.t. a Term Rewriting System $\mathcal{R}$.

Introduction

In the context of the programming language such as ELAN [18], a Term Rewriting System (TRS for short) is a program. We propose here to use tree automata techniques for proving various properties on TRSs and thus on programs. For a given TRS $\mathcal{R}$ and a set of terms E, these proofs are based on the computation of approximations of the set of $\mathcal{R}$-descendants of E and the set of $\mathcal{R}$-normal forms of E. For that, we build an approximation automaton which recognises a superset of the set of $\mathcal{R}$-descendants and $\mathcal{R}$-normal forms of terms in E. Considering $\mathcal{R}$ as a program and E as the set of possible inputs of the program, the set of $\mathcal{R}$-descendants of E represents all intermediate results of the program at every step of its execution on the given set of possible inputs. The set of $\mathcal{R}$-normal forms of E represents the set of all possible results obtained by executing the program $\mathcal{R}$ on the set of possible given inputs E, when the program stops. Thanks to those two sets, we show how to prove sufficient completeness of a program on a set of possible initial inputs, how to achieve some reachability testing on a program, and how to prove termination of a program represented by a TRS and a strategy of application of rewrite rules called sequential reduction strategy.

In Section 1, we recall basic definitions of terms, term rewriting systems, and tree automata. In Section 2, we briefly present sufficient completeness, reachability testing and termination proof under the sequential reduction strategy. Then, in Section 3, we recall some undecidability results on the set of descendants and the set of normal forms motivating our approach by approximation. We also detail the approximation construction which is based on specific matching and rewriting techniques on tree automata, schematising matching and rewriting on sets of terms. Feasibility of the approximation construction and its appropriateness for our purpose is shown in Section 4 on some examples. Some automatic proofs achieved by our prototype are also presented in Section 4. Finally we conclude on this work in Section 5.

1 Preliminaries

We now introduce some notations and basic definitions. Comprehensive surveys can be found in [9] for term rewriting systems, in [11,4] for tree automata and tree language theory, and in [15] for connections between regular tree languages and term rewriting systems.

Terms, Substitutions, Rewriting systems

Let $\mathcal{F}$ be a finite set of symbols associated with an arity function denoted by $ar : \mathcal{F} \mapsto \mathbb{N}$, $\mathcal{X}$ be a countable set of variables, $\mathcal{T}(\mathcal{F}, \mathcal{X})$ the set of terms, and $\mathcal{T}(\mathcal{F})$ the set of ground terms (terms without variables). Positions in a term are represented as sequences of integers. The set of positions in a term t, denoted by $\mathcal{P}os(t)$, is ordered by lexicographic ordering $\prec$. The empty sequence ϵ denotes the top-most position. $\mathcal{R}oot(t)$ denotes the symbol at position ϵ in t. For any term $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we denote by $\mathcal{P}os_{\mathcal{F}}(s)$ the set of functional positions in s, i.e. $\{p \in \mathcal{P}os(s) \mid p \neq \epsilon$ and $\mathcal{R}oot(s|_p) \in \mathcal{F}\}$. If $p \in \mathcal{P}os(t)$, then $t|_p$ denotes the subterm of t at position p and $t[s]_p$ denotes the term obtained by replacement of the subterm $t|_p$ at position p by the term s. A *ground context* is a term of $\mathcal{T}(\mathcal{F} \cup \{\Box\})$ with only one occurrence of $\Box$, where $\Box$ is a special constant not occurring in $\mathcal{F}$. For any term $t \in \mathcal{T}(\mathcal{F})$, $C[t]$ denotes the term obtained after replacement of $\Box$ by t in the ground context $C[\,]$. The set of variables of a term t is denoted by $\mathcal{V}ar(t)$. A term is linear if any variable of $\mathcal{V}ar(t)$ has exactly one occurrence in t. A substitution is a mapping σ from $\mathcal{X}$ into $\mathcal{T}(\mathcal{F}, \mathcal{X})$, which can uniquely be extended to an endomorphism of $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Its domain $\mathcal{D}om(\sigma)$ is $\{x \in X \mid x\sigma \neq x\}$.

A term rewriting system $\mathcal{R}$ is a set of *rewrite rules* $l \to r$, where $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$, and $\mathcal{V}ar(l) \supseteq \mathcal{V}ar(r)$. A rewrite rule $l \to r$ is *left-linear* (resp. *right-linear*) if the left-hand side (resp. right-hand side) of the rule is linear. A rule is linear if it is both left and right-linear. A TRS $\mathcal{R}$ is linear (resp. left-linear, right-linear) if every rewrite rule $l \to r$ of $\mathcal{R}$ is linear (resp. left-linear, right-linear).

The relation $\to_{\mathcal{R}}$ induced by $\mathcal{R}$ is defined as follows: for any $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $s \to_{\mathcal{R}} t$ if there exist a rule $l \to r$ in $\mathcal{R}$, a position $p \in \mathcal{P}os(s)$ and a substitution σ such that $l\sigma = s|_p$ and $t = s[r\sigma]_p$. The transitive (resp. reflexive transitive) closure of $\to_{\mathcal{R}}$ is denoted by $\to_{\mathcal{R}}^{+}$ (resp. $\to_{\mathcal{R}}^{*}$). A term s is *reducible* by $\mathcal{R}$ if there exists t s.t. $s \to_{\mathcal{R}} t$.

A term s is in $\mathcal{R}$-*normal form* (or is $\mathcal{R}$-*irreducible*) if s is not reducible by $\mathcal{R}$. A term s *has a normal form* if there exists a term t in $\mathcal{R}$-normal form s.t. $s \rightarrow^*_{\mathcal{R}} t$. The set of all ground terms in $\mathcal{R}$-normal form is denoted by $IRR(\mathcal{R})$, and $s \rightarrow^*_{\mathcal{R}} t$ with $t \in IRR(\mathcal{R})$ is denoted by $s \rightarrow^!_{\mathcal{R}} t$. The set of $\mathcal{R}$-descendants of a set of ground terms E is denoted by $\mathcal{R}^*(E)$ and $\mathcal{R}^*(E) = \{t \in \mathcal{T}(\mathcal{F}) \mid \exists s \in E \text{ s.t. } s \rightarrow^*_{\mathcal{R}} t\}$. The set of ground $\mathcal{R}$-normal forms of E is denoted by $\mathcal{R}^!(E)$ and $\mathcal{R}^!(E) = \{t \in \mathcal{T}(\mathcal{F}) \mid \exists s \in E \text{ s.t. } s \rightarrow^!_{\mathcal{R}} t\}$. Moreover, $\mathcal{R}^!(E) = \mathcal{R}^*(E) \cap IRR(\mathcal{R})$.

A rewriting system $\mathcal{R}$ is (1) *terminating* or *strongly normalising* if there exists no infinite derivation $s_1 \rightarrow_{\mathcal{R}} s_2 \rightarrow_{\mathcal{R}} \ldots$ where $s_1, s_2, \ldots \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, (2) *weakly normalising* (WN for short) if every s of $\mathcal{T}(\mathcal{F}, \mathcal{X})$ has a normal form, (3) *weakly normalising on* $E \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X})$ (WN on E) if every $s \in E$ has a normal form.

The set of function symbols $\mathcal{F}$ occurring in a TRS $\mathcal{R}$ can be partitioned into the set of *defined symbols* $\mathcal{D} = \{\mathcal{R}oot(l) \mid l \rightarrow r \in \mathcal{R}\}$ and the set of *constructors* $\mathcal{C} = \mathcal{F} \setminus \mathcal{D}$. A *constructor term*, is a ground term with no defined symbol. The set of constructor terms is denoted by $\mathcal{T}(\mathcal{C})$. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be TRSs with respective sets of symbols $\mathcal{F}_1$ and $\mathcal{F}_2$, respective sets of defined symbols $\mathcal{D}_1$ and $\mathcal{D}_2$, and respective sets of constructors $\mathcal{C}_1$ and $\mathcal{C}_2$. TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$ are *hierarchical* if $\mathcal{F}_2 \cap \mathcal{D}_1 = \emptyset$ and $\mathcal{R}_1 \subset \mathcal{T}(\mathcal{F}_1 \setminus \mathcal{D}_2, \mathcal{X}) \times \mathcal{T}(\mathcal{F}_1, \mathcal{X})$.

Automata, Regular Tree Languages

Let $\mathcal{Q}$ be a finite set of symbols, with arity 0, called *states*. $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ is called the set of *configurations*. A *transition* is a rewrite rule $c \rightarrow q$, where $c \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ and $q \in \mathcal{Q}$. A *normalised transition* is a transition $c \rightarrow q$ where $c = f(q_1, \ldots, q_n)$, $f \in \mathcal{F}$, $ar(f) = n$, and $q_1, \ldots, q_n \in \mathcal{Q}$. A bottom-up finite tree automaton (tree automaton for short) is a quadruple $A = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$, where $\mathcal{Q}_f \subseteq \mathcal{Q}$ and Δ is a set of normalised transitions. The rewriting relation induced by Δ is denoted by $\rightarrow_\Delta$. The tree language recognised by A is $\mathcal{L}(A) = \{t \in \mathcal{T}(\mathcal{F}) \mid \exists q \in \mathcal{Q}_f \text{ s.t. } t \rightarrow^*_\Delta q\}$. For a given $q \in \mathcal{Q}$, the tree language recognised by A and q is $\mathcal{L}(A, q) = \{t \in \mathcal{T}(\mathcal{F}) \mid t \rightarrow^*_\Delta q\}$. A tree language (or a set of terms) E is *regular* if there exists a bottom-up tree automaton A such that $\mathcal{L}(A) = E$. The class of regular tree language is closed under boolean operations $\cup, \cap, \setminus$, and inclusion is decidable. A $\mathcal{Q}$-substitution is a substitution σ s.t. $\forall x \in \mathcal{D}om(\sigma)$, $x\sigma \in \mathcal{Q}$. Let $\Sigma(\mathcal{Q}, \mathcal{X})$ be the set of $\mathcal{Q}$-substitutions. For every transition, there exists an equivalent set of normalised transitions. Normalisation consists in decomposing a transition $s \rightarrow q$, into a set $Norm(s \rightarrow q)$ of flat transitions $f(u_1, \ldots, u_n) \rightarrow q'$ where $u_1, \ldots, u_n$, and q' are states, by abstracting subterms $s' \notin \mathcal{Q}$ of s by new states. We first define the abstraction function as follows:

Definition 1. Let $\mathcal{F}$ be a set of symbols, and $\mathcal{Q}$ a set of states. For a given configuration $s \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q}) \setminus \mathcal{Q}$, an abstraction of s is a surjective mapping α:

$$\alpha : \{s|_p \mid p \in \mathcal{P}os_{\mathcal{F}}(s)\} \mapsto \mathcal{Q}$$

The mapping α is extended on $\mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ by defining α as identity on $\mathcal{Q}$.

Definition 2. Let $\mathcal{F}$ be a set of symbols, $\mathcal{Q}$ a set of states, $s \rightarrow q$ a transition s.t. $s \in \mathcal{T}(\mathcal{F} \cup \mathcal{Q})$ and $q \in \mathcal{Q}$, and α an abstraction of s. The set $Norm_\alpha(s \rightarrow q)$ of normalised transitions is inductively defined by:

1. if $s = q$, then $Norm_\alpha(s \to q) = \emptyset$, and
2. if $s \in \mathcal{Q}$ and $s \neq q$, then $Norm_\alpha(s \to q) = \{s \to q\}$, and
3. if $s = f(t_1, \ldots, t_n)$, then $Norm_\alpha(s \to q) = \{f(\alpha(t_1), \ldots, \alpha(t_n)) \to q\} \cup \bigcup_{i=1}^{n} Norm_\alpha(t_i \to \alpha(t_i))$.

Example 3. Let $\mathcal{F} = \{f, g, a\}$ and $A = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$, where $\mathcal{Q} = \{q_0, q_1, q_2, q_3, q_4\}$, $\mathcal{Q}_f = \{q_0\}$, and $\Delta = \{f(q_1) \to q_0, g(q_1, q_1) \to q_1, a \to q_1\}$.

• The languages recognised by q_1 and q_0 are the following: $\mathcal{L}(A, q_1) = \mathcal{T}(\{g, a\})$, and $\mathcal{L}(A, q_0) = \mathcal{L}(A) = \{f(x) \mid x \in \mathcal{L}(A, q_1)\}$.

• Let $s = f(g(q_1, f(a)))$, and α_1 be an abstraction of s, mapping any subterm $s|_p$ with $p \in \mathcal{P}os_{\mathcal{F}}(s)$, to distinct states in $\{q_2, q_3, q_4\}$. A possible normalisation of transition $f(g(q_1, f(a))) \to q_0$ with abstraction α_1 is the following: $Norm_{\alpha_1}(f(g(q_1, f(a))) \to q_0) = \{f(q_2) \to q_0, g(q_1, q_3) \to q_2, f(q_4) \to q_3, a \to q_4\}$.

2 Applications of $\mathcal{R}^*(E)$ and $\mathcal{R}^!(E)$

In this section we present three applications of the set of descendants and the set of normal forms to program verification.

2.1 Sufficient Completeness

This property has already been much investigated [3,19,23,17], in the context of algebraic specifications. We give here a definition of sufficient completeness of a TRS on a subset of the set of ground terms $E \subseteq \mathcal{T}(\mathcal{F})$.

Definition 4. A TRS $\mathcal{R}$ is sufficiently complete on $E \subseteq \mathcal{T}(\mathcal{F})$ if $\forall s \in E$, $\exists t \in \mathcal{T}(\mathcal{C})$ s.t. $s \to_{\mathcal{R}}^* t$, where $\mathcal{C}$ is the set of constructors in $\mathcal{F}$.

Usual methods for checking this property on algebraic specifications are either based on enumeration and testing techniques [19,23,17] or on disunification [3]. We propose, here, to check this property thanks to the set $\mathcal{R}^!(E)$.

Proposition 5. *If the TRS $\mathcal{R}$ is WN on $E \subseteq \mathcal{T}(\mathcal{F})$, and $\mathcal{R}^!(E) \subseteq \mathcal{T}(\mathcal{C})$, then $\mathcal{R}$ is sufficiently complete on E.*

This comes from the fact that since $\mathcal{R}$ is WN on E, for all terms $s \in E$, $\exists t \in IRR(\mathcal{R})$ s.t. $s \to_{\mathcal{R}}^* t$. Moreover, $t \in \mathcal{R}^!(E)$. Since $\mathcal{R}^!(E) \subseteq \mathcal{T}(\mathcal{C})$, we have $t \in \mathcal{T}(\mathcal{C})$.

Example 6. Let $\mathcal{R} = \{app(nil, x) \to x, app(cons(x, y), z) \to cons(x, app(y, z))\}$, $\mathcal{F} = \mathcal{D} \cup \mathcal{C}$, where $\mathcal{D} = \{app\}$ and $\mathcal{C} = \{cons, nil, a\}$, and $E = \{app(nil, nil), app(cons(a, nil), nil), \ldots, app(nil, cons(a, nil)), \ldots\}$. In this context, $\mathcal{R}^!(E) = \{nil, cons(a, nil), cons(a, cons(a, nil)), \ldots\}$. Since $\mathcal{R}$ is terminating, $\mathcal{R}$ is WN on E and since $\mathcal{R}^!(E) \subseteq \mathcal{T}(\mathcal{C})$, then $\mathcal{R}$ is sufficiently complete on E.

On the other hand, sufficient completeness on E does not necessarily imply that $\mathcal{R}^!(E) \subseteq \mathcal{T}(\mathcal{C})$. For example, let $\mathcal{R} = \{f(a) \to a, f(a) \to f(b)\}$, $\mathcal{C} = \{a, b\}$ and let $E = \{f(a)\}$. Then $\mathcal{R}$ is sufficiently complete on E, since $f(a) \to_{\mathcal{R}} a$, but $\mathcal{R}^!(E) = \{a, f(b)\} \not\subseteq \mathcal{T}(\mathcal{C})$.

2.2 Reachability Testing

Reachability testing consists in verifying if a term, or a term containing a pattern, can be reached by rewriting from an initial set E.

Definition 7. Let $\mathcal{R}$ be a TRS, $E \subseteq \mathcal{T}(\mathcal{F})$ and $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. The *pattern t is $\mathcal{R}$-reachable from E* if there exists a ground context $C[\,]$, a term $s \in E$, and a substitution σ s.t. $s \to_{\mathcal{R}}^{*} C[t\sigma]$.

It is clear that

Proposition 8. *A pattern t is $\mathcal{R}$-reachable from E if an instance of t is a subterm of an element of $\mathcal{R}^*(E)$.*

Let us now show what can be the use of reachability testing on a simple example.

Example 9. Assume that we want to compute $A_n^p = \frac{n!}{(n-p)!}$ with the following TRS:

$$\mathcal{R} = \left\{ \begin{array}{ll} A(n,p) \to fact(n)/fact(n-p) & 0 * x \to x \\ fact(0) \to s(0) & s(x) * y \to (x * y) + y \\ fact(s(x)) \to s(x) * fact(x) & 0/s(y) \to 0 \\ x - 0 \to x & s(x)/s(y) \to s((x-y)/s(y)) \\ 0 - x \to 0 & x + 0 \to x \\ s(x) - s(y) \to x - y & x + s(y) \to s(x+y) \end{array} \right\}$$

on the domain $E = \{A(n,p) \mid n, p \in Nat\}$, where $Nat = \{0, s(0), \ldots\}$. Verifying if a division by 0 can occur is equivalent to check whether the pattern $div(x,0)$ is $\mathcal{R}$-reachable from E, i.e. whether $\exists C[\,]$, $\exists \sigma$, s.t. $C[div(x,0)\sigma] \in \mathcal{R}^*(E)$.

2.3 Termination under Sequential Reduction Strategy

Many works are devoted to automatising termination proofs of TRSs [1,14]. On the other hand, it is interesting to study weaker forms of termination, since for many purposes weak normalisation is enough. In theorem provers and programming languages, rules are always applied under a specific strategy, and it is enough to ensure termination under this strategy. In addition, proving termination or WN on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ or on $\mathcal{T}(\mathcal{F})$ is not always needed. In practice, a TRS is often designed to rewrite terms from a subset $E \subseteq \mathcal{T}(\mathcal{F})$, for example logical formulas in disjunctive normal form, flattened lists, or well-typed terms. Moreover, some TRSs are WN on $E \subset \mathcal{T}(\mathcal{F})$, but not on $\mathcal{T}(\mathcal{F})$ [13].

The strategy studied here is called the *Sequential Reduction Strategy* (SRS for short) and consists in separating a TRS $\mathcal{R}$ into several TRSs $\mathcal{R}_1, \ldots, \mathcal{R}_n$ s.t. $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_n$ and in normalising terms successively w.r.t. $\mathcal{R}_1, \ldots, \mathcal{R}_n$. This rewriting relation under SRS is denoted by $\to_{\mathcal{R}_1;\ldots;\mathcal{R}_n}$, and is based on *modular reduction relation* [20].

Definition 10. Let $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_n$ be TRSs. For $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $s \to_{\mathcal{R}_1;\ldots;\mathcal{R}_n} t$ if s is reducible by $\mathcal{R}$ and $\exists s_1, \ldots, s_{n-1} \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ s.t. $s \to_{\mathcal{R}_1}^{!} s_1$ and $s_1 \to_{\mathcal{R}_2}^{!} s_2$ and $\ldots$ and $s_{n-1} \to_{\mathcal{R}_n}^{!} t$.

This kind of strategy is of great interest when normalising terms w.r.t. a TRS splitted into several hierarchical TRSs (or modules) $\mathcal{R}_1, \ldots, \mathcal{R}_n$. In this situation, interleaving of rewriting steps w.r.t. to $\mathcal{R}_1, \ldots, \mathcal{R}_n$ is often not needed. If modules $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are WN and share only constructors, then $\rightarrow_{\mathcal{R}_1;\ldots;\mathcal{R}_n}$ is terminating [13], as a corollary of results of [21,24]. Now let us give an intuition on how to prove termination of $\rightarrow_{\mathcal{R}_1;\ldots;\mathcal{R}_n}$ for WN TRSs $\mathcal{R}_1, \ldots, \mathcal{R}_n$ sharing function symbols. Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$, for example. For proving termination of $\rightarrow_{\mathcal{R}_1;\mathcal{R}_2}$ on a set of initial terms $E \subseteq \mathcal{T}(\mathcal{F})$, we need to prove that for any term $s \in E$, there is no possible infinite derivation $s \rightarrow^!_{\mathcal{R}_1} s'_1 \rightarrow^!_{\mathcal{R}_2} s_1 \rightarrow^!_{\mathcal{R}_1} s'_2 \rightarrow^!_{\mathcal{R}_2} s_2 \rightarrow^!_{\mathcal{R}_1} \ldots$ In that case, a criterion for proving termination of $\rightarrow_{\mathcal{R}_1;\mathcal{R}_2}$ on E is the following: construct the sets $G_1 = \mathcal{R}^!_2(\mathcal{R}^!_1(E))$, $G_2 = \mathcal{R}^!_2(\mathcal{R}^!_1(G_1))$, $\ldots$ until we get a fixpoint G_m s.t. $G_m = \mathcal{R}^!_2(\mathcal{R}^!_1(G_m))$. Then if $\mathcal{R}_1$ WN on $E, G_1, G_2, \ldots,$ and $\mathcal{R}_2$ WN on $\mathcal{R}^!_1(E), \mathcal{R}^!_1(G_1), \ldots$ and if $G_m \subseteq IRR(\mathcal{R}_1 \cup \mathcal{R}_2)$, then $\mathcal{R}$ is WN on E and $\mathcal{R}$ is terminating on E under SRS.

Proposition 11. *If $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are WN resp. on subsets $E_1, \ldots, E_n$ of $\mathcal{T}(\mathcal{F})$, the rewriting relation under SRS $\rightarrow_{\mathcal{R}_1;\ldots;\mathcal{R}_n}$ is terminating on E if the iterated sequence of sets $G_{k+1} = \mathcal{R}^!_n(\ldots \mathcal{R}^!_1(G_k)\ldots)$, starting from $G_0 = E$, has a fixpoint which is a subset of $IRR(\mathcal{R}_1 \cup \ldots \cup \mathcal{R}_n)$, and for all $k \geq 0$, $G_k \subseteq E_1, \mathcal{R}^!_1(G_k) \subseteq E_2, \ldots,$ and $\mathcal{R}^!_{n-1}(\ldots \mathcal{R}^!_1(G_k)\ldots) \subseteq E_n$.*

3 Approximating $\mathcal{R}^*(E)$ and $\mathcal{R}^!(E)$

First, recall that $\mathcal{R}^!(E) = \mathcal{R}^*(E) \cap IRR(\mathcal{R})$. $IRR(\mathcal{R})$ is a regular tree language if $\mathcal{R}$ is left-linear [10], and a procedure for building a regular tree grammar (resp. a tree automaton) producing (resp. recognising) $IRR(\mathcal{R})$ can be found in [5]. However, $\mathcal{R}^*(E)$ is not necessarily a regular tree language, even if E is. The language $\mathcal{R}^*(E)$ is regular if E is regular and if $\mathcal{R}$ is either a ground TRS [7], a right-linear and monadic TRS [25], a linear and semi-monadic TRS [6] or an "inversely-growing" TRS [16], where "inversely-growing" means that every right-hand side is either a variable, or a term $f(t_1, \ldots, t_n)$ where $f \in \mathcal{F}$, $ar(f) = n$, and $\forall i = 1, \ldots, n$, t_i is a variable, a ground term, or a term whose variables do not occur in the left-hand side. On the other hand, for a given regular language E, $\mathcal{R}^*(E)$ is not necessarily regular, even if $\mathcal{R}$ is a confluent and terminating linear TRS [15]. If $\mathcal{R}$ is not "inversely-growing", then $\mathcal{R}^*(E)$ is not necessarily regular [16].

Since our purpose is to deal with TRSs representing programs, we cannot stick to the decidable class of "inversely-growing" TRSs which is not expressive enough. Our goal here is to define, an *approximation of* $\mathcal{R}^*(E)$ i.e. a regular superset of $\mathcal{R}^*(E)$ for left-linear TRSs and regular sets E. Then, since regular langages are closed by intersection, the intersection between the regular superset of $\mathcal{R}^*(E)$ and $IRR(\mathcal{R})$ gives a regular superset of $\mathcal{R}^!(E)$. Before going into details of the construction of the approximation itself, let us first show why an approximation is sufficient for proving properties addressed in Section 2. Let $\mathcal{R}$ be a TRS, and $E \subseteq \mathcal{T}(\mathcal{F})$. For any set G, let $super(G)$ be a regular superset of G. For sufficient completeness: if $super(\mathcal{R}^!(E)) \subseteq \mathcal{T}(\mathcal{C})$ then $\mathcal{R}^!(E) \subseteq \mathcal{T}(\mathcal{C})$.

For reachability testing: if $C[t\sigma] \notin super(\mathcal{R}^*(E))$ then $C[t\sigma] \notin \mathcal{R}^*(E)$. For termination under SRS: if $super(\mathcal{R}_n^!(\ldots super(\mathcal{R}_1^!(G_k))\ldots\ldots))$ is a subset of $IRR(\mathcal{R}_1 \cup \ldots \cup \mathcal{R}_n)$ then so is $\mathcal{R}_n^!(\ldots \mathcal{R}_1^!(G_k)\ldots)$.

Now, starting from a tree automaton A s.t. $\mathcal{L}(A) = E$ and a left-linear TRS $\mathcal{R}$, we show how to build a tree automaton $\mathcal{T}_{\mathcal{R}}\!\uparrow(A)$ s.t. $\mathcal{L}(\mathcal{T}_{\mathcal{R}}\!\uparrow(A)) \supseteq \mathcal{R}^*(\mathcal{L}(A))$. The next proposition gives a sufficient condition for an automaton B to have such a property.

Proposition 12. *Let $\mathcal{R}$ be a left-linear TRS, $A = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$, and $B = \langle \mathcal{F}, \mathcal{Q}', \mathcal{Q}_f, \Delta' \rangle$ two tree automata. $\mathcal{R}^*(\mathcal{L}(A)) \subseteq \mathcal{L}(B)$ if*

1. *$\Delta \subseteq \Delta'$, and*
2. *$\forall l \to r \in \mathcal{R}, \forall q \in \mathcal{Q}', \forall \sigma \in \Sigma(\mathcal{Q}', \mathcal{X})$, $l\sigma \to_{\Delta'}^* q$ implies $r\sigma \to_{\Delta'}^* q$.*

Proof. (sketch) By definition, any term t of $\mathcal{R}^*(\mathcal{L}(A))$ is such that $\exists s \in \mathcal{L}(A)$ s.t. $s \to_{\mathcal{R}}^* t$. By induction on the size of the derivation $s \to_{\mathcal{R}}^* t$, we prove that if $s \to_{\mathcal{R}}^* t$ and $s \to_{\Delta'}^* q$ with $q \in \mathcal{Q}_f$ then $t \to_{\Delta'}^* q$, which implies that $t \in \mathcal{L}(B)$. See [12] for a detailed proof.

For building $\mathcal{T}_{\mathcal{R}}\!\uparrow(A)$, the algorithm we propose starts from the tree automaton A and incrementally adds to Δ the transitions necessary to ensure Condition 2 of Proposition 12, by computing critical peaks between rules of $\mathcal{R}$ and rules of Δ: $r\sigma\ {}_{\mathcal{R}}\!\leftarrow l\sigma \to_{\Delta}^* q$. If $r\sigma \not\to_{\Delta}^* q$, then it is necessary to add the transition $r\sigma \to q$ to Δ. If the transition $r\sigma \to q$ is not normalised, then it has to be normalised according to Definition 2. The choice of new states used to normalise $r\sigma \to q$ is guided by the approximation function γ defined below:

Definition 13. Let $\mathcal{Q}$ be a set of states, $\mathcal{Q}_{new}$ be a set of new states s.t. $\mathcal{Q} \cap \mathcal{Q}_{new} = \emptyset$, and $\mathcal{Q}_{new}^*$ the set of sequences $q_1 \cdots q_k$ of states in $\mathcal{Q}_{new}$. An approximation function is a mapping $\gamma : \mathcal{R} \times (\mathcal{Q} \cup \mathcal{Q}_{new}) \times \Sigma(\mathcal{Q} \cup \mathcal{Q}_{new}, \mathcal{X}) \mapsto \mathcal{Q}_{new}^*$, such that $\gamma(l \to r, q, \sigma) = q_1 \cdots q_k$, where $k = \mathcal{C}ard(\mathcal{P}os_{\mathcal{F}}(r))$.

In the following, for any sequence $S = q_1 \cdots q_k \in \mathcal{Q}_{new}^*$, and for all i s.t. $1 \leq i \leq k$, $\pi_i(S)$ denotes the i-th element of the sequence S, i.e. q_i.

Definition 14. (Approximation Automaton) Let $A = \langle \mathcal{F}, \mathcal{Q}, \mathcal{Q}_f, \Delta \rangle$ be a tree automaton, $\mathcal{R}$ a left-linear TRS, $\mathcal{Q}_{new}$ a set of new states s.t. $\mathcal{Q} \cap \mathcal{Q}_{new} = \emptyset$, and γ an approximation function. An approximation automaton $\mathcal{T}_{\mathcal{R}}\!\uparrow(A)$ is a tree automaton $\langle \mathcal{F}, \mathcal{Q}', \mathcal{Q}_f, \Delta' \rangle$ s.t.
(1) $\mathcal{Q}' = \mathcal{Q} \cup \mathcal{Q}_{new}$, and
(2) $\Delta \subseteq \Delta'$, and
(3)$\forall l \to r \in \mathcal{R}$, $\forall q \in \mathcal{Q}'$, $\forall \sigma \in \Sigma(\mathcal{Q}', \mathcal{X})$, $l\sigma \to_{\Delta'}^* q$ implies

$$Norm_{\alpha}(r\sigma \to q) \subseteq \Delta'$$

where α is the abstraction of $r\sigma$ defined by: $\alpha(r\sigma|_{p_i}) = \pi_i(\gamma(l \to r, q, \sigma))$, for all $p_i \in \mathcal{P}os_{\mathcal{F}}(r) = \{p_1, \ldots, p_k\}$, s.t. $p_i \prec p_{i+1}$ for $i = 1 \ldots k-1$ (where $\prec$ is the lexicographic ordering).

By choosing specific approximation functions γ, we obtain specific approximations.

Theorem 15. *Given a tree automaton A and a left-linear TRS $\mathcal{R}$, every approximation automaton satisfies: for any approximation function γ,*

$$\mathcal{L}(\mathcal{T}_{\mathcal{R}}\!\uparrow(A)) \supseteq \mathcal{R}^*(\mathcal{L}(A))$$

Proof. (sketch) For proving $\mathcal{L}(\mathcal{T}_{\mathcal{R}}\!\uparrow(A)) \supseteq \mathcal{R}^*\mathcal{L}(A)$, it is enough to prove that the approximation automata verifies Conditions 1 and 2 of Proposition 12, for all approximation functions γ. By Definition 14, $\mathcal{T}_{\mathcal{R}}\!\uparrow(A)$ trivially verifies Condition 1. Then, to prove that $\mathcal{T}_{\mathcal{R}}\!\uparrow(A)$ also verifies Condition 2 of Proposition 12, it is enough to prove that $Norm_\alpha(r\sigma \to q) \subseteq \Delta'$ implies $r\sigma \to^*_{\Delta'} q$. See [12] for a detailed proof.

For any rule $l \to r \in \mathcal{R}$, in order to find a $\mathcal{Q}$-substitution σ and a state $q \in \mathcal{Q}$ s.t. $l\sigma \to^*_\Delta q$, it is possible to enumerate every possible combination of σ and q and check whether $l\sigma \to^*_\Delta q$. However, this solution is not usable in practice, especially when $\mathcal{Q}$ is a large set, due to the huge number of possible σ and q to consider. In [13,12], we detail a *matching algorithm* which starts from a *matching problem* $l \trianglelefteq q$ and a set of transitions Δ, and gives every solution $\sigma : \mathcal{X} \mapsto \mathcal{Q}$ s.t. $l\sigma \to^*_\Delta q$. This algorithm is used in our implementation.

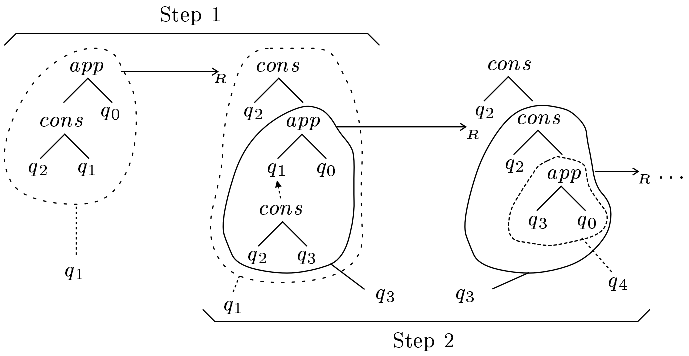
However, adding transitions to Δ may not terminate, depending on the approximation function γ used, as in the following example.

Example 16. Let A be an automaton s.t. $\Delta = \{app(q_0, q_0) \to q_1, cons(q_2, q_1) \to q_0, nil \to q_0, nil \to q_1, a \to q_2\}$, $rl = app(cons(x, y), z) \to cons(x, app(y, z))$, $\mathcal{R} = \{rl\}$, and let γ be the approximation function mapping every tuple (rl, q, σ) to one new state (since $Card(\mathcal{P}os_{\mathcal{F}}(cons(x, app(y, z)))) = 1$).

Step 1 If we apply the matching algorithm on $app(cons(x, y), z) \trianglelefteq q_1$, we obtain a solution $\sigma = \{x \mapsto q_2, y \mapsto q_1, z \to q_0\}$, corresponding to the following critical peak: $cons(q_2, app(q_1, q_0))\ {}_{\mathcal{R}}\!\leftarrow app(cons(q_2, q_1), q_0) \to^*_\Delta q_1$. Thus, the transition to be added is $cons(q_2, app(q_1, q_0)) \to q_1$. Let q_3 be the new state s.t. $\gamma(rl, q_1, \sigma) = q_3$. Then, since $\mathcal{P}os_{\mathcal{F}}(cons(q_2, app(q_1,\ q_0)) = \{p_1\} = \{2\}$, we have $\alpha(app(q_1, q_0)) = \pi_1(\gamma(rl, q_1, \sigma)) = q_3$, and the set of normalised transitions to be added to Δ is:
$Norm_\alpha(cons(q_2, app(q_1, q_0)) \to q_1) = \{cons(q_2, q_3) \to q_1, app(q_1, q_0) \to q_3\}$.

Step 2 Applying the matching algorithm on $app(cons(x, y), z) \trianglelefteq q_3$ gives a solution $\sigma' = \{x \mapsto q_2, y \mapsto q_3, z \mapsto q_0\}$, corresponding to the following critical peak: $cons(q_2, app(q_3, q_0))\ {}_{\mathcal{R}}\!\leftarrow app(cons(q_2, q_3), q_0) \to^*_\Delta q_3$. Thus, the transition to be added is $cons(q_2, app(q_3, q_0)) \to q_3$. Let q_4 be the new state s.t. $\gamma(rl, q_3, \sigma') = q_4$. Then, $\alpha(app(q_3, q_0)) = q_4$, and the set of normalised transitions to be added to Δ is:
$Norm_\alpha(cons(q_2, app(q_3, q_0)) \to q_3) = \{cons(q_2, q_4) \to q_1, app(q_3, q_0) \to q_4\}$.

This process can go on forever and add infinitely many new states. This is due to the fact that we can apply recursively the rule $app(cons(x, y), z) \to cons(x, app(y, z))$ onto infinitely growing terms recognised by the automaton A (with transitions Δ), as shown on the following figure.



In order to have a finite automaton approximating the set $\mathcal{R}^*(\mathcal{L}(A))$, the intuition is to *fold recursive calls* into a unique new state. In the previous example, during Step 1, by applying the rule of $app(cons(x, y), z) \to cons(x, app(y, z))$ on $app(cons(q_2, q_1), q_0)$, we have obtained the configuration $cons(q_2, app(q_1, q_0))$, and we have created a new state q_3 recognising the subterm $app(q_1, q_0)$. During Step 2 we have applied *the same rule* on the subterm $app(q_1, q_0)$ recognised by q_3. In order to fold this recursive call in Step 2, we simply *re-use* the state q_3, instead of creating a new state q_4 for normalising the transition $cons(q_2, app(q_3, q_0)) \to q_3$ obtained in Step 2. Thus we obtain the set of normalised transitions $\{cons(q_2, q_3) \to q_3, app(q_3, q_0) \to q_3\}$ to be added to Δ. No more state nor transition needs to be further added and this automaton recognises a superset of $\mathcal{R}^*(\mathcal{L}(A))$. This is one of the basic idea of the ancestor approximation, which is formalised below.

Informally, every state $q \in \mathcal{Q}' = \mathcal{Q} \cup \mathcal{Q}_{new}$ has a unique ancestor $q_a \in \mathcal{Q}$. The ancestor of any state $q \in \mathcal{Q}$ is q itself, and the ancestor of every new state $q' \in \mathcal{Q}_{new}$ occuring in the sequence $\gamma(l \to r, q, \sigma)$ (used to normalise a new transition $r\sigma \to q$), is the ancestor of q. In the ancestor approximation, (1) the γ function does not depend on the σ parameter and, (2) for any new state $q' \in \mathcal{Q}_{new}$, $\gamma(l \to r, q', \sigma) = \gamma(l \to r, q, \sigma)$, where $q \in \mathcal{Q}$ is the ancestor of q'.

Definition 17. An approximation function γ is called ancestor approximation if

1. $\forall l \to r \in \mathcal{R}$, $\forall q \in \mathcal{Q}'$, $\forall \sigma_1, \sigma_2 \in \Sigma(\mathcal{Q}', \mathcal{X})$,
$$\gamma(l \to r, q, \sigma_1) = \gamma(l \to r, q, \sigma_2), \text{ and}$$
2. $\forall l_1 \to r_1, l_2 \to r_2 \in \mathcal{R}, \forall q \in \mathcal{Q}', \forall q_1, \dots, q_k \in \mathcal{Q}_{new}, \sigma_1, \sigma_2 \in \Sigma(\mathcal{Q}', \mathcal{X})$,
$$\gamma(l_1 \to r_1, q, \sigma_1) = q_1 \dots q_k \Rightarrow \forall i = 1 \dots k, \gamma(l_2 \to r_2, q_i, \sigma_2) = \gamma(l_2 \to r_2, q, \sigma_2).$$

Note that in the particular case of Example 16, using the ancestor approximation, we have $\gamma(rl, q_1, \sigma) = q_3$, and by case 2 of Definition 17 we get $\gamma(rl, q_3, \sigma') = \gamma(rl, q_1, \sigma')$, by case 1 we get that $\gamma(rl, q_1, \sigma') = \gamma(rl, q_1, \sigma) = q_3$, thus we have $\gamma(rl, q_3, \sigma') = q_3$, and the construction of $\mathcal{T}_{\mathcal{R}}\!\uparrow\!(A)$ becomes finite.

Theorem 18. *Approximation automata built using ancestor approximation are finite automata.*

Proof. (sketch) The automaton $\mathcal{T}_\mathcal{R}\!\uparrow\!(A)$ is finite if the set of new states $\mathcal{Q}_{new}$ is finite. Since $\mathcal{Q}$ is finite, $\mathcal{R}$ is finite, and γ does not depend on the σ parameter, there is a finite number of distinct sequences $\gamma(l \to r, q, \sigma)$ for $l \to r \in \mathcal{R}$, $q \in \mathcal{Q}$, and these sequences are finite. On the other hand, every state $q' \in \mathcal{Q}_{new}$ has a unique ancestor $q \in \mathcal{Q}$, and $\gamma(l \to r, q', \sigma) = \gamma(l \to r, q, \sigma)$. Thus, there is a finite number of distinct sequences $\gamma(l \to r, q', \sigma) = q'_1 \ldots q'_n$ with $q', q'_1, \ldots, q'_n \in \mathcal{Q}_{new}$. Hence, there is a finite number of states in $\mathcal{Q}_{new}$, used to normalise transitions. See [12] for a detailed proof.

4 Experiments

Working on tree automaton by hand is always a heavy task. In order to experiment and check feasibility of the method, we have implemented in ELAN [18] a library of usual algorithms on tree automaton: union, intersection, cleaning, inclusion test, as well as algorithms for building the tree automata $\mathcal{T}_\mathcal{R}\!\uparrow\!(A)$, and $A_{IRR(\mathcal{R})}$ (the automaton recognising the set $IRR(\mathcal{R})$) for a given automaton A and a given left-linear TRS $\mathcal{R}$. In all the following examples, we use the same ancestor approximation method. We have experimented with several other approximations: if the γ function does not depend on the rule $l \to r$, on the state q or on the position p, then the approximation automaton is smaller, and faster to compute. However, the recognised language is bigger and sometimes not precise enough for our purpose. On the other hand, if for every σ, the γ function have distinct values, then the construction of the automaton is not necessarily terminating.

4.1 Reachability Testing

Let $\mathcal{R}_1$ be a TRS computing the function $A^p_n = \frac{n!}{(n-p)!}$, and $Aut(0)$ a tree automaton recognising the set $\mathcal{L}(Aut(0)) = \mathcal{L}(Aut(0), q_0) = \{A(n,p) \mid n, p \in \mathcal{L}(Aut(0), q_1)\}$ where $\mathcal{L}(Aut(0), q_1) = Nat = \{0, s(0), \ldots\}$. The TRS $\mathcal{R}_1$ and the automaton $Aut(0)$ are given as input to our prototype in the following syntax:

```
specification Anp
Vars    x y n p
Ops
    A:2 minus:2 div:2 o:0 s:1 fact:1 plus:2 mult:2
R1
    A(n, p) -> div(fact(n), fact(minus(n, p)))
    fact(s(x)) -> mult(s(x), fact(x))
    fact(o) -> s(o)
    mult(o, x) -> o
    mult(s(x), y) -> plus(mult(x, y), y)
    div(o, s(y)) -> o
    div(s(x), s(y)) -> s(div(minus(x, y), s(y)))
    plus(x, o) -> x
    plus(x, s(y)) -> s(plus(x, y))
    minus(x, o) -> x
    minus(o, x) -> o
    minus(s(x), s(y)) -> minus(x, y)
    nil
```

```
Automata
    Description of Aut(0)
    states q|0.q|1.nil
    final states q|0.nil
    transitions A(q|1, q|1) -> q|0
            o-> q|1
            s(q|1) -> q|1
            nil
    End of Description
    nil
```

Computing the automaton $\mathcal{T}_{\mathcal{R}_1}\!\uparrow\!(Aut(0))$ s.t. $\mathcal{L}(\mathcal{T}_{\mathcal{R}_1}\!\uparrow\!(Aut(0))) \supseteq \mathcal{R}_1^*(\mathcal{L}(Aut(0)))$, can be achieved by evaluating the following query: `T_up(R1) on (Aut(0))`, and the result is the automaton $Aut(1)$:

```
[] result term:
   Description of Aut(1) states
   q|12.q|13.q|11.q|10.q|9.q|8.q|6.q|7.q|2.q|5.q|3.q|0.q|1.nil final states q|0.nil
   transitions s(q|11)->q|11.o->q|11.minus(q|10,q|10)->q|12.minus(q|8,q|10)->q|12. s(q|10)->
   q|13.s(q|10)->q|3.minus(q|10,q|8)->q|12.minus(q|8,q|8)->q|12.s(q|8)->q|13.div(q|12,q|13)
   ->q|11.s(q|11)->q|0.plus(q|9,q|7)->q|3.s(q|8)->q|3.mult(q|6,q|7)->q|3.plus(q|9,q|10)->
   q|10.plus(q|9,q|10)->q|12.s(q|10)->q|7.s(q|10)->q|10.s(q|10)->q|12.s(q|10)->q|9.
   plus(q|9,q|8)->q|10. plus(q|9,q|8)->q|12.s(q|10)->q|2.plus(q|9,q|7)->q|2.plus(q|9,q|7)->
   q|9.plus(q|9,q|7)->q|10.plus(q|9,q|7)->q|12.o->q|9.o->q|10.o->q|12.mult(q|1,q|7)->q|9.
   mult(q|1,q|7)->q|10.mult(q|1,q|7)->q|12.plus(q|9,q|7)->q|7.s(q|8)->q|2.o->q|8.s(q|8)->q|7.
   mult(q|6,q|7)->q|7.s(q|1)->q|6.fact(q|1)->q|7.mult(q|6,q|7)->q|2.fact(q|1)->q|2.
   minus(q|1,q|1)->q|5.fact(q|5)->q|3.div(q|2,q|3)->q|0.A(q|1,q|1)->q|0.o->q|1.o->q|5.s(q|1)
   ->q|1.s(q|1)->q|5.nil End of Description
```

The pattern $div(x, 0)$ is not $\mathcal{R}_1$-reachable from $\mathcal{L}(Aut(0))$ if for all ground contexts $C[]$ and all substitutions σ, $C[dix(x, 0)\sigma] \notin \mathcal{L}(\mathcal{T}_{\mathcal{R}_1}\!\uparrow(Aut(0)))$. This can be checked evaluating the following query: `(div(x, o) ?= states) with (Aut(1))`. The result is

```
[] result term:
    nil
```

meaning that there exists no substitution σ and no state $q \in \mathcal{Q}$ s.t. $div(x, 0)\sigma \to_{\Delta}^{*} q$, where $\mathcal{Q}$ and Δ are respectively the set of states and the set of transitions of $Aut(1)$. An interesting aspect of this method is that the automaton $\mathcal{T}_{\mathcal{R}}\!\uparrow(A)$ is computed once for all, and the check itself is a simple and low cost operation. Another advantage is that for computing $\mathcal{T}_{\mathcal{R}}\!\uparrow(A)$, the TRS $\mathcal{R}$ is not supposed to be terminating nor even weakly normalising. This is of great interest when using TRS to encode non-terminating systems, like systems of communicating processes, for example. Note that such non-terminating TRS cannot be handled by induction proof techniques that need a well-founded ordering for proving termination of the TRS.

4.2 Sufficient Completeness

In order to prove sufficient completeness of $A(n, p)$ with $n, p \in Nat$, we first compute the intersection automaton between $Aut(1)$, computed previously, and the automaton recognising the set $IRR(\mathcal{R}_1)$, computed by the function `build_nf(R1)`. The query is `simplify(Aut(1) inter build_nf(R1))`, and the result is:

```
[] result term:
    Description of Aut(2) states q|0.q|1.nil final states q|1.nil
    transitions s(q|0)->q|1.s(q|0)->q|0.o->q|0.nil End of Description
```

Thus, the superset of $\mathcal{R}_1^!(\mathcal{L}(Aut(0)))$ is $\mathcal{L}(Aut(2)) = \mathcal{L}(Aut(2), q_1) = \{s(x) \mid x \in \mathcal{L}(Aut(2), q_0)\}$, and $\mathcal{L}(Aut(2), q_0) = \{0, s(0), \ldots\}$. Thus $\mathcal{L}(Aut(2), q_0) = Nat$, and $\mathcal{L}(Aut(2)) = Nat^*$. Therefore, we trivially have $\mathcal{R}_1^!(\mathcal{L}(Aut(0))) \subseteq Nat^* \subseteq \mathcal{T}(\mathcal{C})$ and if $\mathcal{R}_1$ is weakly normalising on terms $A(n, p)$ with $n, p \in Nat$, then $\mathcal{R}_1$ is also sufficiently complete on those terms. Note that, if $Aut(2)$ is more complex, inclusion between automaton $Aut(2)$ and an automaton recognising exactly $\mathcal{T}(\mathcal{C})$ can also be verified automatically by our prototype.

4.3 Sequential Reduction Strategy

In this third example, we show that sequential reduction strategy is interesting for proving termination of programs combining different methods of termination proof. The following specification defines a function $make\_list(i, j)$, that constructs a list of naturals $(i!, (i+1)!, \ldots, (j-1)!, j!)$. The module $\mathcal{R}_1$ constructs the list and the module $\mathcal{R}_2$ achieves the computation of the factorial function.

```
specification make_list2
Vars    x y z

Ops
  o:0 p:1 s:1 fact:1 plus:2 mult:2 cons:2 int:2
  intlist:1 null:0 fact_list:2 appfact:1

R1
  fact_list(x, y) -> appfact(int(x, y))
  appfact(null) -> null
  appfact(cons(x,y)) -> cons(fact(x),appfact(y))
  intlist(null) -> null
  intlist(cons(x, y)) -> cons(s(x), intlist(y))
  int(o,o) -> cons(o,null)
  int(o,s(y)) -> cons(o, int(s(o), s(y)))
  int(s(x),o) -> null
  int(s(x), s(y)) -> intlist(int(x, y))
  nil
```

```
R2
  p(s(x)) -> x
  mult(o, x) -> x
  mult(s(x), y) -> plus(mult(x, y), y)
  plus(x, o) -> x
  plus(x, s(y)) -> s(plus(x, y))
  fact(s(x)) -> mult(s(x), fact(p(s(x))))
  fact(o) -> s(o)
  nil

Automata
  Description of Aut(0)
  states q|0.q|1.nil
  final states q|0.nil
  transitions
    fact_list(q|1,q|1) -> q|0
    o -> q|1 s(q|1) -> q|1
    nil
  End of Description
  nil
```

Note that neither termination of $\mathcal{R}_1$ nor termination of $\mathcal{R}_2$ can be proven by a simplification ordering. However, termination of $\mathcal{R}_1$ can be proved by the dependency pair method [2], and on the other hand, termination of $\mathcal{R}_2$ can be proved by GPO [8]. Instead of reconsidering the termination of the whole TRS $\mathcal{R}_1 \cup \mathcal{R}_2$, we can automatically verify that the (hierarchical) combination of those two systems is terminating under the sequential reduction strategy, for every initial term from the regular set $\mathcal{L}(Aut(0)) = \mathcal{L}(Aut(0), q_0) = \{fact\_list(n, p) \mid n, p \in \mathcal{L}(Aut(0), q_1)\}$ where $\mathcal{L}(Aut(0), q_1) = \{0, s(0), \ldots\} = Nat$. The query `start(Aut(0))` iterates the process described in Section 2.3, implemented with the `T_up` and `build_nf` operations, until we get a fixpoint. The result of this proof is the following:

```
[] result term:
   [true,Description of nil states q|0.q|1.q|2.q|3.q|4.nil final
   states q|4.nil transitions cons(q|2,q|3)->q|3.null->q|4.null->q|3.cons(q|2,q|3)->q|4.
   s(q|0)->q|1.o->q|0.s(q|1)->q|1.s(q|1)->q|2.s(q|0)->q|2.nil End of Description]
```

where the first field is `true` — the combination is terminating under the sequential reduction strategy — and the second field contains the automaton recognising the superset of the normal forms: lists (possibly empty) of strictly positive natural numbers, which is what was expected by definition of function $make\_list$, and which also proves sufficient completeness of $\mathcal{R}_1 \cup \mathcal{R}_2$ under sequential reduction strategy on $\mathcal{L}(Aut(0))$.

4.4 Testing co-domains of functions

This is a last example showing that computing a superset of the set of normal forms may be of great help also in debugging a functional program. Assume that you have the following program defining a function which reverses a list of elements.

```
specification reverse
Vars    x y z
Ops
    a:0 b:0 rev:1 cons:2 append:2 null:0
R1
    rev(null) -> null
    rev(cons(x, y)) -> append(rev(y), cons(x, null))
    append(null, x) -> null
    append(cons(x, y), z) -> cons(x, append(y, z))
    nil
```

```
Automata
    Description of Aut(0)
    states q|0.q|1.q|2.nil
    final states q|0.nil
    transitions rev(q|1) -> q|0.
            cons(q|2, q|1) -> q|1.
            null -> q|1.
            a -> q|2.
            b -> q|2.
            nil
    End of Description
```

where $\mathcal{L}(Aut(0)) = \mathcal{L}(Aut(0), q_0) = \{rev(l) \mid l \in \mathcal{L}(Aut(0), q_1)\}$, $\mathcal{L}(Aut(0), q_1) = \{null, cons(x, y) \mid x \in \mathcal{L}(Aut(0), q_2), y \in \mathcal{L}(Aut(0), q_1)\}$, and $\mathcal{L}(Aut(0), q_2) = \{a, b\}$. In other words, $\mathcal{L}(Aut(0))$ is of the form $rev(l)$ where l is any flat list of a and b, possibly empty. If we compute the automaton recognising the superset of $\mathcal{R}_1^!(\mathcal{L}(Aut(0)))$, the superset of co-domain, by evaluating the query `simplify(T_up(R1) on(Aut(0)) inter build_nf(R1))`, we obtain:

```
[] result term:
   Description of Aut(1) states q|0.nil final
   states q|0.nil transitions null->q|0.nil End of Description
```

Thus $\mathcal{L}(Aut(1))$, the superset of $\mathcal{R}_1^!(\mathcal{L}(Aut(0)))$, is the singleton $\{null\}$, the empty list. That is clearly not what is expected from the reverse function. If you check TRS $\mathcal{R}_1$ in detail, you will notice that it is wrong: in the third rule of $\mathcal{R}_1$, the right-hand side should be x rather than $null$. The interesting remark here is that $\mathcal{R}_1$ has all usual good properties: it is terminating, confluent, and sufficiently complete on $\mathcal{L}(Aut(0))$. Note also that typing $\mathcal{R}$ would not detect any error. The main interest of the co-domain estimation is to be complementary to usual verification techniques used on TRSs: confluence, termination, sufficient completeness, and typing. After fixing the bug in $\mathcal{R}_1$, we obtain:

```
[] result term:
   Description of Aut(1) states q|0.q|1.q|2.q|3.nil final states q|3.nil
   transitions b->q|1.a->q|1.null->q|3.cons(q|1,q|0)->q|3.cons(q|1,q|0)
   ->q|2.null->q|0.cons(q|1,q|2)->q|2.cons(q|1,q|2)->q|3.nil
   End of Description
```

where $Aut(1)$ recognise any flat list of a and b, possibly empty.

5 Conclusion

We have shown in this work that the computation of regular supersets of $\mathcal{R}$-descendants and $\mathcal{R}$-normal forms using tree automata techniques can provide assistance for checking a few properties of TRSs seen as functional programs.

An important part of this work is devoted to the computation of a regular superset of the set of descendants $\mathcal{R}^*(E)$ for any left-linear TRS $\mathcal{R}$ and any regular set of terms E. The approach proposed here is based on the computation of an approximation automaton recognising a superset of $\mathcal{R}^*(E)$. This approximation seems to be sufficient for our purposes in many practical cases. Approximation of regular language is a notion that was already used in in [16], but in a different way and for a different purpose. In [16], Jacquemard approximates a TRS by another one for which the set of descendants is regular, whereas in our approach, we approximate the set of new states used for normalising transitions, in order to fold recursion when necessary. The set of descendants can be computed exactly thanks to the Tree Tuple Synchronised Grammars (TTSG) approach of non-regular langages proposed in [22]. However, this approach deals with more restricted classes of TRSs; namely linear confluent constructor systems. Moreover, in practice, efficiency of TTSGs for our purposes is not obvious.

A promising application area is the study of non-terminating TRSs encoding the behaviour of systems of communicating processes or systems of parallel processes sharing memory. In this framework, we can prove that there is no deadlock and also some general "reachability" properties: ensure mutual exclusion, ensure that a process never stops, etc. In further research, we intend to compute another regular approximation: a *subset of* $\mathcal{R}^*(E)$ in order to achieve some reachability testing in the other way: for instance to prove that a specific behaviour must occur, we may have to check that a specific pattern *does occur* in the set of $\mathcal{R}$-descendants. We also would like to get rid of the left-linear limitation in order to enlarge the class of programs to be checked, and to compute more precise approximations.

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